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NONLOCAL THEORY OF LONGITUDINAL WAVES IN THERMOELASTIC BARS

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The longitudinal waves in thermoelastic bars are investigated in the context of nonlocal theory.Using integral forms of constitutive equations, balance of momenta and energy, field equations are obtained.Then the frequency equation is found in generalized form. To obtain tangible results, an approximate procedure is applied and numerical results are given for short waves.

1. Introduction.

The purpose of this work is to investigate the longitudinal waves in thermoelastic bars in the context of nonlocal theory. One of the characteristic result of the wave propagation in rods is a geometrical dispersion of waves produced by the presence of the boundaries. This result is obtained by the use of local theory of elasticity.It has been shown by Nowinski [5] that second type of dispersion appears by the application of the nonlocal theory.Present work is an extension of this problem to the nonlocal thermoelasticity.

2.Fundamental Equations.

We consider an infinitely long rod of a circular cross-section of radious R. The material of the rod is homogeneous, isotropic and elastic.Taking into account the thermal effects, the constitutive equations are [1]

$$(2.1) \quad t_{kl} = \lambda e_{mm}\delta_{kl} + 2\mu e_{kl} - \alpha T\delta_{kl} + \int_V [\hat{\lambda}' e'_{mm}\delta_{kl} + 2\hat{\mu}' e'_{kl} - \hat{\alpha}' T'\delta_{kl}]dv'$$

Here λ,μ are Lamé constants,$\hat{\lambda}'$ and $\hat{\mu}'$ are nonlocal elastic moduli which are functions of the distance between an arbitrary point of the body, $\mathbf{x}'$, and the point under consideration,x, i.e.,

$$(2.2, a-b) \qquad \hat{\mu}' = \hat{\mu}'(|\mathbf{x}' - \mathbf{x}|), \qquad \hat{\lambda}' = \hat{\lambda}'(|\mathbf{x}' - \mathbf{x}|)$$

α is a thermal coefficient corresponding to local thermal expansion and $\hat{\alpha}'$ denotes its nonlocal part. T is the temperature function. A more convenient form of (2.1) was given by Eringen [2] by incorporating λ, μ and α into $\hat{\lambda}'$, $\hat{\mu}'$ and $\hat{\alpha}'$ .Then the nonzero stress components refering to a cylindirical coordinate system (r,θ ,z) can be rewritten as follows,

$$(2.3.a-d) \qquad \begin{aligned} t_{rr} &= \int_V \left[(\lambda' + 2\mu')\frac{\partial u'}{\partial r'} + \lambda'\left(\frac{u'}{r'} + \frac{\partial w'}{\partial z'}\right) - \alpha' T'\right] dv' \\ t_{\theta\theta} &= \int_V \left[(\lambda' + 2\mu')\frac{u'}{r'} + \lambda'\left(\frac{\partial u'}{\partial r'} + \frac{\partial w'}{\partial z'}\right) - \alpha' T'\right] dv' \\ t_{rz} &= \int_V \mu'\left(\frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial r'}\right) dv' \\ t_{zz} &= \int_V \left[(\lambda' + 2\mu')\frac{\partial w'}{\partial z'} + \lambda'\left(\frac{\partial u'}{\partial r'} + \frac{u'}{r'}\right) - \alpha' T'\right] dv' \end{aligned}$$

where u and w are radial and longitudinal displacements, respectively. The equations of motions and the equation of balance of energy are

$$(2.4, a-c) \qquad \begin{aligned} \frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{rz}}{\partial z} + \frac{t_{rr} - t_{\theta\theta}}{r} &= \rho\ddot{u} \\ \frac{\partial t_{rz}}{r} + \frac{\partial t_{zz}}{\partial z} + \frac{t_{rz}}{r} &= \rho\ddot{w} \\ \nabla^2 T - \frac{1}{\kappa}\dot{T} - \eta\dot{\theta} + \int_V [\nabla'^2 T' - \frac{1}{\hat{\kappa}'}\dot{T}' - \hat{\eta}'\dot{\theta}']dv' &= 0 \end{aligned}$$

Here κ and η are the thermoelastic constants corresponding to local thermal conducticity and coupling term between the strain and thermal fields respectively and $\hat{\kappa}'$, $\hat{\mu}'$ are their nonlocal parts and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad \theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \tag{2.5}$$

We introduce Fourier transform

$$\overline{f}(r,k,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,z,t)exp[i(kz+\omega t)]dzdt \tag{2.6}$$

and apply the Fourier transformation and Faltung theorems to the equation (2.4,c) to obtain

$$\overline{T}_{,rr} + \frac{1}{r}\overline{T}_{,r} + p^2\overline{T} + \overline{\eta}i\omega\overline{\theta} = 0 \tag{2.7}$$

Here, it is considered that the nonlocal moduli are to be distributions with bounded supports.They are positive functions in a finite domain about the reference point and vanishing rapitly out of this domain.Then we consider that the functions that characterize the particle interactions along radial direction may be approximated in terms of Dirac delta sequences and we may write

$$\int_0^R \delta_n(|r' - r|)\phi(r')dr' \approx \int_0^\infty \delta(|r' - r|)\phi(r')dr' \approx \phi(r) \tag{2.8}$$

Applying the same procedure to the equations (2.3,a-d) and substituting the obtained results into the equations (2.4,a-b) we find

$$\overline{\Omega}_{,rr} + \frac{\overline{\Omega}_{,r}}{r} + \left(l^2 - \frac{1}{r^2}\right)\overline{\Omega} = 0$$
$$\overline{\theta}_{,rr} + \frac{\overline{\theta}_{,r}}{r} + h^2\overline{\theta} - \frac{\Gamma_4}{\Gamma_1}\left(\overline{T}_{,rr} + \frac{\overline{T}_{,r}}{r} - k^2\overline{T}\right) = 0 \tag{2.9, a-b}$$

where

$$\overline{\theta} = \frac{\partial\overline{u}}{\partial r} + \frac{\overline{u}}{r} - ik\overline{w}, \qquad \overline{\Omega} = -(ik\overline{u} + \overline{w}_{,r}) \tag{2.10, a-b}$$

$$\Gamma_1 = \Gamma_2 + 2\Gamma_3, \qquad \Gamma_2 = \overline{\lambda}'(k), \quad \Gamma_3 = \overline{\mu}'(k), \qquad \Gamma_4 = \overline{\alpha}'(k) \tag{2.11, a-d}$$

$$(2.12, a-c) \qquad l^2 = \frac{\rho\omega^2}{\Gamma_3} - k^2, \qquad h^2 = \frac{\rho\omega^2}{\Gamma_1} - k^2, \qquad p^2 = \frac{i\omega}{\kappa} - k^2$$

and bared functions denote the Fourier transforms of the related function. The solution of equation (2.12,a) finite at $r = 0$ is

$$(2.13) \qquad \overline{\Omega} = -B(l^2 + k^2)J_1(lr)$$

and J_1 denotes Bessel function of the first kind. Considering Equations (2.9, a-b) together we find the solutions as

$$(2.14, a-b) \qquad \begin{aligned} \overline{\theta} &= -\left[A_1(\lambda_1^2 + k^2)J_0(\lambda_1 r) + A_2(\lambda_2^2 + k^2)J_0(\lambda_2 r)\right] \\ \overline{T} &= \overline{\eta}\omega i\left[A_1\frac{\lambda_1^2 + k^2}{p^2 - \lambda_1^2}J_0(\lambda_1 r) + A_2\frac{\lambda_2^2 + k^2}{p^2 - \lambda_2^2}J_0(\lambda_2 r)\right] \end{aligned}$$

Here λ_1^2, λ_2^2 are the roots of

$$(2.15) \qquad (\lambda^2 - h^2)(\lambda^2 - p^2) - i\omega\varepsilon(\lambda^2 + k^2) = 0$$

and

$$(2.16) \qquad \varepsilon = \frac{\Gamma_4}{\Gamma_1}\overline{\eta}$$

Assuming the surface of the bar is free of traction and held at constant temperature, we obtain three algebraic equations for three unknowns. For nontrivial solution we set the determinant of the coefficient matrix of these equations to zero which leads to

$$(2.17) \qquad \begin{aligned} &\frac{l^2 - k^2}{2k^2}\left[\frac{\lambda_2^2 + k^2}{p^2 - \lambda_2^2}\frac{\Phi(\lambda_2 R)\kappa_1}{\lambda_2 R J_1(\lambda_1 R)} - \frac{\lambda_1^2 + k^2}{p^2 - \lambda_1^2}\frac{\Phi(\lambda_1 R)\kappa_2}{\lambda_1 R J_1(\lambda_2 R)}\right] \\ &+ \frac{2\Gamma_3}{R^2}[\Phi(lR) - 1]\left[\frac{\lambda_1^2 + k^2}{p^2 - \lambda_1^2}\Phi(\lambda_1 R)\frac{\lambda_2}{\lambda_1} - \frac{\lambda_2^2 + k^2}{p^2 - \lambda_2^2}\Phi(\lambda_2 R)\frac{\lambda_1}{\lambda_2}\right] = 0 \end{aligned}$$

Here

$$(2.18, a-b) \qquad \begin{aligned} \kappa_1 =& 2\Gamma_3\frac{J_1(\lambda_1 R)}{\lambda_1 R}[1 - \Phi(\lambda_1 R)]\lambda_1^2 - \Gamma_2 J_0(\lambda_1 R)\frac{i\omega}{\kappa} \\ &- \varepsilon J_0(\lambda_1 R)\left[\Gamma_2 g(\omega) - \frac{\omega^2}{\kappa}\frac{\Gamma_1}{p^2 - k^2}\right] \\ \kappa_2 =& 2\Gamma_3\frac{J_1(\lambda_2 R)}{\lambda_2 R}[1 - \Phi(\lambda_2 R)]\lambda_2^2 - \Gamma_2 J_0(\lambda_2 R)\frac{\rho\omega^2}{\Gamma_1} \\ &- \varepsilon J_0(\lambda_2 R)\left[\Gamma_2\overline{h}(\omega) + i\omega\frac{\rho\omega^2}{p^2 - k^2}\right] \end{aligned}$$

$$\Phi(\lambda_i R) = \frac{J_0(\lambda_i R)}{J_1(\lambda_i R)}\lambda_i R \tag{2.19}$$

$$g(\omega) = \frac{p^2 + k^2}{p^2 - h^2} i\omega \qquad \overline{h}(\omega) = -\frac{h^2 + k^2}{p^2 - h^2} i\omega \tag{2.20, a - b}$$

$$\lambda_1^2 = p^2 + g(\omega)\varepsilon \qquad \lambda_2^2 = h^2 + \overline{h}(\omega)\varepsilon \tag{2.21, a - b}$$

3.Nonlocal Moduli.

The nonlocal moduli λ', μ', α', η' and κ' are unknown functions of the distance $|\mathbf{x}-\mathbf{x}'|$. λ' and μ', which are nonlocal elastic moduli can be determined by matching the dispersion relation for plane waves with the corresponding equation in lattice dynamics . For one-dimensional lattice model (which is known as Born-Kármán model), dispersion relation within one Brillouin zone is given by

$$\frac{\omega^2}{\omega_0^2} = \frac{\sin^2\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)^2} \tag{3.1}$$

where a is atomic distance.To obtain identical results to the atomic theory, Eringen [2] has expressed the nonlocal moduli for homogeneous and isotropic solids in the following forms:

$$\lambda'(|x - x'|) = \lambda\alpha(|x - x'|), \qquad \mu'(|x - x'|) = \mu\alpha(|x - x'|) \tag{3.2}$$

where

$$\begin{aligned} \alpha(x) &= \frac{1}{a}\left(1 - \frac{|x|}{a}\right), \qquad \frac{|x|}{a} \leq 1 \\ &= 0, \qquad\qquad\qquad \frac{|x|}{a} \geq 1 \end{aligned} \tag{3.3}$$

The coefficient of thermal conductivity κ can be expressed in terms of the heat capacity of phonon, average phonon velocity and the mean free path l. The important point here is that the mean free path can not be shorter than the wavelength of the phonon and the wavelength can not be shorter than a typical distance between two neighbouring atoms [3]. Combining this result with Eringen's approach, which gives perfect agreement with the atomic theory, the nonlocal heat conduction coefficient κ' can be expressed in terms of the local heat conduction coefficient and a kernel $\alpha_l(|x-x'|)$ as

$$\kappa'(|x-x'|) = \kappa\alpha_l(|x-x'|) \tag{3.4}$$

where

$$\alpha_l(x) = \frac{1}{l}\left(1-\frac{|x|}{l}\right), \qquad \frac{|x|}{l} \leq 1$$
$$= 0, \qquad \frac{|x|}{l} \geq 1 \tag{3.5}$$

Now the heat conduction coefficient $\overline{\kappa}'(k)$ is obtained by taking the Fourier transform of equation (3.4). Then we find

$$\frac{\overline{\kappa}'}{\kappa} = \frac{1}{\sqrt{2\pi}}\frac{\sin^2\left(\frac{kl}{2}\right)}{\left(\frac{kl}{2}\right)^2} = \Phi(kl) \tag{3.6}$$

Applying the same idea to the next two coefficients, we write

$$\gamma'(|x-x'|) = \gamma\alpha_b(|x-x'|), \qquad \lambda_t'(|x-x'|) = \lambda_t\alpha_d(|x-x'|) \tag{3.7, a - b}$$

Here, α_b and α_d are defined similar to the form given by equation (3.5) and b and d denote the range of influence of the associated kernels.They can not be shorter than atomic distance a.

As it is known we obtain no frequency higher than the cut-off frequency ω_0. Then we restrict our investigation to the region $\frac{-\pi}{2} \leq \frac{ka}{2} \leq \frac{\pi}{2}$ which is called first Brillouin zone. At the ends of Brillouin zone we write

$$\frac{\partial\omega}{\partial k}\Big|_{\frac{ka}{2}=\pm\frac{\pi}{2}} = 0 \tag{3.8}$$

Considering this fact for the nonlocal thermal waves we determine the range of influence of the thermal coefficients. For the nonlocal thermal waves, the phase velocity c_t is given as [4]

$$c_t = \frac{\omega}{k} = c_1^* \sqrt{\Omega \frac{\Phi(kl)}{\Phi(kb)}} \tag{3.9}$$

Now applying the condition (3.8) to equation (3.9), we find

$$1 \leq \theta \leq 1.29 \tag{3.10}$$

Here θ is defined as $\theta = \frac{l}{b}$. If we apply the same condition to the phase velocity of modified thermoelastic waves , we find the following table for b, d, l [4]:

θ	d/a	b/a
1	1	1.4571
1.1	1	1.2350
1.2	1	1.1000

4. Solution of Frequency Equation.

Just as in the local theory, equation (2.17) shows dispersive character of the wave propagation in bars. A general discussion of this equation presents considerable diffuculties.For this reason we resort to an approximate solution and express the frequency equation in the following form

$$F(\omega, k) + \varepsilon G(\omega, k) = 0 \tag{4.1}$$

Here

$$\begin{aligned} F(\omega, k) &= \Phi(pR)k^2(kR)[(\beta^*\zeta - 1)\sqrt{2\zeta - 1} + (\zeta - 1)^2\sqrt{\beta^*\zeta - 1}] \\ G(\omega, k) &= g(\omega)\Phi(pR)(lR)(hR)G_0(\omega) \end{aligned} \tag{4.2, a - b}$$

$$G_0(\omega) = \left(1 - \frac{1}{lR}\right)\left[\frac{hR}{pR}\left(\frac{1}{pR} - 1\right) - \frac{1}{hR}\frac{1}{\alpha^*\zeta} + \frac{1}{pR}\frac{\beta^*}{\alpha^* - \beta^*}\right.$$

$$\left.\frac{\alpha^*\zeta - 1}{\alpha^*\zeta}\right] + (\zeta - 1)\left[\frac{\beta^*}{\beta^* - \alpha^*}\left\{\frac{1}{pR}\left(\frac{1}{lR} - \frac{hR}{lR}\right)\frac{\alpha^*\zeta - 1}{\alpha^*\zeta}\right.\right.$$

$$(4.3) \qquad \left.\left. - \frac{1}{lR}\frac{1 - \beta^*}{\beta^*}\right\}\right] - (\zeta - 1)\left\{\left[\frac{1}{pR}\left(\frac{1}{pR} - 1\right) + \frac{1}{(p^2 + k^2)R^2}\right]\right.$$

$$\left[\frac{hR}{lR} + \frac{(kR)^2}{(lR)}(1 - \zeta)\right] - \frac{\beta^*}{\alpha^*}(1 - \zeta)\frac{(kR)^2}{lRhR}\left(\frac{1}{hR} - 1\right)$$

$$\left. - \frac{\beta^*}{\alpha^*}\frac{1}{lRhR} + \frac{1}{lR\alpha^*} - \zeta\left(1 - \frac{\beta^*}{\alpha^*}\right)\frac{1}{lR}\frac{1}{\alpha^*\zeta - 2}\right]$$

$$(4.4, a - f) \qquad \beta^* = \frac{2\Gamma_3}{\Gamma_1}, \qquad \alpha^* = \frac{i}{\kappa\omega}\frac{2\Gamma_3}{\rho}, \qquad c_0^2 = \frac{2\Gamma_3}{\rho}, \qquad c^2 = \left(\frac{\omega}{k}\right)^2$$

$$\zeta = \left(\frac{c}{c_0}\right)^2, \qquad \Omega = \frac{\omega\kappa}{c_0^2}$$

$$(4.5, a - c) \quad pR = kR\sqrt{\alpha^*\zeta - 1}, \quad hR = kR\sqrt{\beta^*\zeta - 1}, \quad lR = kR\sqrt{2\zeta - 1}$$

$$(4.6) \qquad \Phi(\lambda_i R) = \Phi(\lambda_{i0} R) + \Phi_\varepsilon(\lambda_{i0} R)\varepsilon \qquad i = 1, 2.$$

For $\varepsilon = 0$ we obtain the nonlocal elastic solution given by Nowinski [5]. It is obvious that $F(\zeta_0) = 0$ for $\varepsilon = 0$. Now we may consider that δ is the increment of the value ζ due to $\varepsilon \neq 0$, then equation (4.1) can be rewritten in the form of

$$(4.7) \qquad F(\zeta_0 + \delta) = -G(\zeta_0 + \delta)\varepsilon$$

Now by the assumption that δ is a small quantity, we may expand both sides of the equation (4.1) into Taylor series in the vicinity of the point ζ_0 and keep only the first two terms.This leads to

$$(4.8) \qquad \delta(\zeta_0, \varepsilon) = -\left[\frac{G(\zeta_0)}{\frac{\partial F}{\partial \zeta}|_{\zeta_0}}\right]\varepsilon$$

In general δ is a complex number. Its real part denotes the change of the phase velocity and imaginary part shows the attenuation constant. Here we are mainly interested in with two limit cases. For long waves one has to consider the limits at $k \to 0$. Calculations give us that the nonlocal and local aspects of the long wave limit become identical as it is expected. For short waves i.e. $k \to \infty$,the arguments of Bessel functions become imaginary , then equation (4.2,a) simplifies to the cubic equation [5]

$$F(\zeta) = \zeta^3 - 4\zeta^2 + (6 - 2\beta^*)\zeta + (\beta^* - 2) \tag{4.9}$$

and equation (4.3) turns out to be

$$\begin{aligned} G_0(\zeta) = -\frac{i\kappa}{c_0\sqrt{\zeta}R}\Big[-\frac{1+\beta^*\zeta}{\sqrt{\beta^*\zeta - 1}} + \frac{\beta^*\sqrt{\Omega}}{i - \beta^*\Omega}\frac{\zeta i - \Omega}{\sqrt{\zeta i - \Omega}} \\ \left[1 + (\zeta - 1)\frac{\sqrt{\beta^*\zeta - 1}}{\sqrt{2\zeta - 1}}\right] + \frac{1-\zeta}{\sqrt{2\zeta - 1}} + \frac{i - \beta^*\Omega}{\sqrt{2\zeta - 1}}\frac{\zeta^2(\zeta - 1)}{i\zeta - 2\Omega}\Big] - \beta^*\Omega i \end{aligned} \tag{4.10}$$

Numerical calculations are carried out for a circular bar made of aliminum for which $a \approx 4.10^{-8}$ cm and for $\theta = 1.1$, $\frac{d}{b} = 1$ and

$\lambda = 5.55\ 10^5 kg/cm^2$, $\mu = 2.612\ 10^5 kg/cm^2$ (at $\mathrm{T}_0 = 20^0\mathrm{C}$)

$\kappa = 0.61\mathrm{cm}^2/\mathrm{sec}$, $\rho = 2.75\mathrm{gr/cm}^3, \mathrm{c}_0 = 3.04\ 10^5\mathrm{cm/sec}$,

$\eta = 939.52\mathrm{sec}^0\mathrm{C/cm}^2$, $\varepsilon = 0.0368$, $\beta^* = 0.4849$, $\zeta_0 = 0.435$.

Ω	δ_r	δ_i	$\frac{\Delta c}{c_o}$
10^{-6}	$2.101\ 10^{-3}$	$0.224\ 10^{-6}$	$2.314\ 10^{-3}$
10^{-5}	$2.013\ 10^{-3}$	$0.172\ 10^{-5}$	$2.313\ 10^{-3}$
10^{-4}	$2.012\ 10^{-3}$	$0.154\ 10^{-4}$	$2.312\ 10^{-3}$
10^{-3}	$2.008\ 10^{-3}$	$0.149\ 10^{-3}$	$2.307\ 10^{-3}$
10^{-2}	$1.991\ 10^{-3}$	$0.147\ 10^{-2}$	$2.288\ 10^{-3}$
10^{-1}	$1.287\ 10^{-3}$	$0.146\ 10^{-1}$	$1.480\ 10^{-3}$

We note that the dimensionless frequency Ω for mechanical vibrations occuring in practice is much smaller than unity ($\Omega \ll 1$). Above table tells us that the change of the phase velocity is decreasing with the increasing value of the frequency. But these changes are less than it is observed in the local theory. The situation is different for attenuation constant. It is increasing with increasing Ω as is expected.

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