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GLOBAL EXISTENCE AND L^1-STABILITY FOR THE DIFFUSION OF THE PARTICLES OF A MIXTURE VIA THE SCATTERING KERNEL FORMULATION OF THE NONLINEAR BOLTZMANN EQUATION

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The global existence and L^1-asymptotic stability of the solutions to a nonlinear evolution problem, in the diffusion of the particles of a mixture, is proved.

1. Introduction.

We consider a mixture of two different kinds of particles, called test particles ($t.p.$) and field particles ($f.p.$). The $t.p.$, having mass m, are injected at time $t = 0$ by a spatially uniform pulsed source $Q^* = QS(v)d(t)$ (the velocity distribution $S(v)$ being nonnegative and normalized to unity) in the interior of an unbounded host medium which consist of the $f.p.$ having mass M and whose total density N is a constant fixed once for all. The $t.p.$ then diffuse in the given host medium by binary collisions either against the $f.p.$ and between themselves. The theory takes into account non only scattering but also removal events in such a way that both scattering and removal collisions frequencies are supposed to be constants. All the mixture can be subject to the action of a general time dependent conservative force $F(t)$.

In the frame of the so-called scattering kernel formulation, the nonlinear integro-differential Boltzmann equation, governing the distribution function $f(v,t)$ of the $t.p.$ for the physical situation illustrated, read as [1, 5, 14]:

$$(1)\quad \begin{aligned}(\partial t + (F(t)/m)\cdot\nabla_{v})f(v,t) = -[N\hat{C} + Cn(t)]f(v,t)+\\ +N\hat{C}_s\int_{\mathbb{R}_3}\hat{\pi}_s(v',v)f(v',t)dv'+\\ +C_s\int_{\mathbb{R}_3}\int_{\mathbb{R}_3}\pi_s(v',v'',v)f(v',t)f(v'',t)dv'dv''\end{aligned}$$

and is to be integrated upon the initial condition

$$(2)\qquad f(v,0) = QS(v).$$

In equation (1)

$$(3)\qquad n(t) = \int_{\mathbb{R}_3} f(v,t)dv$$

is the unknown total density of the $t.p.$ considered, and

$$(4)\qquad \hat{C} = \hat{C}_s + \hat{C}_r,\ C = C_s + C_r$$

are the total microscopic frequencies, scattering plus removal, in the $1/v$ approximation for the relevant cross section, of the $t.p. - f.p.$ and $t.p. - t.p.$ collisions respectively, all being real nonnegative constants. Finally we recall that the scattering probability distributions π_s and π_s are non-negative functions obeying the normalization conditions

$$(5)\qquad \int_{\mathbb{R}_3}\hat{\pi}_s(v',v)dv = 1,\ \int_{\mathbb{R}_3}\pi_s(v',v'',v)dv = 1$$

and π_s is symmetric with respect to the velocities v' and v'' before the collisions, that is

$$(6)\qquad \pi_s(v',v'',v) = \pi_s(v'',v',v).$$

Equation (1) is different, formally, from the homogeneous usual Boltzmann equation but is equivalent to such a formulation by a suitable specification of $\hat{\pi}_s$, π_s, in the case of Maxwellian particles with a cut-off [6, 12, 13].

The physical situation considered above has been recently investigated from the mathematical and physical point of view. We recall that -in the case of no removal- and when the $t.p.-f.p.$ scattering are neglected- the global existence and uniqueness of solutions to equation (1) has been given in the natural space $L^1(\mathrm{IR}^3) \times [0,\infty]$ [2, 3, 9], while in [7, 10, 11] has been studied the problem of stability. In the general case -in which, beside $\dot{C}_s$, also the other three parameters $\hat{C}_s, \hat{C}_r, C_r$ are different from zero- in [1] a local theory of existence and uniqueness for the solution of equation (1) is developed, under some hypothesis on the parameters of the problem which are restrictive and a-priori artificial from the physical point view. In the present paper, by using a weighted norm [9], we prove global existence and uniqueness for the solutions to equation (1) (Sect.2), and its asymptotic stability (Sect.3).

2. Existence and uniqueness of the solution to equation (1).

Let us integrate equation (1) along the general trajectoty of a $t.p.$ between the initial and the reference time t. Setting

$$K(t,\tau) = H(t)/H(\tau) \tag{7}$$

$$H(t) = \exp(-N\hat{C}t)[N\hat{C}_r + QC_r - QC_r \exp(-N\hat{C}_r t)]^{-C/C_r}, \tag{8}$$

indicated by A the nonlinear inhomogeneous operator defined by

$$\begin{aligned} Af(v,t) = {} & Q/(N\hat{C}_r)S[v(0)]\exp(-N\hat{C}t)[N\hat{C}_r \\ & + QC_r(1-\exp(-N\hat{C}_r t)]^{-C/C_r} + \\ & + \int_0^t dt[H(t)/H(\tau)]\left\{N\hat{C}_s \int_{\mathrm{IR}_3} \hat{\pi}_s[(v',v(\tau)]f(v',\tau)dv'\right. \\ & + C_s \int_{\mathrm{IR}_3}\int_{\mathrm{IR}_3} \pi_s[v',v'',v(\tau)]f(v',\tau)f(v'',\tau)dv'dv'' \end{aligned} \tag{9}$$

we can then recast eq.(1) in the operational form

$$Af(v,t) = f(v,t). \tag{10}$$

Let T be any positive constant and denote by E the Banach functional space of the functions $\varphi(v,t)$, defined on $\mathbb{R}_3 \times [0,T]$ which are continuous in t for almost every $v \in \mathbb{R}_3$, and summable in v for any $t \in [0,T]$:

$$E = \{\varphi(v,t) : \mathbb{R}_3 \times [0,T] \to \mathbb{R}; \varphi(v,\cdot) \in L_1(\mathbb{R}_3), \varphi(\cdot,t) \in C[0,T]\} \tag{11}$$

Let denote moreover by

$$\|\varphi(v,t)\|_t = \int_{\mathbb{R}_3} |\varphi(v,t)| dv \tag{12}$$

the L_1-norm with respect to v. For the norm in E we shall take

$$\big|\|\varphi(v,t)\|\big| = \max_{[0,T]} \|\varphi(v,t)\|_t. \tag{13}$$

In the sequel we equip the space E with the weighted norm

$$\big|\|\varphi(v,t)\|\big|w = \max_{[0,T]} \big|\|\varphi(v,t)\| \exp(-\lambda t)\|\big|, \lambda > 0 \tag{14}$$

which is equivalent to the norm (13) because

$$\exp(-\lambda T)\big|\|\varphi(v,t)\|\big| \leq \big\|\varphi(v,t)\|\big|w \leq \big|\|\varphi(v,t)\|\big|. \tag{15}$$

Furthermore, we indicate by B_r the closed set of E

$$B_r = \{\varphi(v,t) \in E : \big\|(v,t)\|\big| \leq r, \big|\|\varphi(v,t)\|\big|w \leq r\}. \tag{16}$$

Indicated by

$$\gamma = \sup_{t\in[0,\infty)} \left[QK(t,0) + (N\hat{C}_s r + C_s r^2) \int_0^t K(t,u)du\right] \tag{17}$$

the following theorem holds

THEOREM 1. *For any positive finite T in the space E equipped with the norm (19) with:*

$$\gamma \leq r \tag{18}$$

one has $AB_r \subseteq B_r$ and moreover the operator A is a contractive mapping on B_r. Hence we obtain global existence and uniqueness for a solution to equation (1).

Proof. We now verify that A maps B into itself, namely $AB_r \subseteq B_r$. Take the modulus of equation (9) and integrate over $v \in \mathbb{R}_3$ for the normalization of S, $\hat{\pi}_s$, π_s we get

$$(19) \qquad ||A\varphi||_t \leq QK(t,0) + N\hat{C}_s \int_0^t K(t,u)||\varphi||_u du + C_s \int_0^t K(t,u)||\varphi||_u^2 du$$

and then, by a further majorization

$$(20) \qquad ||A\varphi|| \leq QK(t,0) + (N\hat{C}_s |||\varphi||| + C_s |||\varphi|||^2) \int_0^t K(t,u)du.$$

Hence we obtain

$$(21) \qquad |||A\varphi|||w \leq |||A\varphi||| \leq \gamma \ \text{ for } \varphi \in Br.$$

Taking into account (18), we easily obtain:

$$(22) \qquad |||A\varphi|||_w \leq r \ \text{ for } \ \varphi \in Br.$$

We prove now that the operator A is a contraction on Br. Accounting for the symmetry of π_s with respect to the velocities before the collisions, for φ, $\psi \in Br$, we have successively

$$(23) \qquad \begin{aligned} |A\varphi - A\psi| \leq\ & N\hat{C}_s \int_0^t du K(t,u) \int_{\mathbb{R}_3} \hat{\pi}_s[v', v(u)] |\varphi(v',u) - \psi(v',u) dv' + \\ & + C_s \int_0^t du K(t,u) \int_{\mathbb{R}_3} \int_{\mathbb{R}_3} \pi_s[v', v'', v(u)] |\varphi(v',u) + \psi(v',u)| \times \\ & |\varphi(v'',u) - \psi(v'',u)| dv' dv'' \end{aligned}$$

where the symmetry of π_s has been used,

$$(24) \qquad \begin{aligned} & ||A\varphi - A\psi||_t \exp(-\lambda t) \leq \\ & \leq N\hat{C}_s \exp(-\lambda t) \int_0^t du K(t,u) \exp(\lambda u) |||\varphi - \psi|||w + \\ & + 2C_s r \exp(-\lambda t) \int_0^t \exp(\lambda u) K(t,u) |||\varphi - \psi|||_w du. \end{aligned}$$

Hence we obtain

$$\|A\varphi - A\psi\|_w \leq (2C_s r + N\hat{C}_s)/(\hat{C}N + \lambda)\big|\|\varphi - \psi\|\big|_w. \tag{25}$$

Choosing

$$\lambda \geq [2C_s r + N(\hat{C}_s - \hat{C})] \tag{26}$$

the theorem is completely acquired by the Caccioppoli-Banach point fix theorem.

We observe that when the scattering of the *t.p.* against the *f.p.* is ignored ($\hat{C}_s = 0$) the existence and uniqueness of the solution to equation (1) follows without any condition. In fact, in this case, following [4], equation (1) can be transformed by introducing a new dependent variable g defined by

$$f(v,t) = n(t)g(v,t). \tag{27}$$

Then, we obtain the following evolution equation for g

$$\begin{aligned} \frac{\partial g(v,t)}{\partial t} + C_s n(t) g(v,t) = C_s n(t) \int_{\mathbb{R}_3}\int_{\mathbb{R}_3} \pi_s(v',v',v) g(v',t)\times \\ g(v'',t)dv'dv''; \quad g(v,0) = S(v). \end{aligned} \tag{28}$$

Then introducing the new independent variable

$$\tau = \int_0^t n(\tau)dt = \frac{C_s}{C_r} ln\{[(N\hat{C}_r + C_r Q) - C_r Q \exp(-N\hat{C}_r t)]/(N\hat{C}_r)\} \tag{29}$$

equation (28) can be rewritten as

$$\begin{cases} \dfrac{\partial g(v,\tau)}{\partial \tau} + g(v,t) = \displaystyle\int_{\mathbb{R}_3}\int_{\mathbb{R}_3} \pi_s(v',v',v) g(v',\tau) g(v'',\tau) \\ g(v,0) = S(v) \end{cases} \tag{30}$$

In [3, 9] the global existence and uniqueness of solutions to equation (30) has been given in the natural space $L_1(\mathbb{R}_3) \times [0,\infty)$. Consequently substituting equation (27) in (30) we obtain the existence and uniqueness of the solution to equation (1) for $\hat{C}_s = 0$. We note finally that in the case of no removal and when the $t.p.-f.p.$ scattering are neglected the results of [3-9] are reproduced.

3. Stability in the L^1-norm.

Let f(v, t) and g(v, t)=f+u be two solutions to equation (1) corresponding respectively to the initial data (2) and

$$g(v,0) = Q[S(v) + S_1(v)], \quad \int_{\mathbb{R}_3} S_1(v)dv = 0 \tag{31}$$

Indicated by

$$||u(v,t)|| = \int_{\mathbb{R}_3} u(v,t)|dv \tag{32}$$

the L^1-norm of the perturbation $u(v,t)$ to the basic solution the following theorem holds.

THEOREM 2. *If $C_r > 0$ the solutions to equation (1) are asymptotically exponentially stable in the L^1-norm.*

Proof. The perturbation "$u(v,t)$" to the "basic" solution $f(v,t)$ shall obey to the nonlinear integral equation

$$\begin{aligned} u(v,t) &= QS_1(v)\exp\left[-\hat{C}Nt - C\int_0^t n(u)du\right] + \\ &+ \int_0^t \exp\left[-\hat{C}N(t-u) - C\int_0^t n(\xi)d\xi\right] \times \\ &\times \left[C_s\int_{\mathbb{R}_3}\int_{\mathbb{R}_3} dv'dv''\pi_s(v',v'',v)(f(v',u) + g(v',u)u(v'',u) + \right. \\ &\left. + N\hat{C}_s\int_{\mathbb{R}_3} dv'\pi_s[v',v(u)]u(v',u)\right] du. \end{aligned} \tag{33}$$

In force of the normalization of $\hat{\pi}_s$ and π_s and of the symmetry condition (6), taking the modulus of equation (33) and integrating over $v \in \mathbb{R}_3$, we obtain

$$\begin{aligned} ||u(v,t)||_t \leq Q||S_1|| \exp\left[-\hat{C}Nt - C\int_0^t n(\xi)d\xi\right] + \\ + \int_0^t (2C_s n(u) + N\hat{C}_s)||u||_u \\ \exp\left[-\hat{C}N(t-u) - C\int_u^t n(\xi)d\xi\right] du \end{aligned} \tag{34}$$

then, setting

$$||\Psi||_t = \exp\left[\hat{C}Nt + C\int_0^t n(\xi)d\xi\right] ||u||_t \tag{35}$$

it follows

$$||\Psi||_t \leq Q||S_1|| + \left[\int_0^t (2C_s n(u) + N\hat{C}_s)||\Psi||_u du\right]. \tag{36}$$

Taking into account the Gronwall-lemma we obtain

$$||\Psi||_t \leq Q||S_1|| \exp\left[\int_0^t 2C_s n(u) + N\hat{C}_s)du\right] \tag{37}$$

which implies

$$\begin{aligned} ||u||_t \leq Q||S_1|| \exp(-\hat{C}_r Nt) \times \\ \times [(N\hat{C}_r + C_r Q) - C_r Q \exp(-N\hat{C}_r t)]/(N\hat{C}_r)]^{(C_s - C_r)/C_r}. \end{aligned} \tag{38}$$

Then setting

$$A = \max[(N\hat{C}_r + C_r Q)/(N\hat{C}_r)^{(C_s - C_r)/C_r}, 1] \tag{39}$$

it follows

$$||u||_t \leq AQ||S_1|| \exp(-\hat{C}_r Nt) \tag{40}$$

and the theorem is proved.

Remark. If $C_r = 0$ from (1) we obtain

$$n(t) = Q\exp(-N\hat{C}_r t). \tag{41}$$

Then, accounting for (37), it follows

$$\|\psi\|_t \le Q\|S_1\|\exp[2C_s Q(1-\exp(-N\hat{C}_r t))/N\hat{C}_r]\exp(N\hat{C}_s t) \tag{42}$$

and therefore

$$\|u\|_t \le Q\|S_1\|\exp(2CsQ/N\hat{C}_r)\exp(-N\hat{C}rt) \tag{43}$$

which proves theorem 2 for $C_r = 0$.

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