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SOME PROPERTIES OF SOLITON SOLUTIONS OF THE GENERALIZED ZAKHAROV SYSTEM

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A generalization of the well known Zakharov system of ion-acoustic waves (Langmuir solitons) has been obtained while studying the coupling between shear-horizontal surface waves and Rayleigh surface waves propagating on a structure made of a nonlinear elastic substrate and a superimposed thin elastic film. We obtain thus a nearly integrable system made of a nonlinear Schrödinger equation coupled to two wave equations for the secondary acoustic system. Here we present essentially some comments and results of numerical simulations (pure SH mode, coupled case, influence of dissipation in the Rayleigh subsystem, collisions of solitons).

1. Introduction.

The problem consists in studying the possible propagation of *surface* solitary waves, eventually *solitons* of the true surface-wave type (amplitude decreasing in the substrate) in a structure made of a *nonlinear* elastic isotropic *substrate* (half-space $X_2 > 0$) and a superimposed *linear* elastic isotropic thin film, the latter being perfectly bonded to the former (Figure 1). The *nonlinearity* originates thus from the substrate while the *dispersion* is induced by the film which plays the role of a *wave guide* [15]. In the mathematical description the thin film is reduced to an *interface* of vanishing thickness which, however, still carries a mass density (hence inertia) and membrane elasticity in agreement with a general continuum approach [2]. A general surface wave problem in this structure involves both an SH (shear horizontal) elastic scalar component (polarized along X_3) and a Rayleigh two-component displacement polarized parallel to the so-called sagittal plane P_S [16], [19]. The complete coupled nonlinear problem is a tedious one which is shown to be tractable in several steps. First, in the *linear* approximation an SH *dispersive* surface mode of the type of Murdoch [25], and a classical (non dispersive) Rayleigh mode propagate independently of one another as a consequence of the assumed isotropy of the materials. At the *next order,* both modes couple through the nonlinearityy [17]. However, if the primary signal entered in the system through a transducer is of the SH type and is $O(\varepsilon)$, then the Rayleigh subsystem will develop an $O(\varepsilon^2)$ component. This nonlinear mutual coupling [17] is neglected in the first instance where the pure *nonlinear* SH mode is shown to be governed by a single cubic Schrödinger (NLS) equation at the *interface* for modulated signals with slowly varying envelope [7], [21]. Then the problem accounting for the nonlinear coupling with the Rayleigh components is shown to be reducible to the announced generalized Zakharov system [10], when the main field still is of the SH type. Here essentially numerical simulations are presented, further analytical and numerical results being found in other works [8]-[9], [5]-[6].

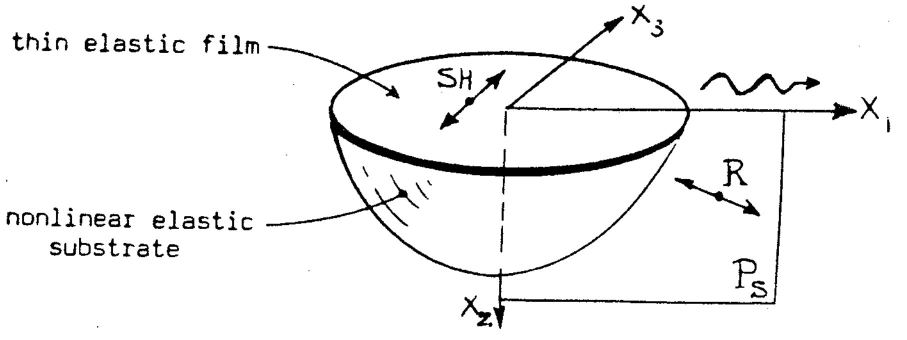

Fig. 1 - Setting of the surface elastic - wave problem.

2. Pure SH surface-wave problem.

In the simplified case, using the boundary condition provided by the theory of material interfaces [2], we find that the initial mechanical (two-space-dimensional) problem is governed by the following set of equations rewritten in nondimensional units [7], [21]:

$$(2.1)\qquad \begin{cases} \beta^2 U_{tt} - (U_{xx} + U_{yy}) = \beta^2 \Delta[(U_x \Phi)_x + (U_y \Phi)_y] \\ \qquad\qquad\qquad\qquad \text{for } X_2 = y > 0, \\ \hat{U}_{tt} - \hat{U}_{xx} = U_y(1 + \beta^2 \Delta \Phi), \\ U = \hat{U} \text{ at } X_2 = y = 0, \\ U(x, y \to +\infty, t) = 0, \end{cases}$$

where $\Phi = U_x^2 + U_y^2$, subscripts indicate space (x and y) and time (t) differentiations, Δ is the *nonlinearity* parameter and β is the *dispersion* parameter. In the absence of nonlinearity $\Delta = 0$, the above system yields Murdoch's linear surface waves [25]; in the absence of dispersion (zero left-handside in $(2.1)_2$), it yields *Mozhaev's nonlinear surface waves* [24], [4]. The *full* system, however, possesses all good ingredients to exhibit *solitary waves* of the surface-wave type (so as to satisfy the last of (2.1)). This is proven analytically by using the Whitham-Newell [1], [26] technique of treatment of nonlinear dispersive small amplitude, almost monochromatic waves [8]-[21]. In the process, «*waves action*» conservation laws and «dispersive» nonlinear dispersion relations are established for this type of surface waves that could also be approached by using Whitham's averaged-Lagrangian technique [29] as modified by Hayes [12] to account for a transverse modal behavior. The full analysis [21] is conducted simultaneously in the bulk ($y > 0$) and at the interface ($y = 0$); Combining then the two *at the interface* results in a single nonlinear Schrödinger (NLS) equation for the envelope of complex amplitude a for the SH mode (in reduced coordinates):

$$(2.2)\qquad i a_t + p a_{xx} + q|a|^2 a = 0,$$

where p and q are real and depend on the working regime (ω_0, k_0) of the carrier wave along the linear dispersion relation of Murdoch's waves. Explicitly,

$$(2.3)\qquad p(\omega_0, k_0) = \frac{1}{2}\omega_0'', \; q(\omega_0, k_0) = \frac{3}{8}\Delta\beta^4 \omega_0 \frac{\beta^2 \omega_0^2 - 2k_0^2}{\beta^2 + 2(w_0^2 - k_0^2)},$$

where ω_0'' is the curvature of the linear dispersion relation. The NLS equation (2.2) is *exactly integrable* [31] and admits *bright* and *dark* envelope (true) solitons depending on the sign of the product pq as is well known in nonlinear optics [3], [11]. If the nonlinear material making up the substrate is known (e.g., $LiNbO_3$ [17] for which $\Delta > 0$) then this criterion allows one to select the thin-film material to guarantee the existence of the desired stable surface solitary wave. In the present case with $\Delta > 0$, $1/2 < \beta^2 < 1$ (film of aluminium) and $\beta^2 < 1/2$ (film of gold) provide stable bright and dark solitons, respectively [21]. The analytical solutions thus obtained are used as initial boundary value conditions in characteristic coordinates in *direct* numerical simulations performed on the original (obviously not exactly integrable) two-space dimension system (2.1). Explicit and implicit numerical finite-difference schemes in three-dimensinal Euclidean space-time grids were used for this (see [9] for technical details). The surface waves indeed propagate as *solitary* waves along the $X_1 = x$ direction with a nice exponential decrease of amplitude with depth (along $X_2 = y$), Figure 2. A lack of accurary in numerical

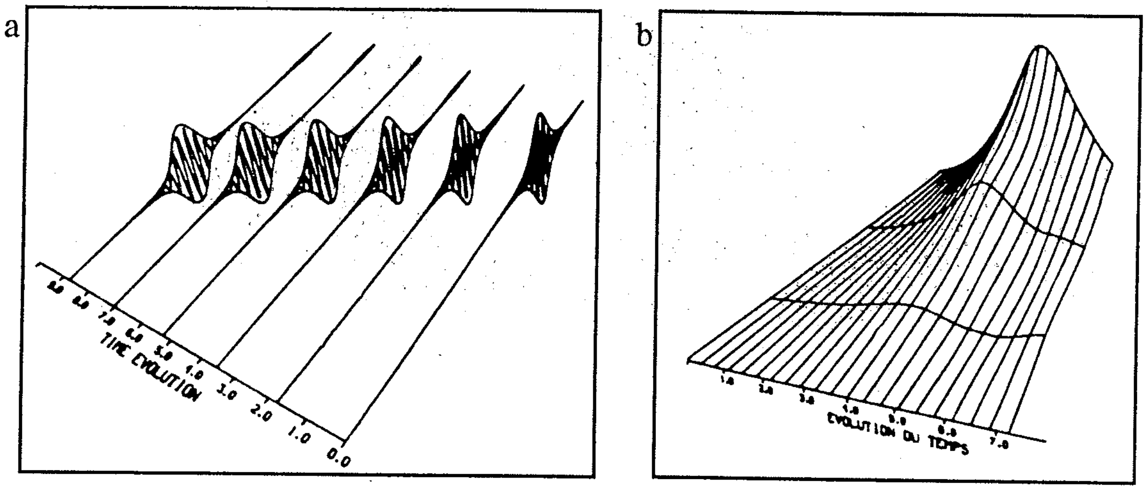

Fig. 2 - Propagation of a bright enevelope soliton in system (2.1): (a) true signal at the interface, (b) decrease of amplitude $|a|^2$ with depth.

solutions in depth will show up at the interface sooner or later. A predominance of nonlinear effects over dispersion may yield the formation of *surface shock waves* after a typical steepening [9]. While (2.2) obviously exhibits a true *solitons* behaviour in soliton interactions, the *rather pure* solitonic behavior of system (2.1), for small amplitudes, can only be checked numerically. This indeed is practically the case as shown in Figure 3. which exhibit the interaction of two (head-on) colliding unequal solitons in the SH system at different depths in the substrate (fifty layers are accounted for in the computation along depth); At this point it should be noted that there is no difficulty to account for the *viscosity* in the pure SH system and the subsequent alteration in (2.2).

3. Coupled SH-rayleigh problem.

In agreement with Section 1 the main displacement field $O(\varepsilon)$ is the SH component, in which case the Rayleigh components are $O(\varepsilon^2)$. Considering slow-varying envelope solution for the SH component, a long asymptotic evaluation [10] allows one to show that, after integration along the transverse coordinate y, and appropriate scaling, the whole problem is governed at $y = 0$ by the following system of equations for the *complex* amplitude a of the SH mode and the real fields n_1 and n_2 for which the components v and w of elastic displacement along x and y, at $y = 0$, are longitudinal potentials ($n_1 = v_x$, $n_2 = w_x$), respectively (Figure 1)

$$\begin{cases} ia_t + a_{xx} \pm 2\lambda|a|^2 a + 2a(\alpha_L n_1 + \alpha_T n_2) = 0, \\ (n_1)_{tt} - c_L^2(n_1)_{xx} - \eta_L(n_1)_{xxt} = -\mu_L(|a|^2)_{xx}, \\ (n_2)_{tt} - c_T^2(n_2)_{xx} - \eta_T(n_2)_{xxt} = -\mu_T(|a|^2)_{xx}, \end{cases} \tag{3.1}$$

where viscosity has been introduced for the Rayleigh components only (see above made remark); System (3.1), a *nearly* integrable system only (in contrast to (2.2)), is a system which generalizes the system of Zakharov [30], [13] for which $\lambda = 0$, $n_2 = 0$, $\alpha_T = \mu_T = 0$, $\eta_L = 0$, that appears in ion-acoustic systems in plasmas (Langmuir solitons for which a and n_1 pertains to the electric field and ion density, respectively). This system has been extensively studied analytically. The general system (3.1) obviously is richer and presents many interesting features. Two of these are more particularly examined below.

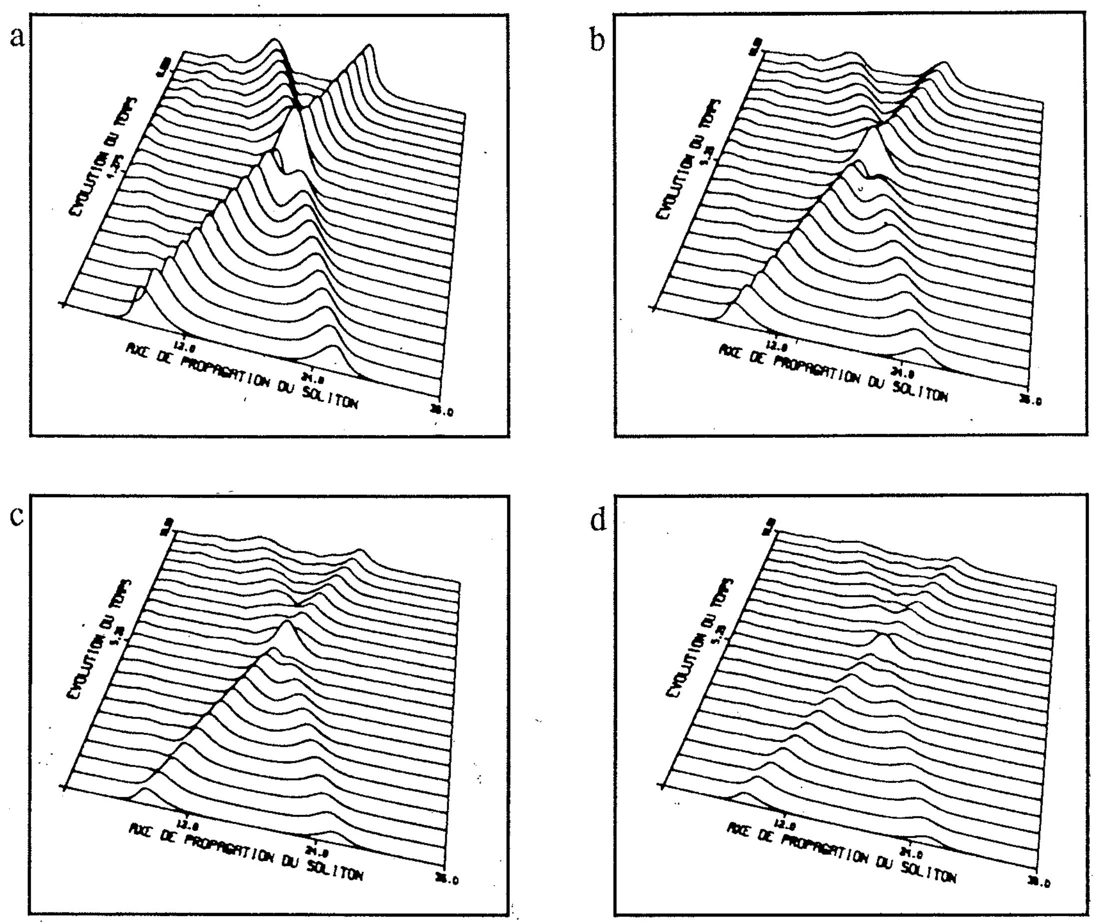

Fig. 3 - Head - on collision of unequal envelope solitons of the surface wave type in system (2.1): (a) 1st layer (top - interface), (b) 15th layer from top, (c) 25th layer from top, (d) 35th layer from top.

4. Dissipation-induced evolution of solitons.

We consider first the evolution of envelope solitary waves in the SH-dispersive a-system of (3.1) under the influence of dissipation (viscosities η_L and η_T) in the nondispersive (n_1, n_2) Rayleigh subsystem. The two are coupled through the coupling coefficients μ_L and μ_T. In spite of its appearance, system (3.1) still conserve the number of (SH) surface phonons (or wave action)

$$N = \int_{-\infty}^{+\infty} |a|^2 dx. \tag{4.1}$$

In the analytical treatment [5] which applies the *balance-equation analysis* [14] to the *slow* dissipation-induced evolution of the *exact* one-soliton solution of the Zakharov system (for the sake of simplicity $w = 0$, $\mu_T = 0$ in (3.1), but this is justified [10]) although this system is *not* exactly integrale, three different scenarii of evolution are shown to be possible: (i) adiabatic (slow) transformation of a moving *subsonic* soliton into the stable quiescent one, (ii) complete adiabatic decay of a *transsonic* soliton with a small amplitude, and (iii) coming of the *transsonic* soliton with a large amplitude into

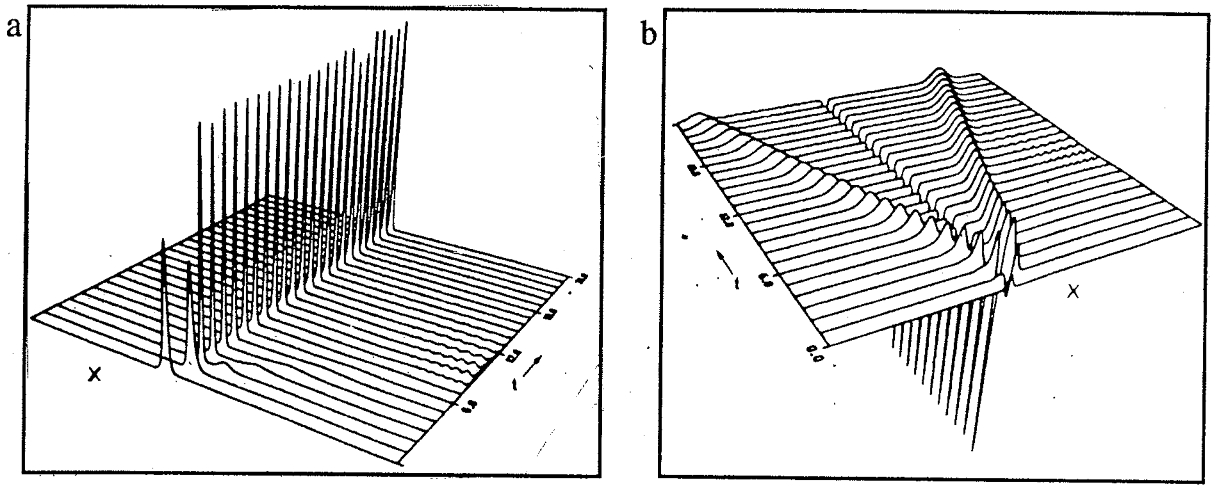

Fig. 4 - Dissipation - induced evolution of the exact one - soliton solution of Zakharov's system; abrupt split into three pulses in the n - system (large wave action, large velocity): (a) $|u|^2$, (b) $-n$.

a critical state, from which further adiabatic evolution is *not* possible. In the latter case a numerical investigation of the further evolution of the soliton is particularly enlightening. In a general case, it is shown that it abruptly splits into the stable quiescent soliton, the slowly decaying small-amplitude transsonic one, and a pair of left- and right-running acoustic pulses slowly fading under the action of the weak dissipation; This is exhibited in Figures 4 and 5. The abrupt splitting seems to be a new type of *inelastic* process for a soliton, induced by small perturbations (see the review in [14]). This concludes our brief excursion in the evolution of *one* soliton in the *damped* generalized Zakarov system.

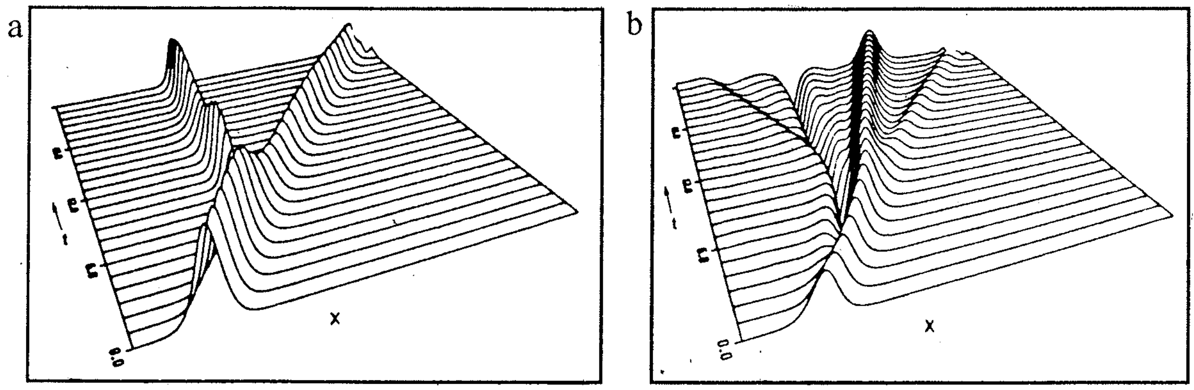

Fig. 5 - Dissipation - induced evolution of the exact one - soliton solution of Zakharov's system; rearrangement (perestroïka) of the soliton in the intermediate case (smaller values of wave action and velocity than in Figure 4).

5. Soliton-Soliton collision in the generalized zakharov system.

As seen above the generalized Zakharov system (3.1), in the absence of viscosity in the Rayleigh subsystem, admits both *subsonic* and *transsonic* one-soliton solutions. The question naturally arises of the interaction (collision) of such solitons (i.e., whether they are *indeed* solitons), for instance in symmetric soliton-soliton collisions. In the analytical study given elsewhere [6], the collision-induced emission of acoustic waves (in the Rayleigh subsystem) was treated for soliton velocities much larger than their amplitudes. In particular, it was shown that the acoustic losses are exponentially small unless the velocities are much larger than the characteristic sound velocity (c_L) in the Rayleigh sybsystem. The numerical simulation of the head-on soliton-soliton collision brings up two *basic phenomena*: (i) the collision of *subsonic* solitons always lead to their *fusion into a breather*, provided the system is sufficiently far from the integrable limit (i.e., the NLS case), and (ii) the collision between *transsonic* solitons gives rise to a *multiple production* of solitons (both subsonic *and* transsonic) and the *quasi-elastic* character of the collision is recovered in the limit of large velocities. This is illustrated in Figures 6, 7 and 8.

6. Conclusion.

It appears that the initial, purely mechanical, surface-wave problem considered yields, on the hand, a very interesting physical application which may be of interest in signal processing (we have a mechanical analog of light solitons guided by optical fibers and expect the experimental device to be realized soon) and, on the

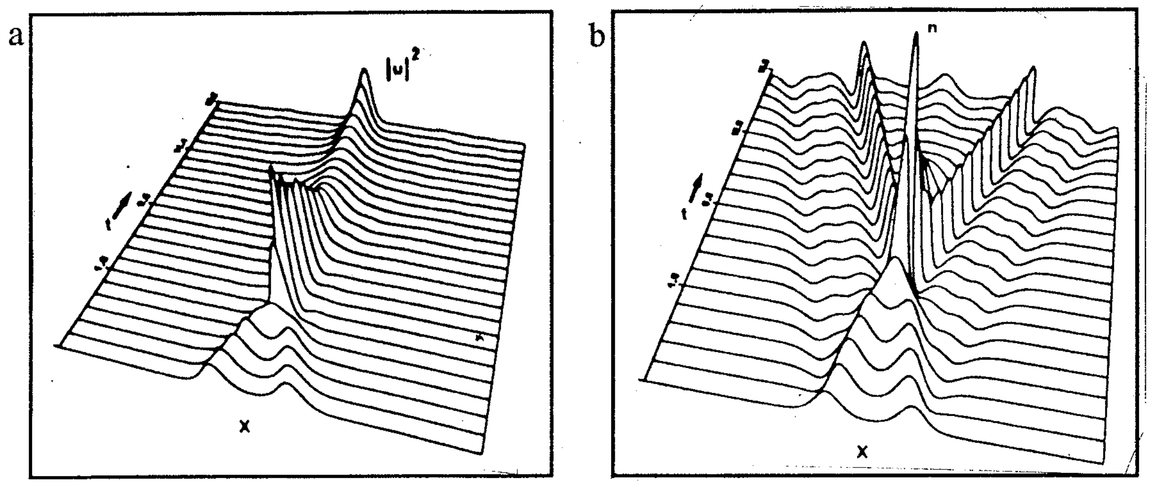

Fig. 6 - Collision - induced fusion of subsonic solitons into a breather with acoustic emission in the Rayleigh subsystem: (a) $|u|^2$, (b) n.

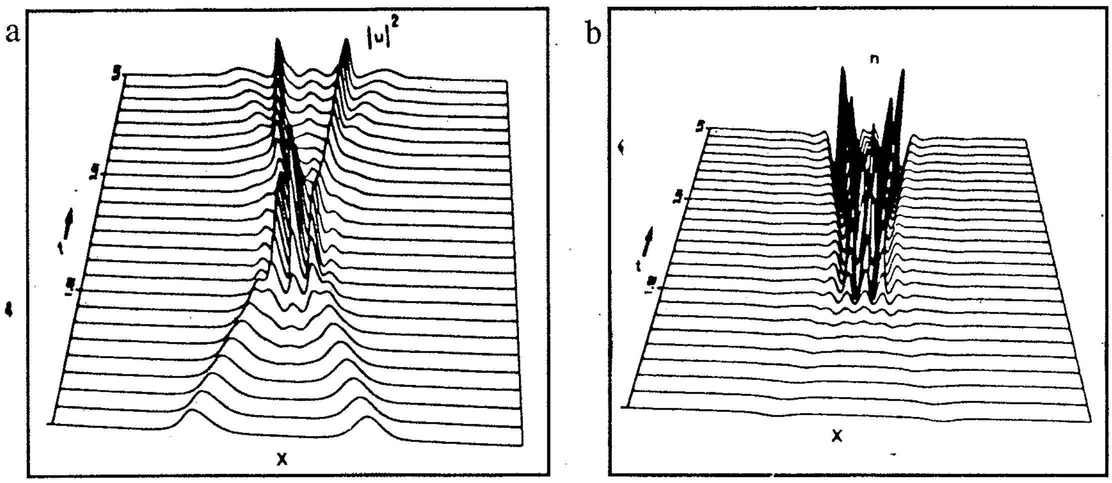

Fig. 7 - Collision of two transsonic soliton at moderate velocities : (a) $|a|^2$, (b) n.

other hand, a class of paradigmatic problems in soliton theory for *nearly integrable* systems made of an exactly integrable equation

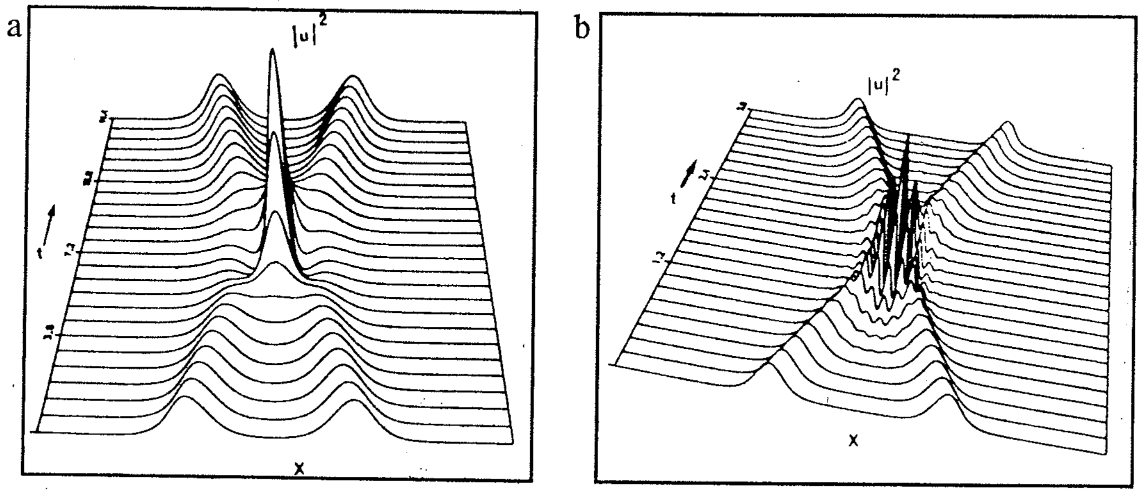

Fig. 8 - comparison between the soliton - soliton collision for system (3.1) close to the integrable NLS limit (a), and the collision of two transsonic solitons at high velocities in the generalized Zakharov system (b).

coupled nonlinearly to one or two d'Alembert equations. The *sine-Gordon d'Alembert* systems introduced previously by Maugin and Pouget in elastic ferromagnets [22], elastic ferroelectrics of the molecular group type [27], and more generally (and abstractly) in oriented elastic solids [23] [28], belong to the same class. The *modified-boussinesq-d'Alembert* system introduced recently by Maugin and Cadet [20] in martensitic (shape-memory) alloys appears to be even richer in so far as the variety of dynamical behaviors is concerned (see works by C.I. Christov and G.A. Maugin).

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