SHOCK STRUCTURE IN MASSLESS GASES

A. MAJORANA (Catania) - O. MUSCATO (Catania)

The shock structure problem is investigated in the framework of the Eckart theory of irreversible thermodynamics in the ultrarelativistic limit. It is considered a neutrino gas and a gas in the approximation of hard sphere model.

1. Introduction.

Relativistic shock waves are a subject of importance for various areas of astrophysics and laboratory plasma physics [1]. A mathematical idealization treats shock waves as discontinuities; a more realistic description is in terms of a thin layer where the fluid variables vary rapidly but smoothly. The thickness of the shock layer is determined by dissipative process and is usually of the same order of a mean free path.

We shall study the shock structure problem in the framework of the Eckart theory of irreversible thermodynamics [2] in the ultrarelativistic limit. We shall limit our analysis for two kind of particles, e.g. a neutrino gas and an hard sphere model gas: they may be considered test problems for the hydrodynamical model of Eckart. A qualitative analysis of this problem was performed by
Koch [3]; Kosowski [4] studied the structure of shock waves in an ultrarelativistic case for a constant differential cross section, but the results are incorrect because the transport coefficients are wrong.

2. Basic equations.

We shall consider a simple neutral gas in the flat space-time of special relativity with inertial coordinates $x^\alpha = (ct, x, y, z)$ and metric tensor $g^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. A simple gas is described by the conservation laws:

\begin{equation}
\partial_\alpha N^\alpha = 0 \quad \partial_\alpha T^{\alpha\beta} = 0
\end{equation}

where $N^\alpha$ is the particle flux vector, and $T^{\alpha\beta}$ is the energy-momentum tensor. As usually we split the energy momentum tensor in the non dissipative part $T_0^{\alpha\beta}$ (which depends only by the hydrodynamical velocity, the particle density $n$ and the absolute temperature $T$) and in the irreversible part $T_1^{\alpha\beta}$ (which contains also the heat flux $q^\alpha$, the shear stress $\Pi^{\alpha\beta}$ and the bulk stress $\pi$), such that

\[ T^{\alpha\beta} = T_0^{\alpha\beta} + T_1^{\alpha\beta}. \]

In the particle frame of reference $N^\alpha$ and $T^{\alpha\beta}$ take the form:

\begin{equation}
N^\alpha = cnu^\alpha
\end{equation}

\begin{equation}
T_0^{\alpha\beta} = mc^2 n G(z) u^\alpha u^\beta + nk_B T g^{\alpha\beta}
\end{equation}

\begin{equation}
T_1^{\alpha\beta} = \frac{1}{c} \left( q^\alpha u^\beta + q^\beta u^\alpha \right) + \Pi^{\alpha\beta} + \pi h^{\alpha\beta}
\end{equation}

where $c$ is the light speed, $cu^\alpha$ is the fluid four-velocity, $m$ the mass particle, $k_B$ the Boltzmann constant, $h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ and

\[ G(z) = \frac{K_3(z)}{K_2(z)} \quad z = \frac{mc^2}{k_B T} \]

$K_n(z)$ being the modified Bessel function of the second kind.
3. The shock structure problem.

Let us consider a steady one-dimensional flow such that, in an appropriate coordinate frame, all the thermodynamical quantities depend only on a single spatial coordinate \( x \). We also assume that

\[
u^\alpha = \Gamma(c,v,0,0)\]

\( v \) being the \( x \)-velocity and \( \Gamma \) the Lorentz factor. Hence from Eqs. (1) we obtain the conservation equations

\[
T^{\alpha 1} = K^\alpha , \quad N^1 = M
\]

where \( K^\alpha \) and \( M \) are constants.

The plane shock structure problem [5] consists of finding an unique solution (apart from translations) of hydrodynamic equations (1), with the boundary conditions given by assuming that the gas reaches equilibrium states as \( x \to -\infty \) and \( x \to +\infty \), i.e.

\[
\lim_{x \to \pm \infty} T^{\alpha \beta}_{1} = 0.
\]

These conditions imply

\[
\lim_{x \to \pm \infty} q^\alpha = 0 , \quad \lim_{x \to \pm \infty} \Pi^{\alpha \beta} = 0 , \quad \lim_{x \to \pm \infty} \pi = 0.
\]

The equilibrium states are completely described by the particle density, the temperature and the fluid four-velocity: in the case of one-dimensional flow these quantities reduce to \( n, T, v \). We denote the limit values by \( n_+, T_+, v_+ \) and \( n_-, T_-, v_- \), where the subscripts indicate \( x \to +\infty \) or \( x \to -\infty \) respectively.

The boundary conditions (6) with Eqs.(5), give some restrictions (the Rankine-Hugoniot equations) for the above six quantities: these relations are described in a complete way by one of the author [6], showing that explicit analytic solutions for the Rankine-Hugoniot conditions in term of \( T_+, T_- \) and \( n_- \) exist. Our analysis will restrict to the ultrarelativistic limit, where

\[
\frac{mc^2}{k_BT} \ll 1
\]
and we can assume that
\[ G(z) = \frac{4}{z}. \]

It is useful to introduce dimensionless variables: we choose
\[ X = n_\sigma_\sigma x, \quad \tau = \frac{\Gamma T}{\Gamma_\sigma T} \]
where \( \sigma_- = \sigma(T_-) \) is the characteristic differential cross section depending only on \( T \) and \( (n_\sigma \sigma_-)^{-1} \) is the limit mean free path as \( x \to -\infty \).

Therefore from Eqs.(5) and the Rankine-Hugoniot relations we obtain
\[
\begin{align*}
\lambda Q \frac{d\tau}{dX} &= 3\tau(1 - v^2) + \left( 3v_- + \frac{1}{v_-} \right) v - 4 \\
\lambda S \frac{dv}{dX} &= \tau(1 - v^2) \left( \frac{1}{v} - 3v \right) + 8v - (1 + v^2) \left( 3v_- + \frac{1}{v_-} \right)
\end{align*}
\]

where
\[
\begin{align*}
\lambda Q &= \frac{\lambda}{Mk_B n_\sigma} , \quad \lambda S = \frac{4}{3 M k_B T_\sigma \Gamma} \Gamma^\sigma n_\sigma. 
\end{align*}
\]

Elementary qualitative theory of ordinary differential equations shows that a necessary condition of the existence of solutions of the shock structure is that the boundary points must be singular points of (7), i.e.
\[
\begin{align*}
\lim_{x \to -\infty} (v, \tau) &= (v_-, 1) , \quad \lim_{x \to +\infty} (v, \tau) = \left( \frac{1}{3v_-}, 1 \right).
\end{align*}
\]

It is proved that the boundary points are respectively a saddle point and a node, and than an unique solution exists. We have studied Eqs.(7) allowing \( v_- \) to vary in the range \( (1/\sqrt{3}, 1) \) and we have evaluated the shock waves thickness \( S \) defined by
\[
S = \frac{|v_- - v_+|}{\left| \frac{dv}{dX} \right|_{\text{max}}}.
\]
for two type of particles: a neutrino gas and a gas in the approximation of hard sphere model. For the neutrino gas [2]:

\[ \lambda_Q = \frac{3}{320 \pi \Gamma_\nu^3 v_\nu} \left( \frac{\Gamma}{\tau} \right)^2, \quad \lambda_S = \frac{1}{184 \pi \Gamma_\nu^3 v_\nu} \frac{\Gamma^4}{\tau} \]

whereas in the other case [2,7]

\[ \lambda_Q = \frac{1}{\pi \Gamma_\nu v_\nu}, \quad \lambda_S = \frac{4}{5 \pi \Gamma_\nu^3 v_\nu} \Gamma^2 \tau. \]

We remark that the Eqs.(7) do not contain the parameter \( T_- \), as one expects in the ultrarelativistic framework: therefore the shock thickness will depend only on the velocity \( v_- \). We have integrated the differential equation by using standard numerical procedures and we evaluated \( S \) as a function of \( v_- \). The results are presented in Ref. 8.

REFERENCES


A. Majorana
Dipartimento di Matematica,
Università di Catania

O. Muscato
Dipartimento di Matematica,
Università di Catania