LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 283-294

ON THE LYAPUNOV STABILITY OF A PLANE PARALLEL CONVECTIVE FLOW OF A BINARY MIXTURE

G. MULONE (Catania)

The nonlinear stability of plane parallel convective flows of a binary fluid mixture in the Oberbeck-Boussinesq scheme is studied in the stress-free boundary case. Nonlinear stability conditions independent of Reynolds number are proved.

1. Introduction.

In recent papers [1,2] the Lyapunov direct method has been used to study the nonlinear conditional stability of parallel shear flows and of plane parallel convective flows of a homogeneous fluid.

Here, we generalize the results obtained in [2] to a binary fluid mixture in the Oberbeck-Boussinesq scheme, in the stress-free boundary case.

We consider Couette and Poiseuille flows of a mixture: (a) heated from below and salty from above, (b) heated from above and salty below, (c) heated and salty below. The case heated and salty from above is obatined by exchanging the roles of the Rayleigh number $\mathcal{R}_a{}^2$ for heat and the Rayleigh number $\mathcal{C}^2$ for solute. Then we set the linear stability problem and show a Squire theorem: we prove that the three-dimensional problem is equivalent to a two-dimensional one at a smaller Reynolds number R.

Then we study nonlinear stability. We show that the classical energy method gives nonlinear stability conditions which *depend* on the Reynolds number.

In order to prove a nonlinear stability condition which *do not depend* on the Reynolds number, following the guidelines given in [3-8], we introduce a new Lyapunov function and prove nonlinear stability conditions which are *independent* on the Reynolds number.

2. Basic equations.

Let us consider a layer of a binary fluid mixture in the Oberbeck-Boussinesq scheme, bounded by two horizontal parallel planes. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ and $Oxyz$ be a cartesian frame of reference with unit vectors **i**, **j**, **k** respectively. Let us assume that the layer is parallel to the plane xy.

The stationary Oberbeck-Boussinesq equations for a fluid mixture with equation of state

$$\rho = \rho_o[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)] \tag{2.1}$$

are:

$$\begin{cases} \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \dfrac{p_1}{\rho_o} + [1 - \alpha_T(T - T_0) \\ \qquad\qquad + \alpha_C(C - C_0)]\mathbf{g} + \nu \Delta \mathbf{U} \\ \nabla \cdot \mathbf{U} = 0 \\ \mathbf{U} \cdot \nabla T = k_T \Delta T \\ \mathbf{U} \cdot \nabla C = k_C \Delta C \end{cases} \quad \text{in } \Omega_d \tag{2.2}$$

where $\mathbf{U}, T, C$ and p_1 are the velocity, temperature, concentration and pressure fields; ρ_0, T_0, C_0 are reference density, temperature and concentration, respectively; α_T and α_C are volume expansion coefficients, ν is the kinematic viscosity, k_T and k_C are the thermal and solute diffusivity coefficients and $\mathbf{g} = -g\mathbf{k}$ is the acceleration of gravity; finally ∇ is the nabla operator and Δ is the three-dimensional laplacian.

We shall assume the following boundary conditions:

$$(2.3)\qquad \begin{cases} \mathbf{U}(x,y,-\frac{d}{2}) = -V\mathbf{i} & \mathbf{U}(x,y,\frac{d}{2}) = V\mathbf{i} \\ T(x,y,-\frac{d}{2}) = T_1 & T(x,y,\frac{d}{2}) = T_2 \qquad (x,y)\in \mathbb{R}^2 \\ C(x,y,-\frac{d}{2}) = C_1 & C(x,y,\frac{d}{2}) = C_2 \end{cases}$$

where V,T_1,T_2,C_1 and C_2, are assigned real constants.

Let us consider a *basic* laminar solution $m_o = (\mathbf{U}, T, C, p_1)$ of the boundary value problem (2.2) - (2.3).

The non-dimensional equations which govern the evolution disturbance $(\mathbf{u}, \vartheta, \gamma, p)$ of the basic motion m_o are:

$$(2.4)\qquad \begin{cases} \mathbf{u}_t + RU(z)\mathbf{u}_x + RwU'(z)\mathbf{i} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \\ \qquad + (\mathcal{R}_a\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ P_T\big(\vartheta_t + RU(z)\vartheta_x + \mathbf{u}\cdot\nabla\vartheta\big) = -\mathcal{R}_a\mathbf{u}\cdot\mathbf{k}_T + \Delta\vartheta \\ P_C\big(\gamma_t + RU(z)\gamma_x + \mathbf{u}\cdot\nabla\gamma\big) = -\mathcal{C}\mathbf{u}\cdot\mathbf{k}_C + \Delta\gamma, \end{cases}$$

in $\Omega_1 \times (0,\infty)$, where $\Omega_1 = \mathbb{R}^2 \times (-\frac{1}{2},\frac{1}{2})$, with initial condition

$$(2.5)\qquad \begin{cases} \mathbf{u}(x,y,z,0) = \mathbf{u}_o(x,y,z) \\ \vartheta(x,y,z,0) = \vartheta_o(x,y,z) \qquad (x,y,z)\in\Omega_1 \\ \gamma(x,y,z,0) = \gamma_o(x,y,z) \end{cases}$$

and boundary conditions

$$(2.6)\qquad \mathbf{u}(x,y,\pm\frac{1}{2},t) = 0,\ \vartheta(x,y,\pm\frac{1}{2},t) = \gamma(x,y,\pm\frac{1}{2},t) = 0, \quad t>0,$$

in the rigid-rigid case, and

$$(2.7)\qquad \begin{array}{l} w(x,y,\pm\frac{1}{2},t) = 0,\ u_z(x,y,\pm\frac{1}{2},t) = v_z(x,y,\pm\frac{1}{2},t) = 0, \\ \vartheta(x,y,\pm\frac{1}{2},t) = \gamma(x,y,\pm\frac{1}{2},t) = 0, \quad t>0 \end{array}$$

in the stress-free boundary case.

The subscripts x, z and t denote partial derivatives, the prime denotes derivative with respect to z; $\mathbf{u}_0, \vartheta_o, \gamma_o$ are assigned regular fields with $\nabla \cdot \mathbf{u}_0(\mathbf{x}) = 0$, $\mathbf{u} = (u, v, w)$ and

$$U(z) = \begin{cases} 2z & \text{for } \textit{Couette flow}, \\ 1 - 4z^2 & \text{for } \textit{Poiseuille flow}. \end{cases}$$

The stability parameters in (2.4) are the Rayleigh numbers for heat and solute and a Reynolds number R given by

$$\mathcal{R}_a{}^2 = \frac{g\beta_1\alpha_T d^4}{\nu k_T}, \qquad \mathcal{C}^2 = \frac{g\beta_2\alpha_C d^4}{\nu k_C}$$

and $R = \dfrac{Ud}{\nu}$, where β_1 and β_2 are the constant gradients of temperature and concentration, respectively. Moreover $P_T = \dfrac{\nu}{k_T}$ and $P_C = \dfrac{\nu}{k_C}$ are the Prandtl and Schmidt numbers,

$$\mathbf{k}_T = \begin{cases} -\mathbf{k} & \text{(heated below)} \\ \mathbf{k} & \text{(heated above)}, \end{cases} \qquad \mathbf{k}_C = \begin{cases} -\mathbf{k} & \text{(salty below)} \\ \mathbf{k} & \text{(salty above)}. \end{cases}$$

As is usual, we assume that the perturbations are periodic functions of x and y of periods $\dfrac{2\pi}{a_x}$, $\dfrac{2\pi}{a_y}$, respectively, $(a_x > 0,\ a_y > 0)$ and denote by Ω the periodicity cell $\Omega = \left[0, \dfrac{2\pi}{a_x}\right] \times \left[0, \dfrac{2\pi}{a_x}\right] \times \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ and by $a = (a_x^2 + a_y^2)^{\frac{1}{2}}$ the wave number. Moreover we require the "average velocity condition".

3. The linear problem.

If we now assume that all the disturbances have a dependence on x, y and t of the form $\exp\{i(\alpha x + \beta y) - i\alpha c t\}$ then the linearized problem associated to (2.4) gives:

$$(3.1) \qquad \begin{cases} [D^2 - (\alpha^2 + \beta^2) - i\alpha(RU - c)]u = RwU' + i\alpha p \\ [D^2 - (\alpha^2 + \beta^2) - i\alpha(RU - c)]v = i\beta p \\ [D^2 - (\alpha^2 + \beta^2) - i\alpha(RU - c)]w = Dp - \mathcal{R}_a\vartheta + \mathcal{C}\gamma \\ i(\alpha u + \beta v) + Dw = 0 \\ [D^2 - (\alpha^2 + \beta^2) - i\alpha P_T(RU - c)]\vartheta = -\mathcal{R}_a h_1 w \\ [D^2 - (\alpha^2 + \beta^2) - i\alpha P_C(RU - c)]\gamma = -\mathcal{C} h_2 w \end{cases}$$

where $D = \dfrac{d}{dz}$,

$$h_1 = \begin{cases} -1 & \text{(heated below)} \\ 1 & \text{(heated above),} \end{cases} \qquad h_2 = \begin{cases} -1 & \text{(salty below)} \\ 1 & \text{(salty above),} \end{cases} \tag{3.2}$$

and now $u, v, w, p, \vartheta, \gamma$ are function of z.

These equations together with the boundary conditions

$$u = v = w = \vartheta = \gamma = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{3.3}$$

or

$$Du = Dv = w = \vartheta = \gamma = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{3.4}$$

define the three-dimensional linear stability problem (in the rigid-rigid case and in the stress-free case, respectively).

We now wish to show that this three-dimensional problem can be reduced, by means of Squire's transformation, to an equivalent two-dimensional problem.

For this purpose we let

$$\begin{cases} \tilde{\alpha}\tilde{u} = \alpha u + \beta v, \quad \tilde{w} = w, \quad \tilde{p} = p, \quad \tilde{\vartheta} = \vartheta \\ \tilde{\gamma} = \gamma, \quad \tilde{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}}, \quad \tilde{c} = c\frac{\alpha}{\tilde{\alpha}}, \\ \tilde{P_T} = P_T, \quad \tilde{P_C} = P_C, \quad \tilde{R} = \frac{\alpha}{\tilde{\alpha}}R, \quad \tilde{\mathcal{R}}_a = \mathcal{R}_a, \quad \tilde{\mathcal{C}} = \mathcal{C}. \end{cases} \tag{3.5}$$

The equations (3.1) are thereby reduced to the following form:

$$\begin{cases} [D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}(\tilde{R}U - \tilde{c})]\tilde{u} = \tilde{R}\tilde{w}U' + i\tilde{\alpha}\tilde{p} \\ [D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}(\tilde{R}U - \tilde{c})]\tilde{w} = D\tilde{p} - \tilde{\mathcal{R}}_a\tilde{\vartheta} + \tilde{\mathcal{C}}\tilde{\gamma} \\ i\tilde{\alpha}\tilde{u} + D\tilde{w} = 0 \\ [D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{P}_T(\tilde{R}U - \tilde{c})]\tilde{\vartheta} = -\tilde{\mathcal{R}}_a h_1 \tilde{w} \\ [D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{P}_C(\tilde{R}U - \tilde{c})]\tilde{\gamma} = -\tilde{\mathcal{C}} h_2 \tilde{w}. \end{cases} \tag{3.6}$$

These equations together with the boundary conditions

$$\tilde{u} = \tilde{w} = \tilde{\vartheta} = \tilde{\gamma} = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{3.7}$$

or

$$D\tilde{u} = \tilde{w} = \tilde{\vartheta} = \tilde{\gamma} = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{3.8}$$

have the same mathematical structure as equations (3.1)-(3.4) with $\beta = v = 0$ and they thus define the equivalent two-dimensional problem. From this it follows the Squire Theorem:

THEOREM 3.1. *The three-dimensional problem (3.1)-(3.4) is equivalent to the two-dimensional problem (3.6)-(3.8) at a smaller Reynolds number.*

4. Non-linear stability.

The nonlinear stability has been studied with the classical energy method by Joseph [9]. He used the "energy" $E(t)$

$$E(t) = \frac{\|\mathbf{u}\|^2 + P_T\|\vartheta\|^2 + P_C\|\gamma\|^2}{2}, \tag{4.1}$$

with $\|\cdot\|$ the $L_2(\Omega)$ - norm.

Denoting by

$$\begin{aligned} &\mathrm{IR}_T = (R^2 + \mathcal{R}_a{}^2 + \mathcal{C}^2)^{\frac{1}{2}}, \quad \mathcal{A}_R = \frac{R}{\mathrm{IR}_T}, \\ &\mathcal{A}_{\mathcal{R}_a} = \frac{\mathcal{R}}{\mathrm{IR}_T} \quad \mathcal{A}_C = \frac{\mathcal{C}}{\mathrm{IR}_T} \end{aligned} \tag{4.2}$$

he showed that the basic flow is globally and monotonically stable if

$$\mathrm{IR}_T^2 < \mathrm{IR}_C^2 \tag{4.3}$$

where IR_C is given by the maximum problem

$$\frac{1}{\mathrm{IR}_C} = \max_{\mathcal{H}} \frac{(U'(z)\mathcal{A}_R u - 2\mathcal{A}_{\mathcal{R}_a}\theta + 2\mathcal{A}_C\gamma, w)}{\|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 + \|\nabla\gamma\|^2}, \tag{4.4}$$

where $\mathcal{H}$ is the space of "admissible functions" :

$$\begin{aligned} \mathcal{H} = \{&\mathbf{u},\ \theta,\ \gamma \text{ regular fields, periodic in } x \text{ and } y, \\ &\nabla\cdot\mathbf{u} = 0, \quad \text{satisfying (2.7) or (2.6) on } z = \pm\tfrac{1}{2}, \\ &0 < \|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 + \|\nabla\gamma\|^2 < \infty\}. \end{aligned}$$

Solving the maximum problem (4.4) with the method used by Joseph for a homogeneous fluid [9-10], it is easy to prove that - for example in the Couette case with stress-free boundaries - we have

$$\mathbb{R}_C{}^2 = \frac{27}{4}\pi^4. \tag{4.5}$$

The stability result (4.3) obtained in this way *depend* on the Reynolds number in all cases: Couette flow, Poiseuille flow and different boundary conditions.

In order to obtain stability conditions *independent of the Reynolds number R*, in this section, we shall study the nonlinear stability with the Lyapunov second method, using a Lyapunov functional different from energy (4.1) according to the methods given in [3-8]. The Lyapunov function V is the sum of two terms $V_0(t)$ and $V_1(t)$. V_0 is a Lyapunov function for the linear stability problem and depends on the essential variables w, ζ, θ, γ, [1,2], while $V_1(t)$ must dominate the nonlinear terms.

The evolution equation of ζ and w are obtained in the usual way, [3-8].

First we give linear stability conditions with Lyapunov method. For this, we consider the Lyapunov function

$$V_0(t) = \frac{1}{2}[||\zeta||^2 + \beta_o||\Delta w||^2 + \gamma_o P_T||\theta||^2 + \gamma_1 P_C||\gamma||^2], \tag{4.6}$$

where β_o, γ_o and γ_1 are positive constants which will be chosen later.

The evolution equation of $V_0(t)$ is:

$$\begin{aligned} \dot{V}_0 = {} & R\int_\Omega U'(z) w_y \zeta \, d\Omega + \mathcal{R}\int_\Omega [\beta_o \Delta_1 \theta \Delta w + \gamma_o h \theta w] \, d\Omega \\ & - \mathcal{C}\int_\Omega [\beta_o \Delta_1 \gamma \Delta w + \gamma_1 k \gamma w] \, d\Omega \\ & - [||\nabla\zeta||^2 + \beta_o||\nabla\Delta w||^2 + \gamma_o||\nabla\theta||^2 + \gamma_1||\nabla\gamma||^2]. \end{aligned} \tag{4.7}$$

By using the Schwarz inequality, from (4.7) we have:

$$\dot{V}_0 \le Rm||w_y|| \, ||\zeta|| - ||\nabla\zeta||^2 - \mathcal{R}_a \beta_o \mathcal{D}\left(\frac{1}{\mathcal{R}_a} - \frac{\mathcal{I}}{\mathcal{D}}\right), \tag{4.8}$$

with

$$m = \max_{\frac{1}{2} \le z \le \frac{1}{2}} |U'(z)|, \tag{4.9}$$

$$\mathcal{I} = [(\Delta_1\theta, \Delta w) + \sigma_o h(\theta, w)] - \alpha_r[(\Delta_1\gamma, \Delta w) + \sigma_1 k(\gamma, w)], \tag{4.10}$$

$$\mathcal{D} = ||\nabla\Delta w||^2 + \sigma_o||\nabla\theta||^2 + \sigma_1||\nabla\gamma||^2, \tag{4.11}$$

$\sigma_o = \dfrac{\gamma_o}{\beta_o}, \quad \sigma_1 = \dfrac{\gamma_1}{\beta_o}, \quad \alpha_r = \dfrac{\mathcal{C}}{\mathcal{R}_a}.$

Defining

$$\frac{1}{\mathcal{R}_c} = \max_{\mathcal{G}} \frac{\mathcal{I}}{\mathcal{D}} \tag{4.12}$$

with

$$\begin{aligned} &\mathcal{G} = \{\ w, \theta, \gamma \text{ regular fields, periodic in } x \text{ and } y, \\ &0 < \mathcal{D} < \infty, w = w_{zz} = w_{zzzz} = 0 \\ &\theta = \theta_{zz} = \theta_{zzzz} = \gamma = \gamma_{zz} = \gamma_{zzzz} = 0 \quad \text{on } z = \pm\frac{1}{2}\}, \end{aligned} \tag{4.13}$$

from (4.8) we deduce:

$$\dot{V}_0 \leq Rm||w_y||\,||\zeta|| - ||\nabla\zeta||^2 - \mathcal{R}_a\beta_o\mathcal{D}\left(\frac{1}{\mathcal{R}_a} - \frac{1}{\mathcal{R}_c}\right), \tag{4.14}$$

A simple calculation gives:

$(i)\quad \dfrac{1}{\mathcal{R}_c{}^2} = \dfrac{4(1+\alpha_r{}^2)}{27\pi^4}$ (heated below and salty above),

$(ii)\quad \dfrac{1}{\mathcal{R}_c{}^2} = \dfrac{4}{27\pi^4}$ (heated and salty below),

$(iii)\quad \dfrac{1}{\mathcal{R}_c{}^2} = 0$ (heated above and salty below).

Now we assume that

$$\mathcal{R}_a{}^2 < \mathcal{R}_c{}^2, \tag{4.15}$$

(in the cases (i) and (ii)), from (4.14), by using the following well known inequalities (Poincaré, Wirtinger, Young), after some manipulation we get:

$$\dot{V}_0 \leq -\frac{\pi^2}{c_o}\left(1 - \frac{\mathcal{R}_a}{\mathcal{R}_c}\right) V_o(t) \tag{4.16}$$

with $c_o = \max(1, P_T, P_C)$. Integrating this last inequality, it follows

$$V_o(t) \leq V_o(0) \exp\{-\frac{\pi^2}{c_o}\left(1 - \frac{\mathcal{R}_a}{\mathcal{R}_c}\right) t\} . \tag{4.17}$$

So we have proved the

THEOREM 4.1. *If (4.15) holds, with $\mathcal{R}_c^2$ given by (4.12), then the basic motion m_o is linearly stable in the norm $V_o(t)$ for any Reynolds number.*

Remark. We observe that the same result holds if the maximum given by (4.12) is 0 (i.e. in the case (*iii*)). In this case, it is sufficient to put *formally* $\frac{1}{\mathcal{R}_c{}^2} = 0$ in the previous calculations.

Then we have the following linear stability results in the cases (*i*), (*ii*), (*iii*):

(*iii*) gives a stability condition which holds for every Rayleigh numbers (both for heat and solute) *independent of the Reynolds number*, and this may physically be expected.

(*ii*) gives stability results for every Rayleigh number for solute whenever

$$\mathcal{R}_a{}^2 < 657.511. \tag{4.18}$$

In this case we obtain a stability condition *independent of Reynolds number* and we get a non-destabilizing effect of the concentration. Now in the case of the motionless state of a mixture heated and salty from below, Joseph [9,10] proved a stabilizing effect of concentration on the onset of convection. Then a similar result may possibly be expected also for parallel shear flows.

From (*i*) we deduce:

$$\mathcal{R}_c{}^2 = \frac{27\pi^4}{4(1 + \alpha_r{}^2)} \tag{4.19}$$

and we get the stability condition:

$$\mathcal{R}_a{}^2 + \mathcal{C}^2 < 657.511 \tag{4.20}$$

for *any Reynolds* number R.

In order to study the nonlinear stability, we introduce the Lyapunov function

$$V(t) = V_o(t) + bV_1(t) \tag{4.21}$$

(b is a positive constant), with

$$V_1(t) = \frac{1}{2}(\|\nabla \mathbf{u}\|^2 + \|\nabla(\nabla \times \mathbf{u})\|^2) \tag{4.22}$$

and write the evolution equation of V(t). To this end, following [2, §4], we have:

$$\dot{V}(t) = I_o - D_o + N_o + bI_1 - bD_1 + bN_1 \,. \tag{4.23}$$

Defining

$$M = \frac{\mathcal{R}_a}{\mathcal{R}_c}\,, \tag{4.24}$$

and following [2,§4] it is possible to choose b such that:

$$bI_1 \leq D_2 \tag{4.25}$$

where

$$D_2 = \frac{1-M}{4}D_o + \frac{b}{2}D_1 \,. \tag{4.26}$$

Moreover, with the same arguments of [2,§4], and [6,§12], it is easy to see that there exist two positive constants A and B, depending on M and on the parameters of the basic motion, such that

$$N_o + bN_1 \leq AD_2V^{\frac{1}{2}}, \qquad BV \leq D_2 \,. \tag{4.27}$$

Then we have:

$$\dot{V} \leq -D_2(1 - AV^{\frac{1}{2}}) \leq -BV(1 - AV^{\frac{1}{2}}). \tag{4.28}$$

This last inequality implies the following nonlinear stability theorem.

THEOREM 4.2. *Let $V(0) < A^{-2}$. Assuming (4.18) or (4.20) in the cases of a mixture heated and salty from below or heated from below* *and salty above, respectively, then the basic motion is nonlinearly asymptotically stable:*

$$V(t) \leq V(0)\exp\{-B[1 - AV(0)^{\frac{1}{2}}]t\}. \tag{4.29}$$

The proof follows from (4.28) by a recursive argument (cf. [6],[7]).

Moreover, by classical imbedding theorems, we have the following corollary:

COROLLARY. *In the hypotheses of Theorem 5.2 we have the following pointwise (exponential) decay:*

$$\sup_{\mathbb{R}^2\times[-\frac{1}{2},\frac{1}{2}]} |\mathbf{u}(\mathbf{x},t)| \to 0 \quad as \quad t \to \infty.$$

Remark. In the case of a mixture heated above and salty below, the only condition $V(0) < A^{-2}$ assures the exponential stability.

Acknowledgement.

This research has been partially supported by the italian Ministry for University (M.U.R.S.T.) under 40% and 60% contracts, and by S.P. MMAI of italian C.N.R.

REFERENCES

[1] Rionero S., Mulone G., *On the Nonlinear Stability of Parallel Shear flows,* Continuum Mech. Thermodyn., **3**, (1991), 1-11.

[2] Mulone G., *On the Stability of Plane Parallel Convective Flows,* Acta Mechanica, **87**, (1991) 153-162.

[3] Rionero S., *On the Choice of the Lyapunov Functional in the Stability of Fluid Motions*, Proc. Meeting "Energy Stability and Convection", G.P. Galdi and B. Straughan Eds. Pitman Research Notes in Mathematics, **168** (1988).

[4] Galdi G.P., *Non-linear Stability of the Magnetic Bénard Problem via a Generalized Energy Method,* Arch. Rational Mech. Anal. **87**, (1985) 167-186.

[5] Galdi G.P., Straughan B., *A Nonlinear Analysis of the Stabilizing Effect of Rotation in the Bénard Problem,* Proc.R. Soc. Lond. A **402**, (1985) 257-283.

[6] Rionero S., Mulone G., *A Nonlinear Stability Analysis of the Magnetic Bénard Problem through the Lyapunov Direct Method,* Arch. Rational Mech. Anal. **103**, (1988) 347-368.

[7] Mulone G., Rionero S., *On the Non-linear Stability of the Rotating Bénard Problem via the Lyapunov Direct Method,* J. Mat. Anal App. **144**, (1989) 109-127.

[8] Galdi G.P., Padula M., *A New Approach to Energy Theory in the Stability of Fluid Motion,* Arch. Rational Mech. Anal. **110**, (1990) 187-286.

[9] Joseph D.D., *Global Stability of Conduction Diffusion Solution,* Arch. Rational Mech. Anal. **36**, (1970) 285-292.

[10] JosephD.D., *Stability of Fluid Motions* (2 vols.) Springer Tracts in Natural Philosophy, vols. 27 and 28. Berlin, Springer-Verlag, (1976).

Dipartimento di Matematica
V.le A. Doria 6
95125 Catania
Italy