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SHOCK STRUCTURE IN MASSLESS GASES

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The shock structure problem is investigated in the framework of the Eckart theory of irreversible thermodynamics in the ultrarelativistic limit. It is considered a neutrino gas and a gas in the approximation of hard sphere model.

1. Introduction.

Relativistic shock waves are a subject of importance for various areas of astrophysics and laboratory plasma physics [1]. A mathematical idealization treats shock waves as discontinuities; a more realistic description is in terms of a thin layer where the fluid variables vary rapidly but smoothly. The thickness of the shock layer is determined by dissipative process and is usually of the same order of a mean free path.

We shall study the shock structure problem in the framework of the Eckart theory of irreversible thermodynamics [2] in the ultrarelativistic limit. We shall limit our analysis for two kind of particles, e.g. a neutrino gas and an *hard sphere model* gas: they may be considered test problems for the hydrodynamical model of Eckart. A qualitative analysis of this problem was performed by

Koch [3] ; Kosowski [4] studied the structure of shock waves in an ultrarelativistic case for a constant differential cross section, but the results are incorrect because the transport coefficients are wrong.

2. Basic equations.

We shall consider a simple neutral gas in the flat space-time of special relativity with inertial coordinates $x^\alpha = (ct, x, y, z)$ and metric tensor $g^{\alpha\beta} = diag(-1, 1, 1, 1)$. A simple gas is described by the conservation laws:

(1) $$\partial_\alpha N^\alpha = 0 \ , \ \partial_\alpha T^{\alpha\beta} = 0$$

where N^α is the particle flux vector, and $T^{\alpha\beta}$ is the energy-momentum tensor. As usually we split the energy momentum tensor in the non dissipative part $T_0^{\alpha\beta}$ (which depends only by the hydrodynamical velocity, the particle density n and the absolute temperature T) and in the irreversible part $T_1^{\alpha\beta}$ (which contains also the heat flux q^α, the shear stress $\Pi^{\alpha\beta}$ and the bulk stress π), such that

$$T^{\alpha\beta} = T_0^{\alpha\beta} + T_1^{\alpha\beta}.$$

In the particle frame of reference N^α and $T^{\alpha\beta}$ take the form:

(2) $$N^\alpha = cnu^\alpha$$

(3) $$T_0^{\alpha\beta} = mc^2 nG(z) u^\alpha u^\beta + nk_B T g^{\alpha\beta}$$

(4) $$T_1^{\alpha\beta} = \frac{1}{c}\left(q^\alpha u^\beta + q^\beta u^\alpha\right) + \Pi^{\alpha\beta} + \pi h^{\alpha\beta}$$

where c is the light speed, cu^α is the fluid four-velocity, m the mass particle, k_B the Boltzmann constant, $h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ and

$$G(z) = \frac{K_3(z)}{K_2(z)} \ , \ z = \frac{mc^2}{k_B T}$$

$K_n(z)$ being the modified Bessel function of the second kind.

3. The shock structure problem.

Let us consider a steady one-dimensional flow such that, in an appropriate coordinate frame, all the thermodynamical quantities depend only on a single spatial coordinate x. We also assume that

$$u^\alpha = \Gamma(c, v, 0, 0)$$

v being the x-velocity and Γ the Lorentz factor. Hence from Eqs. (1) we obtain the conservation equations

$$T^{\alpha 1} = K^\alpha \ , \ N^1 = M \tag{5}$$

where K^α and M are constants.

The plane shock structure problem [5] consists of finding an unique solution (apart from translations) of hydrodynamic equations (1), with the boundary conditions given by assuming that the gas reaches equilibrium states as $x \to -\infty$ and $x \to +\infty$, i.e.

$$\lim_{x \to \pm\infty} T_1^{\alpha\beta} = 0. \tag{6}$$

These conditions imply

$$\lim_{x \to \pm\infty} q^\alpha = 0 \ , \ \lim_{x \to \pm\infty} \Pi^{\alpha\beta} = 0 \ , \ \lim_{x \to \pm\infty} \pi = 0.$$

The equilibrium states are completely described by the particle density, the temperature and the fluid four-velocity: in the case of one-dimensional flow these quantities reduce to n, T, v. We denote the limit values by n_+, T_+, v_+ and n_-, T_-, v_-, where the subscripts indicate $x \to +\infty$ or $x \to -\infty$ respectively .

The boundary conditions (6) with Eqs.(5), give some restrictions (the Rankine-Hugoniot equations) for the above six quantities: these relations are described in a complete way by one of the author [6], showing that explicit analytic solutions for the Rankine-Hugoniot conditions in term of T_+, T_- and n_- exist. Our analysis will restrict to the ultrarelativistic limit, where

$$\frac{mc^2}{k_B T} \ll 1$$

and we can assume that

$$G(z) = \frac{4}{z}.$$

It is useful to introduce dimensionless variables: we choose

$$X = n_- \sigma_- x, \ \tau = \frac{\Gamma T}{\Gamma_- T_-}$$

where $\sigma_- = \sigma(T_-)$ is the characteristic differential cross section depending only on T and $(n_- \sigma_-)^{-1}$ is the limit mean free path as $x \to -\infty$.

Therefore from Eqs.(5) and the Rankine-Hugoniot relations we obtain

$$(7) \qquad \begin{cases} \lambda_Q \dfrac{d\tau}{dX} = 3\tau(1 - v^2) + \left(3v_- + \dfrac{1}{v_-}\right) v - 4 \\ \lambda_S \dfrac{dv}{dX} = \tau(1 - v^2)\left(\dfrac{1}{v} - 3v\right) + 8v - (1 + v^2)\left(3v_- + \dfrac{1}{v_-}\right) \end{cases}$$

where

$$(8) \qquad \lambda_Q = \frac{\lambda}{M k_B} n_- \sigma_- \ , \ \lambda_S = \frac{4}{3} \frac{\eta c^2}{M k_B T_- \Gamma_-} \Gamma^3 n_- \sigma_- .$$

Elementary qualitative theory of ordinary differential equations shows that a necessary condition of the existence of solutions of the shock structure is that the boundary points must be singular points of (7), i.e.

$$(9) \qquad \lim_{x \to -\infty} (v, \tau) = (v_-, 1) \ , \ \lim_{x \to +\infty} (v, \tau) = (\frac{1}{3v_-}, 1).$$

It is proved that the boundary points are respectively a saddle point and a node, and than an unique solution exists. We have studied Eqs.(7) allowing v_- to vary in the range $(1/\sqrt{3}, 1)$ and we have evaluated the shock waves thickness S defined by

$$(10) \qquad S = \frac{|v_- - v_+|}{\left|\dfrac{dv}{dX}\right|_{max}}.$$

for two type of particles: a neutrino gas and a gas in the approximation of *hard sphere* model. For the neutrino gas [2]:

$$\lambda_Q = \frac{3}{320\pi\Gamma_-^3 v_-}\left(\frac{\Gamma}{\tau}\right)^2 , \lambda_S = \frac{1}{184\pi\Gamma_-^3 v_-}\frac{\Gamma^4}{\tau} \tag{11}$$

whereas in the other case [2,7]

$$\lambda_Q = \frac{1}{\pi\Gamma_- v_-} , \lambda_S = \frac{4}{5\pi\Gamma_- v_-}\Gamma^2\tau. \tag{12}$$

We remark that the Eqs.(7) do not contain the parameter T_-, as one expects in the ultrarelativistic framework: therefore the shock thickness will depend only on the velocity v_-. We have integrated the differential equation by using standard numerical procedures and we evaluated S as a function of v_-. The results are presented in Ref. 8.

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