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THERMODYNAMICS OF LIGHT AND SOUND

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Light in a cavity and sound in a solid may be considered as gases of quasiparticles, the photons and photons respectively. It is then possible to treat them thermodynamically. The first successful attempt in that direction was Boltzmann's derivation of what we now call the Stefan-Boltzmann law. That law relates the energy density of cavity radiation in equilibrium with the wall to the temperature of the wall.

Radiation hydrodynamics or radiation thermodynamics is important for the theory of stellar structure, because the radiation pressure in stars equilibrates the gravitational pull. Eddington, a pioneer of stellar physics, was able to derive the general form of the stress tensor of radiation which he wrote as

$$P_{ij} = e\left(\frac{1-\chi}{2}\delta_{ij} + \frac{3\chi-1}{2}\frac{p_i p_j}{p^2}\right).$$

e and p_i are the densities of energy and momentum. χ is called the Eddington factor, it is a function of e and p^2.

The form of that function is much discussed in the literature. As recently as 1984 Levermore [1] has compared several propositions for the Eddington factors. More recently Anile, Pennisi and Sammartino [2] have approached the problem by use of extended thermodynamics. They showed that χ must have the form

$$\chi = \frac{5}{3} - \frac{2}{3}\sqrt{4 - 3\frac{c^2 p^2}{e^2}}.$$

That form was also confirmed by Kremer and Müller [3] by a different approach.

This paper presents a thermodynamic theory of light and sound. It demonstrates that extended thermodynamics permits the explicit calculation of the main part of the equations of balance of energy for photons and phonons. Wave speeds are calculated and the limiting cases of near-equilibrium and free streaming are discussed.

1. Phonons and Photons.

1.1. *Cavity Radiation and Sound in Single Crystals.*

Light and Sound are waves, electro-magnetic and elastic respectively. But in some ways, which the physicists understand they may be considered as a gas of particles, the photons and phonons.

Thus a wave of frequency ω and wave number $\mathbf{k}$ corresponds to a particle of energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$. The frequency and the wave number are not independent, of course. For light we have

$$\omega = ck, \tag{1.1}$$

where c is the speed of light. For sound the *dispersion relation* $\omega = \omega(\mathbf{k})$ is more complex, but it reduces to (1.1) – with c as the speed of sound – for longitudinal sound waves of small frequencies. This is the only case we shall consider here so that equation (1.1) holds for both photons and phonons. We shall refer to these particles as photons.

To fix the ideas about the systems under consideration we have drawn Figure 1. On the left hand side we show a cylinder closed off by a piston and filled with radiation, i.e. a gas of photons.

In equilibrium the temperature of the wall of this cavity is T everywhere and the photons exert a pressure p on the piston. Kirchhoff's experiments – more than 100 years ago – have shown that the energy density of the radiation is a function only of temperature

$$e = e(T) \tag{1.2}$$

On the right hand side of Figure 1 we see a single crystal specimen of some solid «filled with sound», a gas of phonons. In equilibrium the temperature is uniformly equal to T and specific heat measurements show that the energy density is only a function of T, at least if we can assume incompressibility of the body.

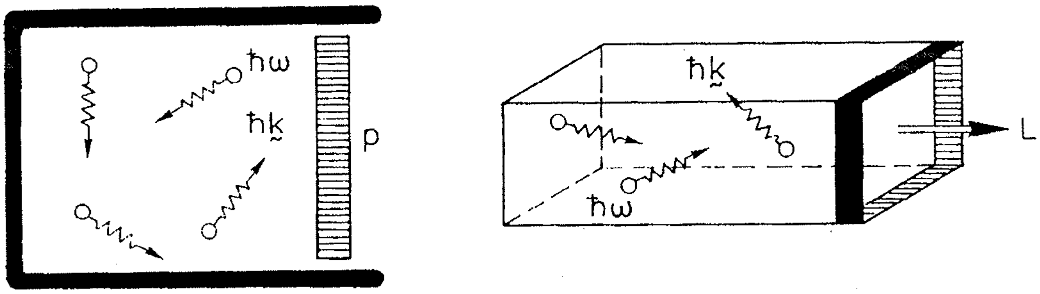

Fig. 1 - A photon gas and a phonon gas.

1.2. *Thermodynamic Processes.*

The objective of thermodynamics of photons is the determination of 8 fields, viz.

(1.3) photon number density n photon flux j_k
energy density e energy flux Q_k.

These quantities are objective scalars and vectors respectively. The necessary field equations are based upon the balance equations for

(1.4)

$$\text{photon number} \quad \frac{\partial n}{\partial t} + \frac{\partial j_k}{\partial x_k} = P_n \quad \text{photon flux} \quad \frac{\partial j_i}{\partial t} + \frac{\partial J_{ik}}{\partial x_k} = P_{j_i}$$

$$\text{energy} \quad \frac{\partial e}{\partial t} + \frac{\partial Q_k}{\partial x_k} = P_e \quad \text{momentum} \quad \frac{\partial p_i}{\partial t} + \frac{\partial P_{ik}}{\partial x_k} = P_{p_i}.$$

We make some specific assumptions relating the quantities in (1.4). We assume

(1.5) J_{ik}, P_{ik} – symmetric tensors, $J_{ii} = nc^2, P_{ii} = e,\ Q_i = p_i c^2.$

These assumptions seem arbitrary at this stage but they are really well-motivated as we shall see in the next section. First, however, we complete the discussion of the structure of the theory. The system of equations (1.4) is closed by considering the flux tensors J_{ik}, P_{ik} and the productions P_n through P_{pi} as constitutive quantities. As usual in extended thermodynamics the generic form of the constitutive equations reads

$$C = \hat{C}(n, j_i, e, p_i) \tag{1.6}$$

so that the value of a constitutive quantity at a point and time depends only on the values of the fields (1.3) at that point and time.

If the constitutive functions $\hat{C}$ were known, we could eliminate the consitutive quantities from the balance and obtain a specific set of field equations. Every solution of these equations is called a *thermodynamic process.*

1.3. *Motivations.*

There are two sources of motivation for the assumptions (1.5), the first only valid for photons. The Maxwell equations and the Maxwell-Lorentz aether relations

$$D_i = \varepsilon_0 E_i \quad H_i = \frac{1}{\mu_0} B_i \tag{1.7}$$

imply equations of balance for energy and momentum of the electro-magnetic field. In terms of the electric field **E**, the dielectric displacement **D**, the magnetic filed **H** and the magnetic flux density **B**, we have,

$$\begin{aligned} &\overset{e.m}{e} = \frac{1}{2}(\mathbf{ED} + \mathbf{BH}) && \overset{e.m}{p_k} = (\mathbf{D} \times \mathbf{B})_k \\ &\overset{e.m}{Q_k} = (\mathbf{E} \times \mathbf{H})_k && \overset{e.m}{P_{ik}} = \tfrac{1}{2}(\mathbf{ED} \times \mathbf{BH})\delta_{ik} - E_i D_k - B_i H_k \end{aligned} \tag{1.8}$$

Inspection shows that $\overset{e.m}{P_{ik}}$ is symmetric and obviously we have $\overset{e.m}{P_{ii}} = \overset{e.m}{e}$. Also, by (1.7) we see that $\overset{e.m}{Q_k} = \overset{e.m}{p_k}\, c^2$ holds, since $c^2 = \dfrac{1}{\varepsilon_0 \mu_0}$. Thus the Maxwell theory of electrodynamics supports some of the assumptions (1.5).

The second source of motivation for those assumptions may be found in the kinetic theory of photons. In that theory one introduces a phase density $f(\mathbf{x},\mathbf{k},t)$ such that $f(\mathbf{x},\mathbf{k},t)d\mathbf{x}d\mathbf{k}$ is the number of photons with $\mathbf{k}$ at $\mathbf{x},t$. This phase density obeys a balance law, the *Boltzmann-Peierls equation.*

$$\frac{\partial f}{\partial t}+\frac{\partial w_i f}{\partial x_i}=P \quad \text{with} \quad w_i=\frac{\partial \omega}{\partial k_i} \quad \text{or with (1.1)} : \quad w_i=c\frac{k_i}{k}. \tag{1.9}$$

P is the production density of photons in the phase space spanned by $\mathbf{k}$ and $\mathbf{x}$.

Integration over $\mathbf{k}$ after multiplication by 1, $\frac{\partial \omega}{\partial k_i}=c\frac{k_i}{k}$, $\hbar\omega$ and $\hbar k_i$ leads to the following four balance laws

$$\begin{aligned}
&\frac{\partial}{\partial t}\int f d\mathbf{k}+\frac{\partial}{\partial x_l}\int c\frac{k_l}{k}f d\mathbf{k}=\int P d\mathbf{k}\\
&\frac{\partial}{\partial t}\int c\frac{k_i}{k}f d\mathbf{k}+\frac{\partial}{\partial x_l}\int c^2\frac{k_j k_l}{k^2}f d\mathbf{k}=\int c\frac{k_i}{k}P d\mathbf{k}\\
&\frac{\partial}{\partial t}\int \hbar c k f d\mathbf{k}+\frac{\partial}{\partial x_l}\int \hbar c^2 k_l f d\mathbf{k}=\int \hbar\omega P d\mathbf{k}\\
&\frac{\partial}{\partial t}\int \hbar k_i f d\mathbf{k}+\frac{\partial}{\partial x_l}\int \hbar c\frac{k_i k_l}{k}f d\mathbf{k}=\int \hbar k_i P d\mathbf{k}
\end{aligned} \tag{1.10}$$

$\int f d\mathbf{k}$ is obviously the number density of photons while $\int \hbar\omega f d\mathbf{k}$ and $\int \hbar k_i f d\mathbf{k}$ are the densities of energy and momentum. Therefore the equations (1.10) must be interpreted as the balance equations for photon number, photon flux, photon energy and photon momentum.

Comparison with (1.4) provides the following interpretations

$$\begin{aligned}
n&=\int f d\mathbf{k} & j_i&=\int c\frac{k_i}{k}f d\mathbf{k} & P_n&=\int P d\mathbf{k}\\
& & J_{ik}&=\int c^2\frac{k_i k_k}{k^2}f d\mathbf{k} & P_{j_i}&=\int c\frac{k_i}{k}P d\mathbf{k}\\
e&=\int \hbar c k f d\mathbf{k} & Q_i&=\int \hbar c^2 k_i f d\mathbf{k} & P_e&=\int \hbar\omega P d\mathbf{k}\\
p_i&=\int \hbar k_i f d\mathbf{k} & P_{ik}&=\int \hbar c\frac{k_i k_k}{k}f d\mathbf{k} & P_{p_i}&=\int \hbar k_i P d\mathbf{k}
\end{aligned} \tag{1.11}$$

Inspection shows that the assumptions (1.5) are confirmed by the equations (1.11). Also, since ω and k_i are an objective scalar and vector respectively – at least in a non-relativistic theory – we conclude that the quantities n through P_{p_i} in (1.3) are all objective.

1.4. *Principles of the Constitutive Theory.*

In order to restrict the generality of the constitutive functions we exploit the three restrictive principles of the constitutive theory of extended thermodynamics, viz.

the principle of relativity

the entropy principle, and

the requirement of convexity and causality.

The principle of relativity requires that the field equations be invariant under changes of frame.

In the present case where the equations (1.4) are themselves invariant, this implies that the constitutive functions $\hat{C}$ in (1.6) are isotropic functions.

The entropy principle requires that the inequality

$$\frac{\partial h}{\partial t} + \frac{\partial \Phi_i}{\partial x_i} \geq 0. \tag{1.12}$$

holds for all thermodynamic processes. The entropy density h and the entropy flux Φ_i are constitutive quantities of the generic form (1.6).

The requirement of convexity and causality states that the matrix of second derivatives of h with respect to its variables n, j_i, e, p_i be negative definite.

1.5. *Exploitation of the Entropy Principle.*

The key to the exploitation of the entropy inequality is the statement that (1.12) must hold for all thermodynamic processes rather than for all fields. We may eliminate that constraint by the use of Lagrange multipliers. Indeed the larger inequality

$$\frac{\partial h}{\partial t}+\frac{\partial \Phi_e}{\partial x_e}-\lambda\left(\frac{\partial n}{\partial t}+\frac{\partial j_e}{\partial x_e}-P_n\right)-\lambda_i\left(\frac{\partial j_i}{\partial t}+\frac{\partial J_{ie}}{\partial x_e}-P_{j_i}\right)-$$
$$-\Lambda\left(\frac{\partial e}{\partial t}+\frac{\partial Q_e}{\partial x_e}-P_e\right)-\Lambda_i\left(\frac{\partial p_i}{\partial t}+\frac{\partial P_{ie}}{\partial x_e}-P_{p_i}\right)\geq 0 \tag{1.13}$$

must hold for *all* fields n, j_i, e, p_i. The Lagrange multipliers $\lambda, \lambda_i, \Lambda, \Lambda_i$ may be functions of n, j_i, e, and p_i; they must be isotropic functions because of the principle of relativity.

We insert the constitutive relations (1.6) and employ the chain rule to write the inequality (1.13) in a more explicit form, viz.

(1.14)

$$\underline{\left(\frac{\partial h}{\partial n}-\lambda\right)}\frac{\partial n}{\partial t}+\underline{\left(\frac{\partial h}{\partial j_i}-\lambda_i\right)}\frac{\partial j_i}{\partial t}+\underline{\left(\frac{\partial \Phi_e}{\partial n}-\lambda_i\frac{\partial J_{ie}}{\partial n}-\Lambda_i\frac{\partial P_{ie}}{\partial n}\right)}\frac{\partial n}{\partial x_e}+$$
$$+\underline{\left(\frac{\partial \Phi_e}{\partial j_n}-\lambda\delta_{en}-\lambda_i\frac{\partial J_{ie}}{\partial j_n}-\Lambda_i\frac{\partial P_{ie}}{\partial j_n}\right)}\frac{\partial j_n}{\partial x_e}+$$
$$\underline{\left(\frac{\partial h}{\partial e}-\Lambda\right)}\frac{\partial e}{\partial t}+\underline{\left(\frac{\partial h}{\partial p_i}-\Lambda_i\right)}\frac{\partial p_i}{\partial t}+\underline{\left(\frac{\partial \Phi_e}{\partial e}-\lambda_i\frac{\partial J_{ie}}{\partial e}-\Lambda_i\frac{\partial P_{ie}}{\partial e}\right)}\frac{\partial e}{\partial x_e}+$$
$$+\underline{\left(\frac{\partial \Phi_e}{\partial p_n}-c^2\Lambda\delta_{en}-\lambda_i\frac{\partial J_{ie}}{\partial p_n}-\Lambda_i\frac{\partial P_{ie}}{\partial p_n}\right)}\frac{\partial p_n}{\partial x_e}-$$
$$-\lambda P_n-\lambda_i P_{j_i}-\Lambda P_e-\Lambda_i P_{p_i}\geq 0.$$

The left hand side of this inequality is explicitly linear in the derivatives

$$\frac{\partial n}{\partial t},\frac{\partial j_i}{\partial t},\frac{\partial n}{\partial x_e},\frac{\partial j_n}{\partial x_e},\frac{\partial e}{\partial t},\frac{\partial p_i}{\partial t},\frac{\partial e}{\partial x_e},\frac{\partial p_n}{\partial x_z}. \tag{1.15}$$

Since these derivatives are arbitrary we could violate the inequality if they were to contribute to its left hand side. Therefore the underlined quantities in (1.14) must vanish, a requirement which we summarize as follows.

$$dh=\lambda dn+\lambda_i dj_i+\Lambda de+\Lambda_i dp_i$$
$$d\Phi_e=\lambda dj_e+\lambda_i dJ_{ie}+\Lambda c^2 dp_e+\Lambda_i dP_{ie} \tag{1.16}$$

There remains the residual inequality

$$\lambda P_n + \lambda_i P_{j_i} + \Lambda P_e + \Lambda_i P_{p_i} \geq 0. \tag{1.17}$$

In equilibrium we expect the photon flux and the momentum density to vanish. Therefore the Lagrange multipliers λ_i and Λ_i must also vanish in equilibrium and (1.16) implies

$$dh_E = \lambda_E dn + \Lambda_E de. \tag{1.18}$$

By comparison with the Gibbs equation of thermostatics we conclude

$$\lambda_E = -\frac{g}{nT} \quad \text{and} \quad \Lambda_E = \frac{1}{T}, \tag{1.19}$$

where $g = e - Th_E + \frac{1}{3}P_{ii}$ is the free enthalpy of the photon gas. The integrability condition implied by (1.18) reads

$$\left(\frac{\partial \lambda_E}{\partial e}\right)_n = \left(\frac{\partial \Lambda_E}{\partial n}\right)_e \quad \text{or} \quad -\frac{1}{n}\left(\frac{\partial g/T}{\partial e}\right)_n = \left(\frac{\partial 1/T}{\partial n}\right)_e$$

with $P_{ii} = e$ and $\left(\frac{\partial h_E}{\partial e}\right)_n = \frac{1}{T}$ we have

$$T - 4e\left(\frac{\partial T}{\partial e}\right)_n = -3n\left(\frac{\partial T}{\partial e}\right)_n\left(\frac{\partial e}{\partial n}\right)_T.$$

Since by (1.2) e depends on T only we obtain

$$e = \sigma T^4. \tag{1.20}$$

This is the well-known *law of Stefan-Boltzmann*, found empirically by Stefan and derived – from the Gibbs equation (1.18) in the above manner – by Boltzmann.

For the exploitation of the equations (1.16) it is convenient to introduce the *potentials*

$$\begin{aligned} h' &= -h + \lambda n + \lambda_i j_i + \Lambda e + \Lambda_i p_i \quad \text{and} \\ \Phi'_i &= -\Phi_i + \lambda j_i + \lambda_e J_{ei} + \Lambda c^2 p_i + \Lambda_e P_{ei} \end{aligned} \tag{1.21}$$

so that (1.16) assumes the form

$$dh' = n d\lambda + j_j d\lambda_j + e d\Lambda + p_i d\Lambda_i$$
(1.22)
$$d\Phi'_i = j_i d\lambda + J_{ij} d\lambda_j + p_i c^2 d\Lambda + P_{ij} d\Lambda_j .$$

It follows that the knowledge of h' and Φ'_i as functions of the variables λ, λ_j, Λ, Λ_j suffices to calculate n, j_j, e, p_i, J_{ij} and P_{ij} as functions of these variables. Therefore h' and Φ'_i are properly called potentials.

The conditions (1.22) have not been exploited yet in full generality, nor will this be done here.

We shall, however, proceed to illustrate their restrictive character by investigating a special case, the case where e and p_i are the only fields.

2. Energy and Momentum of Light and Sound.

2.1. *Eddington Factor, Entropy and Entropy Flux.*

We restrict the scope of the theory by considering its objective to be the determination of 4 fields, namely

(2.1) energy density e momentum density p_i.

In this case the relevant balance laws are those for energy and momentum, viz.

(2.2) $$\frac{\partial e}{\partial t} + \frac{\partial Q_k}{\partial x_k} = P_e \quad \text{and} \quad \frac{\partial p_i}{\partial t} + \frac{\partial P_{ik}}{\partial x_k} = P_{p_i} .$$

As before we assume

(2.3) $$P_{ik} \sim \text{symmetric}, \; P_{ii} = e, \; Q_i = p_i c^2 .$$

We close the system (2.2) by the formulation of constitutive equations for P_{ik} and P_e, P_{p_i} of the generic form

(2.4) $$C = \hat{C}(e, p_i).$$

The functions $\hat{C}$ must be isotropic functions so that we have the representations

$$(2.5) \qquad P_{ik} = e\left(\frac{1-\chi}{2}\delta_{ik} + \frac{3\chi-1}{2}\frac{p_i p_k}{p^2}\right) \quad \text{where} \quad \chi = \chi(e, p^2).$$

There is only one scalar coefficient in (2.5), because of the constraint $(2.3)_2$. χ is called the Eddington factor, because Eddington has first introduced it in connection with the radiative stress in stars.

The entropy principle will restrict the function χ. The restrictions have been given in Section 1.5 for the general case. In the present case there remain only those parts of the conditions that refer to the fields e and p. Therefore we have

$$(2.6) \qquad \begin{aligned} dh' &= e\,d\Lambda + p_j\,d\Lambda_j \\ d\Phi'_i &= p_i c^2 d\Lambda + P_{ij}\,d\Lambda_j \end{aligned} \qquad \text{where} \qquad \begin{aligned} h &= -h' + \Lambda e + \lambda_j p_j \\ \Phi_i &= -\Phi'_i + \Lambda c^2 p_i + \Lambda_j P_{ji} \end{aligned}$$

and the residual inequality reads

$$(2.7) \qquad \Lambda P_e + \Lambda_i P_{p_i} \geq 0.$$

h' and Φ'_i are functions of Λ and Λ_i so that we have

$$(2.8) \qquad h' = h'(\Lambda, L) \quad \text{and} \quad \Phi'_i = \varphi(\Lambda, L)\Lambda_i$$

where L stands for $\Lambda_i\Lambda_i$. More explicitly the equations (2.6) read therefore

$$(2.9) \qquad \frac{\partial h'}{\partial \Lambda} = e,\ 2\frac{\partial h'}{\partial L}\Lambda_i = p_i,\ \frac{\partial \varphi}{\partial \Lambda}\Lambda_i = c^2 p_i,\ 2\frac{\partial \varphi}{\partial L}\Lambda_i\Lambda_j + \varphi\delta_{ij} = P_{ij}.$$

The exploitation proceeds as follows. Elimination of p_i between $(2.9)_{2,3}$ gives

$$(2.10) \qquad \frac{\partial h'}{\partial L} = \frac{1}{2c^2}\frac{\partial \varphi}{\partial \Lambda}, \quad \text{hence by } (2.9)_1 \quad \frac{\partial e}{\partial L} = \frac{1}{2c^2}\frac{\partial^2 \varphi}{\partial \Lambda^2}.$$

The trace of $(2.9)_4$ reads with

$$(2.11) \qquad 2\frac{\partial \varphi}{\partial L}L + 3\varphi = e$$

and if this is combined with $(2.10)_2$ we obtain a differential equation for φ, viz.

$$4\frac{\partial^2\varphi}{\partial L^2}L + 10\frac{\partial\varphi}{\partial L} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial\Lambda^2} \tag{2.12}$$

It is possible to find the general solution of this differential equation for a given function $\varphi_0(\Lambda, 0)$.

The solution can be found in [4]. Here we shall follow [3] and derive the solution by making a power Ansatz of the form

$$\varphi = \sum_{r=0}^{\infty}\varphi_r(\Lambda)L^r. \tag{2.13}$$

Insertion into the differential equation (2.12) leads to a recurrence relation of the form

$$\varphi_{r+1} = \frac{1}{2c^2}\frac{1}{(r+1)(2r+5)}\frac{\partial^2\varphi_r}{\partial\Lambda^2}. \tag{2.14}$$

The initiation for the solution of this relation is provided by (2.11) and $(1.19)_2$, (1.20) from which formulae we obtain

$$\varphi_0(\Lambda) = \frac{1}{3}e_E(\Lambda) = \frac{1}{3}\sigma\frac{1}{\Lambda^4}. \tag{2.15}$$

A simple calculation will then show that the solution of the recurrence relation (2.14) reads

$$\varphi_r = \frac{\sigma}{3}\frac{1}{\Lambda^4}(r+1)\left(\frac{1}{c^2\Lambda^2}\right)^r. \tag{2.16}$$

Hence we obtain

$$\varphi = \frac{\sigma}{3}\frac{1}{\Lambda^4}\sum_{r=0}^{\infty}(r+1)\left(\frac{L}{c^2\Lambda^2}\right)^r = \frac{\sigma}{3}\frac{1}{\left[\Lambda^2 - \frac{L}{c^2}\right]^2}. \tag{2.17}$$

Now that φ is known as an explicit function of Λ and L, we may calculate e and p^2 from (2.11) and $(2.9)_3$ respectively

$$e = \frac{\sigma}{3}\frac{3\Lambda^2 + \frac{L}{c^2}}{\left[\Lambda^2 - \frac{L}{c^2}\right]^3}, \quad c^2p^2 = \frac{\sigma^2}{9}\frac{16\Lambda^2\frac{L}{c^2}}{\left[\Lambda^2 - \frac{L}{c^2}\right]^6}. \tag{2.18}$$

We may invert these equations and obtain Λ and L as functions of e and p^2. We obtain with $\surd = \left(1 - \frac{3}{4}\frac{c^2p^2}{e^2}\right)^{1/2}$

$$\Lambda = \frac{\sigma^{1/4}}{2^{1/2}}\frac{1}{e^{1/4}}\frac{(1+\surd)^{5/4}}{\left[1+\surd - \frac{3}{2}\frac{c^2p^2}{e^2}\right]^{3/4}}$$

(2.19)

$$\frac{L}{c^2} = \frac{3^2\sigma^{1/2}}{2^3}\frac{1}{e^{1/2}}\frac{c^2p^2}{e^2}\frac{(1+\surd)^{1/2}}{\left[1+\surd - \frac{1}{2}\frac{c^2p^2}{e^2}\right]^{3/2}}.$$

It follows from (2.18) that

$$(2.20) \qquad \varphi = \frac{1}{3}(\sqrt{4e^2 - 3c^2p^2} - e), \qquad \frac{\partial\varphi}{\partial L}L = e - \frac{1}{2}\sqrt{4e^2 - 3c^2p^2}$$

so that, by $(2.9)_4$ we have

$$(2.21) \qquad P_{ij} = e\left(\frac{1}{3}\left(\sqrt{4 - 3\frac{c^2p^2}{e^2}} - 1\right)\delta_{ij} + \left(2 - \sqrt{4 - 3\frac{c^2p^2}{e^2}}\right)\frac{p_ip_j}{p^2}\right).$$

Comparison with (2.5) shows that the Eddington factor has thus been identified as

$$(2.22) \qquad \chi = \frac{5}{3} - \frac{2}{3}\sqrt{4 - 3\frac{c^2p^2}{e^2}}.$$

We shall represent all subsequent results in terms of the Eddington factor and for that purpose we list the following identities that follows from (2.22) and (2.19)

$$\frac{c^2p^2}{e^2} = \frac{3}{4}(3-\chi)\left(\chi - \frac{1}{3}\right), \Lambda = \frac{1}{2^{3/4}}\frac{1}{3^{1/4}}\frac{\sigma^{1/4}}{e^{1/4}}\frac{(3-\chi)^{1/2}}{(1-\chi)^{3/4}},$$

(2.23)

$$\Lambda^2 - \frac{L}{c^2} = \frac{2^{1/2}}{3^{1/2}}\frac{\sigma^{1/2}}{e^{1/2}}\frac{1}{(1-\chi)^{1/2}}.$$

We proceed to calculate the entropic quantities. First the entropy density and its flux. By $(2.10)_1$, $(2.9)_1$ and $(2.8)_2$ we have

$$(2.24) \qquad h' = -\frac{\sigma}{3}\frac{\Lambda}{\left[\Lambda^2 - \frac{L}{c^2}\right]^2}, \quad \text{hence by } (2.6)_1 \; h = -\frac{4}{3}\sigma\frac{\Lambda}{\left[\Lambda^2 - \frac{L}{c^2}\right]^2}$$

$\Phi'_i = \varphi \Lambda_i$, hence by $(2.6)_2$

(2.25) $$\Phi_i = -\frac{4}{3}\sigma \frac{\Lambda_i}{\left[\Lambda^2 - \frac{L}{c^2}\right]^2} = \frac{\left[\Lambda^2 - \frac{L}{c^2}\right]^2}{\Lambda} c^2 p_i.$$

By use of (2.23) we therefore obtain for the entropy density and the entropy flux as functions of e and p:

(2.26) $$h = \frac{4}{3}\sigma^{1/4} e^{3/4} \frac{3^{3/4}}{2^{7/4}} (3-\chi)^{1/2}(1-\chi)^{1/4},$$
$$\Phi_i = \frac{2^{5/4}}{3^{1/4}} \frac{(1-\chi)^{1/4}}{(3-\chi)^{1/2}} \left(\frac{\sigma}{e}\right)^{1/4} c^2 p_i.$$

2.2. *Summary of Results and Limiting Cases.*

The field equations for the determination of energy and momentum of the photons read

(2.27) $$\frac{\partial e}{\partial t} + c^2 \frac{\partial p_k}{\partial x_k} = P_e \qquad \frac{\partial p_i}{\partial t} + \frac{\partial P_{ik}}{\partial x_k} = P_{p_i}.$$

The main part of these equations has become fully explicit by the application of the entropy principle. We have obtained

(2.28) $$P_{ik} = e\left(\frac{1-\chi}{2}\delta_{ik} + \frac{3\chi - 1}{2}\frac{p_i p_k}{p^2}\right), \text{ where } \chi = \frac{5}{3} - \frac{2}{3}\sqrt{4 - 3\frac{c^2 p^2}{e^2}}$$

is the Eddington factor.

Moreover we have determined the entropy density and its flux, viz.

(2.29) $$h = \frac{4}{3}\sigma^{1/4} e^{3/4} \frac{3^{3/4}}{2^{7/4}} (3-\chi)^{1/2}(1-\chi)^{1/4},$$
$$\Phi_i = \frac{2^{5/4}}{3^{1/4}} \frac{(1-\chi)^{1/4}}{(3-\chi)^{1/2}} \left(\frac{\sigma}{e}\right)^{1/4} c^2 p_i.$$

We distinguish two limiting cases as follows

i.) the near-equilibrium case $cp \ll e$.

In that case $\chi \to \frac{1}{3}$ and the momentum flux becomes isotropic

$$P_{ik} = \frac{e}{3}\delta_{ik}. \tag{2.30}$$

The entropy density and entropy flux assume the forms

$$h = \frac{4}{3}\sigma^{1/4}e^{3/4} \quad \Phi_i = \left(\frac{\sigma}{e}\right)^{1/4} c^2 p_i \tag{2.31}$$

which by use of the equilibrium relation (1.20) we may write in the form

$$h = \frac{4}{3}\sigma T^3 \quad \Phi_i = \frac{c^2 p_i}{T}. \tag{2.32}$$

It is remarkable that the entropy flux for photons near equilibrium has the same form as in a particle gas near equilibrium, namely energy flux divided by absolute temperature.

ii.) the free-streaming case $cp_i \lesssim en_i$ where n_i is a unit vector in the streaming direction. In that case $\chi \to 1$ and the momentum flux reads

$$P_{ik} = en_i n_k. \tag{2.33}$$

The entropy density is constant and the entropy flux vanishes.

2.3. *Waves.*

It is instructive to investigate linear wave propagation in the two limiting cases that we have discussed.

In the near equilibrium case, we linearize the equations (2.27) in p and obtain with (2.30)

$$\frac{\partial e}{\partial t} + c^2\frac{\partial p_k}{\partial x_k} = 0 \quad \frac{\partial p_i}{\partial t} + \frac{1}{3}\frac{\partial e}{\partial x_i} = B_0 p_i \tag{2.34}$$

where B_0 is the equilibrium value of the coefficient B that determines the momentum production $P_{p_i} = B_{p_i}$.

We eliminate p from these equations and obtain a telegraph equation of the form

$$\frac{\partial^2 e}{\partial t^2} - B_0\frac{\partial e}{\partial t} - \frac{c^2}{3}\frac{\partial^2 e}{\partial x_i \partial x_i} = 0. \tag{2.35}$$

This equation describes the propagation of a damped wave in all directions with the speed

$$V = \frac{c}{\sqrt{3}}. \tag{2.36}$$

In the case of phonons this mode of propagation is often called the *second sound.*

In the free streaming case we have $P_e = 0$ and $P_{p_i} = 0$. The two equations (2.27) read with $p_i = \frac{e}{c} n_i$ and $P_{ik} = e n_i n_k$, see (2.37)

$$\frac{\partial e}{\partial t} + c n_k \frac{\partial e}{\partial x_k} = 0 \quad \text{and} \quad \frac{1}{c} n_i \frac{\partial e}{\partial t} + n_i n_k \frac{\partial e}{\partial x_k} = 0. \tag{2.37}$$

These equations are identical. They describe the propagation of a wave in the direction of n_i with the speed

$$V = c. \tag{2.38}$$

This mode of propagation is called ballistic in the case of phonons while in the case of light it represents the free streaming of photons in a laser beam (say).

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