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ON ASYMPTOTIC TIME DECAY OF SOLUTIONS TO A PARABOLIC EQUATION IN UNBOUNDED DOMAINS

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Estimates on the asymptotic behavior of solutions to a parabolic equation are given, when the I.B.V.P. is posed in particular domains. More precisaly, the domain Ω is unbounded, unbounded in any direction, and Ω is enclosed in a wedge or in a cone of two or three-dimensional Euclidean space. It is proved that the order of decay is increasing for decreasing opening of the wedge or of the cone.

1. Introduction.

The present paper is concerned with time asymptotic behavior of solutions to a linear parabolic equation in unbounded domains. This problem is of relevant interest either from the physical point of view, since, as is well-known, the diffusion phenomena are governed by such a kind of equations, or from the mathematical point of view, since the time-decay at infinity may be used to deduce some compactness results for solutions [4,5,7,10]. The purpose of this paper is to improve some rates of decay of solutions to a parabolic equation in the case where the domain is contained in a wedge or in a cone of a proper opening.

2. Statement of the problem.

Before giving the statement of our Theorem, to fix idea, we start by making some considerations about the Cauchy problem associated to the heat equation:

$$
\begin{aligned}
&u_t(x,t) = \Delta u(x,t), && \text{on } R^3 \times [0,\infty), \\
&u(x,t) \to 0 \text{ for } |x| \to \infty, && \forall t \geq 0, \\
&u(x,0) = u_0(x), && \text{on } R^3 \times \{0\}.
\end{aligned} \tag{2.1}
$$

As is well-known, if $u_0(x)$ is a smooth function with compact support, then the solution to system (2.1) decays as $t \to \infty$ according to the following estimate [1]

$$\sup_{R^3} |u(x,t)| \leq Ct^{-\frac{3}{2}}, \tag{2.2}$$

for some positive constant C indipendent of t. Moreover, the rate of decay expressed by (2.2) is sharp (cf. Remarks 1,2 in [1]). On the other hand, we can not excluded that inequality (2.2) might be improved, provided $u_0(x)$ is required to meet a further (suitable) condition. Indeed, the following proposition shows that this is possible.

PROPOSITION. *Let $u_0(x)$ be a smooth function with a compact support on R^3 such that*

$$\int_{R^3} u_0(x)\, dx = 0. \tag{2.3}$$

Then the solution $u(x,t)$ to System (2.2) corresponding to $u_0(x)$ obeys the inequality

$$\sup_{R^3} |u(x,t)| \leq Ct^{-2}, \tag{2.4}$$

where C is a positive constant indipendent of t.

Proof. See Proposition 2.1 of [8]

The above result suggest that the asymptotic behavior (2.2) can be improved if one properly limit the choice of initial data $(2.1)_2$. On the other hand, the choice of suitable initial data does not appear to be the only approch we could follow to improve the rate of decay of solutions to equations of type $(2.1)_1$. Indeed, as is stated in [2], as far as the two-dimensional equation $(2.1)_1$ in a wedge of opening α is concerned, the decay of the solutions is connected to α in the sense that it grows as α decreases. Therefore, we can argue that the order of decay $t^{-\frac{n}{2}}$ $(n = 2,3)$ could be improved for suitable unbounded domains. In a quite different context, the connection between the opening α of the cone and the stability of the solutions, was argued in [9]. Indeed, for particular α in [9] it is proved that the Reynolds number associated to some basic fluid motion can be more large to obtain stability. The aim of the current paper is just to present a technique based on a proper use of a weight Poincaré's inequality which allows us to improve for particular domains, i.e., those contained in a wedge or a cone of opening less then $\pi/2$, the order of decay $t^{-\frac{n}{2}}$. To this end we consider the following initial boundary value problem

$$(2.5)\quad \begin{aligned} &\mathcal{L}(u) = u_t(x,t) - \nabla\cdot(\mathbf{A}(\mathbf{x},\mathbf{t})\nabla\mathbf{u}(\mathbf{x},\mathbf{t})) = 0, && \text{on } \Omega\times(0,\infty),\\ &u(x,t) = 0, && \text{on } \partial\Omega\times(0,\infty),\\ &u(x,t)\to 0 \text{ for } |x|\to\infty, && \forall t\ge 0,\\ &u(x,0) = u_0(x), && \text{on } \Omega\times\{0\}, \end{aligned}$$

where Ω is an unbounded domain of R^n $(n = 2,3)$, $u_t(x,t) = \dfrac{\partial}{\partial t}u(x,t)$ and $\mathbf{A}(\mathbf{x},\mathbf{t})$ is, $\forall(x,t)\in\Omega\times(0,\infty)$, a second-order tensor, i.e., a linear transformation from R^n into R^n such that

$$(2.6)\quad \begin{aligned} &\sup_{\Omega\times(0,T)} |\mathbf{A}(x,t)| \le \beta,\\ &\mathbf{a}\cdot\mathbf{A}(x,t)\mathbf{a} \ge \alpha\,\mathbf{a}^2,\ \forall\mathbf{a},\ \forall(x,t)\in\Omega\times(0,\infty), \end{aligned}$$

for some positive constants α and β independent of (x,t).

We are concerned with solutions to system (2.5) such that

$$u(x,t)\in L^2(0,T;W^{2,2}(\Omega))\cap C(0,T;\overset{\circ}{W}{}^{1,2}(\Omega))$$

$$u_t(x,t)\in L^2(0,T;L^2(\Omega)),\ u(x,t) \text{ satisfies (2.5) a.e.}$$

As far as the existence of such a kind of solutions we refer to [3].
We are now in a position to give our main result.

THEOREM. *Let Ω be an unbounded domain of R^n $(n = 2, 3)$ contained in a wedge $\mathcal{W}$ or in a cone $\mathcal{C}$ of opening $2\varphi_0$, $\varphi_0 \leq \frac{\pi}{2}$. Let $u_0(x) \in L^2(\Omega, r^k) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$, for some $k \in \left(0, \frac{\alpha}{\beta\delta}\right]$, where*

$$\delta = \begin{cases} \sqrt{2}\varphi_0 & \text{if } \Omega \subset \mathcal{W} \\ 2\varphi_0(1+\cos\varphi_0)^{-1} & \text{if } \Omega \subset \mathcal{C}. \end{cases} \tag{2.7}$$

Let $u(x,t)$ the solution to (2.5) corresponding to $u_0(x)$, then

$$|u(t)|_p \leq C|r^{\frac{k}{2}}u_0|t^{-\gamma}, \; \forall p \geq 2, \; \forall t > 0, \tag{2.8}$$

where

$$\gamma = \frac{n}{2}(\frac{1}{2} - \frac{1}{p}) + \frac{k}{4},$$

and

$$|r^{\frac{h}{2}}u(t)| \leq C|r^{\frac{k}{2}}u_0|t^{-\gamma_1}, \; \forall h \in [0,k], \; \forall t > 0, \tag{2.9}$$

where

$$\gamma_1 = \frac{k-h}{4}.$$

Proof. See Section 4 of [8].

Remark 2.1 It is obvious that from (2.7)-(2.8)- (2.9) it follows that $k \to \infty$ as $\delta \to 0$.

It is worth noting as the constants appearing in (2.7)- (2.8)-(2.9) specialize when $A_{ij} = \delta_{ij}$, so that $(2.5)_1$ reduces to the heat equation. In this case a simple calculation shows that the improvement of the rate of decay (2.2) is achieved for $n = 3$ when the opening θ of the wedge $\mathcal{W}$ is less than $\frac{\pi}{6}$ and in the case of the cone $\mathcal{C}$ when the opening θ is less then $\frac{\pi}{4}$.

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