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PURE-JUMP PROCESSES AND CONSTITUTIVE EQUATIONS FOR SIMPLE THERMODYNAMIC BODIES WITH FADING MEMORY

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Can it be useful to use discontinuous jump-processes in order to formulate a somewhat different thermodynamic theory for a general simple body with fading memory? In this communication *I* will present the results of paper [9], where the above question is investigated. By means of a certain well defined class of *quasi-processes of local pure-jump*, there *I* set up a thermodynamic theory $\mathbf{T}^*$ for such a body in which only the dynamic part of entropy is assumed to exist. The postulated dissipation inequality does not involve the pure-jump part of entropy, where this part is defined through its rate-of-change along the process of local pure-jump associated with the actual process, at any time and material point. Theory $\mathbf{T}^*$ is more general than the corresponding ordinary theory $\mathbf{T}$ with the Clausius-Duhem inequality, because in $\mathbf{T}^*$ one can conceive materials for which the pure-jump entropy has no constitutive equation. A certain integrability condition, involving only the pure-jump parts of stress and internal energy, is equivalent to the existence of a response function for the pure-jump entropy. By adding to theory $\mathbf{T}^*$ this integrability condition, as an axiom, one obtains a theory $\mathbf{T}^*_c$ which is equivalent to the classical theory $\mathbf{T}$, in the sense that any theorem of $\mathbf{T}$ can be proved in $\mathbf{T}^*_c$.

In paper [9], by means of a certain well defined class of quasi-processes of local pure-jump, *I* set up a theory $\mathbf{T}^*$, for a general simple body with fading memory, in which only the dynamic part of entropy is primitive. Following [3,7], there I consider the possibility for the constitutive equations of a given simple body not to be uniquely determined by balance laws, dissipation inequality and boundary conditions for stress and heat flux; that is, *possible indeterminations in the response functionals are allowed, at least a priori.* For detailed informations on this point of view read the introductions of [3,7].

Briefly, a simple body $\mathcal{B}$ with fading memory (in Coleman-Noll's sense) is one for which there is some *(generalized) set of response functionals*, connected with a given reference configuration of it. This is formed by fucntionals $\bar{\mathbf{P}}$, $\bar{q}$, $\bar{e}$ and $\bar{\eta}$, for which the balance laws of linear momentum, angular momentum, of energy, and the Clausius-Duhem inequality, hold along any process (smooth enough and) physically possible for $\mathcal{B}$. In these laws $\bar{\mathbf{P}}$ to $\bar{\eta}$ are used as $\mathcal{B}$'s functionals for stress, heat flux, specific internal energy, and specific entropy, respectively.

In this definition an essential use of the physical possibility notion is made. Incidentally note that, following [1, 2, 3, 7], in paper [9] I regard the concept of *physical possibility* as primitive. I do not use the assertion "the process $\hat{p}$ is physically possible" as an equivalent of "the process $\hat{p}$ is compatible with the physical laws". Rather that assertion means here that "$\hat{p}$ can be carried out by ideal experiments". I use *physical possibility* in the second sense and regard the physical possibility of a process not to be equivalent to its compatibility with axioms (1).

The key difference with usual treatments of continua is that – in accordance with the Mach-Painlevé point of view – the balance laws and the entropy inequality are not postulated as axioms, before giving constitutive equations; rather, they are conditions which are imposed in the definition of a simple body. In this way the possibility for the constitutive equations not to be unique is admitted.

That some indeterminations in constitutive equations of some continuous media can really occur has been shown e.g. (i) in [5] for the entropy functional of a rigid heat conductor with memory, (ii) in [6] for the heat flux functional of any simple body, and (iii) in [10] for the response functions of entropy, internal energy, and heat flux, in

(1) Thus processes of body $\mathcal{B}$ compatible with physical laws may fail to be physically possible. In particular some conceivable thermokinetic processes of $\mathcal{B}$ may fail to be possible.

certain differential bodies.

An *ordinary functional for the stress* is a generalized one for which the normal stress determined by it at the boundary of the body necessarily vanishes whenever this "touches" an empty space region – see N. 1 and def. 6.1 in [3]. In [3] it has been proved a uniqueness theorem for the generalized function for the stress of any hyper-elastic body. This theorem asserts that *any two generalized functions for the Cauchy stress differ by a Eulerian pressure, which is constant on the whole body.* Then, by using the possibility of putting some small part of the boundary of the body in contact with vacuum, the strict uniqueness of the ordinary response function for the stress in proved. In paper [7] these theorems have been extended to thermodynamics, as far as elastic bodies are concerned.

Let $p(\cdot) = (\mathbf{x}(\cdot), \theta(\cdot))$ be any physically possible thermokinetic process for $\mathcal{B}$, where $\mathbf{x}(X,t)$ is the position and $\theta(X,t)$ is the absolute temperature at the material point X and at time t, and set $\Lambda = (\mathbf{F}, \theta, \mathbf{G})$, with $\mathbf{F} = Grad\mathbf{x}(X,t)$, $\theta = \theta(X,t)$, and $\mathbf{G} = Grad\theta(X,t)$.

The pure-jump response functions, associated with the response functionals of any general simple body with fading memory, are defined by considering certain families $\Lambda_0 := \{\Lambda_\varepsilon(\cdot,\cdot)\}_{\varepsilon>0}$ of possible smooth (local) processes, $\Lambda_\varepsilon = \Lambda_\varepsilon(X,t)$, which are called (possible) *quasi-processes of local pure-jump.* At a given material point X_0 and time t the histories of Λ_ε, of its rate $\dot{\Lambda}_\varepsilon$, and of its gradient $GRAD\Lambda_\varepsilon$, are close to histories which are constant in the past and which jump at time t. The pure-jump response functions at X_0 are just the response functionals evaluated along histories which are the limit at X_0, as $\varepsilon \to 0$, of the histories of Λ_ε, $\dot{\Lambda}_\varepsilon$ and $GRAD\Lambda_\varepsilon$, where $\Lambda_\varepsilon(\cdot,\cdot) \in \Lambda_0$.

Then in [9] *I* prove uniqueness theorems for the pure-jump response functions associated with the response functionals of generalized stress, ordinary stress, and internal energy. For the pure-jump stresses the results exactly agree with the afore-mentioned results in the thermo-elastic case. Instead the uniqueness theorem for the pure-jump internal energy e^* differs considerably from its analogue in that case. In fact any two response functions $\bar{e}^*$ and $\tilde{e}^*$ for the pure-jump internal energy differ by a function of the material point and the (constant) past history:

$$\bar{e}^* - \tilde{e}^* \equiv \bar{E}(\Lambda^*(\cdot), X), \quad \text{where} \quad \Lambda^*(s) \equiv \Lambda^* \quad \text{for each} \quad s > 0.$$

Fortunately the rate-of-change of this indetermination function identically vanishes, and thus the rate-of-change $\dot{e}^*$ is unique:

$$\dot{E} = \dot{\bar{e}}^* - \dot{\tilde{e}}^* \equiv 0 \qquad \text{– see [9, Theorem 7.1].}$$

This fact gives a physically satisfactory character to the rate-of-change of the pure-jump internal energy.

Hence in [9, N. 9] one can prove the validity of the Gibbs relation

$$\theta\dot{\eta}^* - \dot{e}^* + \mathbf{P}^* \cdot \dot{\mathbf{F}}/\rho \equiv 0$$

(ρ =mass-density, $\dot{\mathbf{F}} = (\partial/\partial t)Grad\mathbf{x}(X,t)$), stating the vanishing of the internal dissipation, in the limit, along any quasi-process of local pure-jump. This relation allows us to easily prove that also $\dot{\eta}^*$ is unique, and to easily ascertain that η^* suffers an indetermination exactly coinciding with the one suffered by e^*. Incidentally note that this Gibbs relation formally agrees with the one in [4] – see (10.21) there –, which holds to within an error of magnitude $o(\alpha)$, along any α-retarded process ($0 < \alpha < 1$).

In [9, N. 10] it is pointed out that the difference $Q^* = \bar{\mathbf{q}}^*_\kappa - \tilde{\mathbf{q}}^*_\kappa$ between any two response functions for the pure-jump heat flux has a form which exaclty agrees with the one giving the maximal indetermination for the response function of the heat flux in a thermo-elastic body:

$$Q^* = \mathbf{G} \times Grad\varphi \quad (Q^{*A} = \varepsilon^{ABC} G_B \varphi_{/C},\ A = 1,2,3)$$

for some function $\varphi = \varphi(\theta, X)$ – see [6].

It is often asserted that *a simple body with fading memory behaves like an elastic one in sufficiently slow (or static) processes*. In [9, Part 1] this assertion is in effect extended to quasi-processes of local pure-jump.

Lastey, in [9, Part 2] I set up a thermodynamic theory $\mathbf{T}^*$ for the above body with fading memory in which (i) *the Clausius-Duhem inequality is replaced with the dissipation inequality*

$$\rho[\theta\dot{\eta}^d - (\dot{e} - \dot{e}^*)] + (\mathbf{P} - \mathbf{P}^*) \cdot \dot{F} \geq \theta^{-1}\mathbf{q}_\kappa \cdot \mathbf{G},$$

not involving the pure-jump part of entropy; (ii) only the notion of dynamic entropy is primitive, and (iii) *entropic rate-of-change of pure-jump can be defined, through the above Gibbs relation, in terms of pure-jump stress and internal energy.* Hence the induced notion of rate-of-change of entropy – defined as the sum of the rate-changes for pure-jump entropy and dynamic entropy – exactly coincides with the one in the corresponding classical theory **T**, based on the Clausius-Duhem inequality.

Then in analogy with the theory developed in [8], where differential bodies of complexity one are treated, it is pointed out that the existence of a response function for the pure-jump entropy is equivalent to the following integrability conditions

$$\frac{{}'\partial}{{}'\partial\theta}\left[\theta^{-1}\left(\frac{{}'\partial e^*}{{}'\partial \mathbf{F}}-\rho^{-1}\mathbf{P}^*\right)\right]=\frac{{}'\partial}{{}'\partial \mathbf{F}}\left(\theta^{-1}\frac{{}'\partial e^*}{{}'\partial\theta}\right),$$
$$\frac{{}'\partial}{{}'\partial \mathbf{G}}\left(\theta^{-1}\frac{{}'\partial e^*}{{}'\partial\theta}\right)=\frac{{}'\partial}{{}'\partial\theta}\left(\theta^{-1}\frac{{}'\partial e^*}{{}'\partial \mathbf{G}}\right),$$
$$\frac{{}'\partial}{{}'\partial \mathbf{G}}\left[\theta^{-1}\left(\frac{{}'\partial e^*}{{}'\partial \mathbf{F}}-\rho^{-1}\mathbf{P}^*\right)\right]=\frac{{}'\partial}{{}'\partial \mathbf{F}}\left(\theta^{-1}\frac{{}'\partial e^*}{{}'\partial \mathbf{G}}\right),$$

for the system of PDEs below, which involve the response functions of pure-jump internal energy and stress.

$$\frac{{}'\partial\eta^*}{{}'\partial \mathbf{F}}=\theta^{-1}\left(\frac{{}'\partial e^*}{{}'\partial \mathbf{F}}-\rho^{-1}\mathbf{P}^*\right),$$
$$\frac{{}'\partial\eta^*}{{}'\partial\theta}=\theta^{-1}\frac{{}'\partial e^*}{{}'\partial\theta},$$
$$\frac{{}'\partial\eta^*}{{}'\partial \mathbf{G}}=\theta^{-1}\frac{{}'\partial e^*}{{}'\partial \mathbf{G}}.$$

Hence *one can assume the existence of both materials for which these integrability conditions hold and others* – see [9, N. 16]. *Thus theory* $\mathbf{T}^\star$ *is more general than theory* **T**. By postulating these integrability conditions, theory $\mathbf{T}^\star$ becomes a theory, $\mathbf{T}^\star_c$, which is equivalent – i.e. it has the same theorems – to the ordinary theory **T** based on the Clausius-Duhem inequality. Hence, by the afore-mentioned results of [9, Part 2], one can assert that

The static part of entropy needs not be used as a primitive concept in any classical thermodynamic theory for simple bodies with fading memory. Furthermore, with regard to these bodies thermostatics can be set up using the entropy notion as defined.

All the afore-mentioned results have been deduced under a very weak assumption of frame-indifference: we only need that the pure-jump response functions are invariant under Galilean transformations of space-time.

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