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WAVE SPEEDS IN RELATIVISTIC WARM PLASMAS

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A recently obtained warm plasma model is here considered and the speeds of propagation of the discontinuity waves are calculated. These speeds, relative to the fluid, are 0, $\pm\sqrt{7/15}\varepsilon$, $\pm\sqrt{3/5}\varepsilon$, $\pm\sqrt{1/3}$, where ε is a smallness parameter invariantly related to temperature. Explicit expressions are also shown for the eigenvectors.

1. Introduction.

Anile and I have recently proposed the following set of field equations to describe warm plasmas (see refs [1,2])

$$(1.1)\qquad \begin{cases} \partial_\alpha(h\omega^\alpha)=0 \\ \partial_\alpha(h\omega^\alpha\omega^\beta+\vartheta^{\alpha\beta})=ehF^{\beta\alpha}\omega_\alpha \\ \partial_\alpha(h\omega^\alpha\omega^\beta\omega^\gamma+3\omega^{(\alpha}\vartheta^{\beta\gamma)}+S^{\alpha\beta\gamma})=2e(\vartheta^{(\beta}{}_\mu F^{\gamma)\mu}+h\omega^{(\beta}F^{\gamma)\mu}\omega_\mu) \end{cases}$$

where $\omega^\alpha=\sqrt{1+\varepsilon^2}u^\alpha$, $g^{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$ (the metric tensor), $h^{\alpha\beta}=g^{\alpha\beta}+u^\alpha u^\beta$;

$$\vartheta^{\alpha\beta}=h\varepsilon^2K^{\langle\alpha\beta\rangle}+2h\varepsilon^3K^{(\alpha}u^{\beta)}+h(\tilde{K}+g(\varepsilon))\varepsilon^4\left(\frac{1}{3}h^{\alpha\beta}+u^\alpha u^\beta\right)+\frac{1}{3}h\varepsilon^3h^{\alpha\beta},$$

$$S^{\alpha\beta\gamma} = 2h\varepsilon^4\sqrt{1+\varepsilon^2}(\tilde{K}+g(\varepsilon))u^{(\alpha}h^{\beta\gamma)} + \frac{6}{5}h\varepsilon^3\sqrt{1+\varepsilon^2}K^{(\alpha}h^{\beta\gamma)} +$$
$$+2h\varepsilon^4 K^{(\langle\alpha\beta\rangle}u^{\gamma)} + 4h\varepsilon^5 K^{(\alpha}(u^{\beta}u^{\gamma)} + \frac{1}{5}h^{\beta\gamma)}) +$$
$$+h\varepsilon^6\left[\tilde{K}\left(\frac{5}{2}\tilde{\phi}_2(h) + \frac{571}{45}\right) + \frac{1}{9}\right](u^{\alpha}u^{\beta}u^{\gamma} + u^{(\alpha}h^{\beta\gamma)});$$

$g(\varepsilon)$ is a given function of ε whose expansion around $\varepsilon = 0$ is

$$g(\varepsilon) = \frac{1}{6} - \frac{1}{12}\varepsilon^2 + \frac{5}{4\cdot 27}\varepsilon^4 - \frac{253}{3\cdot 2^6}\varepsilon^6 + 0(\varepsilon^8), \tag{1.2}$$

$F^{\alpha\beta}$ is the electromagnetic field tensor, $\tilde{\phi}_2(h)$ is an arbitrary function of h, e is the electric charge, units are chosen so that the speed of light $c = 1$ and the particle mass $m = 1$, while h, ε, $\tilde{K}$, u^{α}, K^{α}, $K^{\langle\alpha\beta\rangle}$ are the independent variables constrained only by $u_{\alpha}u^{\alpha} = -1$; $u_{\alpha}K^{\alpha} = 0$; $u_{\alpha}K^{\langle\alpha\beta\rangle} = 0$; $K^{\langle\alpha\beta\rangle} = K^{\langle\beta\alpha\rangle}$; $g_{\alpha\beta}K^{\langle\alpha\beta\rangle} = 0$; so that we have 14 independent components.

I give here a direct proof of hyperbolicity of the field equations (1.1), calculating also the speeds of propagation of the discontinuity waves. This proof follows the general guidelines exposed in ref [3] and is parallel to that relative to the case of relativistic extended thermodynamics as exposed in ref [4].

2. proof of hyperbolicity.

A given system of equation of the form

$$\sum_{B=1}^{N} A_B^{\alpha A}\nabla_{\alpha}u^B = p^A \tag{2.1}$$

in the N unknowns u, is hyperbolic in the time-like direction ξ_{α} iff the system

$$\varphi_{\alpha}A_B^{\alpha A}\delta u^B = 0 \tag{2.2}$$

in the unknowns δu^B and with

$$\phi_{\alpha} = \phi(\zeta_{\alpha} - \lambda\xi_{\alpha});\ \xi_{\alpha}\xi^{\alpha} = -1;\ \xi_{\alpha}\zeta^{\alpha} = 0;\ \zeta_{\alpha}\zeta^{\alpha} = 1 \tag{2.3}$$

has real eigenvalues λ and N linearly independent eingenvectors δu^B and moreover det $\xi_\alpha A_B^{\alpha A} \neq 0$ (ref [5]); λ is the speed of propagation of the discontinuity wave, relative to ξ_α.

Let us define $\varphi = -\varphi_\alpha u^\alpha$; $\phi_\alpha = h_{\alpha\beta}\varphi^\beta$ (from which $\varphi_\alpha = \varphi u_\alpha + \phi_\alpha$ and $\phi_\alpha u^\alpha = 0$) and use the expressions of ω^α, $\vartheta^{\alpha\beta}$, $S^{\alpha\beta\gamma}$ reported in sect. 1; we can express the system (2.1) corresponding to (1.1) in the frame where $u^\alpha \equiv (1,0,0,0)$; $\phi^\alpha \equiv (0,\phi^1,0,0)$ and calculate the coefficients in the reference state defined by $\tilde{K} = 0$, $K^\alpha = 0$, $K^{\langle\alpha\beta\rangle} = 0$. In this way the system splits into

$$(2.4) \qquad A\delta\bar{K}^{23} = 0; \sum_{j=1}^{7} A_{ij}X^j = 0; \sum_{j=1}^{3} B_{ij}Y^J = 0; \sum_{j=1}^{3} B_{ij}Z^j = 0$$

where $\bar{K}^{\alpha\beta} = K^{\langle\alpha\beta\rangle} + \varepsilon^2\tilde{K}h^{\alpha\beta}$ whose inverse are

$$\varepsilon^2\tilde{K} = \frac{1}{3}\bar{K}^\alpha_\alpha; \; K^{\langle\alpha\beta\rangle} = \bar{K}^{\alpha\beta} - \frac{1}{3}\bar{K}^\mu_\mu h^{\alpha\beta};$$

moreover

$$X^j \equiv \left(\delta h + \delta\varepsilon\frac{\varepsilon}{1+\varepsilon^2},\ \delta\varepsilon,\ \delta\bar{K}^{11},\ \delta\bar{K}^{11} + \delta\bar{K}^{22} + \delta\bar{K}^{33}, \delta\bar{K}^{33}, \delta u_1, \delta K_1\right)$$

$$Y^j \equiv (\delta u^2, \delta K^2, \delta\bar{K}_{12}); \; 2^j \equiv (\delta u^3, \delta K^3, \delta\bar{K}_{13})$$

while A, A_{ij}, B_{ij} are explicit functions of ε, h, φ, ϕ^1; I omit their expressions for the sake of brevity.

We can now proceed to calculate the eigenvalues and the corresponding eigenvectors from eqs. (2.4). To this purpose let us consider the following three cases.

I) Solution of $A = 0$.

It is $\varphi = 0$. The corresponding eigenvectors $(\delta K_{23}; X^J; Y^j; Z^j)$ for the system (2.4) are $(1; 0^J; 0^j; 0^j)$; $(0; 0,0,0,0,1,0,0; 0^j; 0^j)$; $(0; A^1, A^2, A^3, A^4, 0, 0, 0; 0^j; 0^j)$; $(0; B^1, B^2, B^3, B^4, 0, 0, 0; 0^j; 0^j)$; $(0; 0^J; C^1, C^2, 0; 0^j)$; $(0; 0^J; 0^j; C^1, C^2, 0)$ where (A^1, A^2, A^3, A^4) and (B^1, B^2, B^3, B^4) are two independent solutions of the system

$$(\varepsilon^4 g + \varepsilon^2)X^1 + h[(\varepsilon^4 g + \varepsilon^2)' - \varepsilon(\varepsilon^4 g + \varepsilon^2)/(1+\varepsilon^2)]X^2 + 3h\varepsilon^2 X^3 -$$

$$-\frac{2}{3}h\varepsilon^2 X^4 = 0$$

$$\left[(3\varepsilon^4 g + \varepsilon^2)\sqrt{1+\varepsilon^2} + \frac{1}{9}\varepsilon^6\right] X^1 + h\left\{\left[(3\varepsilon^4 + g + \varepsilon^2)\sqrt{1+\varepsilon^2} + \frac{1}{9}\varepsilon^6\right]' - \right.$$
$$\left. - \varepsilon\left[(3\varepsilon^4 g + \varepsilon^2)\sqrt{1+\varepsilon^2} + \frac{1}{9}\varepsilon^6\right](1+\varepsilon^2)^{-1}\right\} X^2 +$$
$$+ h\varepsilon^2(3\sqrt{1+\varepsilon^2} + 2\varepsilon^2)X^3 + \frac{1}{3}h\varepsilon^4\left(\frac{5}{2}\tilde{\phi}_2 + \frac{481}{45}\right) X^4 = 0$$

while $C^1 = -\frac{2}{5}\varepsilon^3(3\sqrt{1+\varepsilon^2} + 2\varepsilon^2)$; $C^2 = \sqrt{1+\varepsilon^2}(3\varepsilon^4 g + \varepsilon^2) + \frac{1}{9}\varepsilon^6$.

II) Other solutions of $\det(B_{ij}) = 0$.

We have that the values φ such that $\det(B_{ij}) = 0$ are the above considered $\varphi = 0$ and moreover the solutions of

$$\varphi^2 = \left[1 + \left(\sqrt{1+\varepsilon^2} - 1\right) + \sqrt{1+\varepsilon^2}\left(\varepsilon^2 - \frac{7}{3}\varepsilon^4 g\right) - \right.$$
$$\left. - \frac{5}{27}\varepsilon^6 + \frac{4}{3}\varepsilon^2\left(\frac{4}{3}\varepsilon^4 g + \frac{4}{3}\varepsilon^2 + 1\right)\right]^{-1} \cdot \left[\frac{7}{15} + \right.$$
$$+ \frac{1}{5}(\sqrt{1+\varepsilon^2} - 1) - \frac{7}{15}\varepsilon^2 g\sqrt{1+\varepsilon^2} - \frac{1}{27}\varepsilon^4 +$$
$$\left. + \frac{16}{45}(\varepsilon^4 g + \varepsilon^2)\right]\varepsilon^2(\phi^1)^2;$$

When ε is sufficiently little the dominant parts in ε of these new roots are

$$\varphi = \pm\sqrt{\frac{7}{15}}\phi^1\varepsilon. \tag{2.5}$$

To each of these values φ_1, φ_2 of φ the following eigenvectors correspond $(0; 0^J; D^j(\varphi_k); 0^j)$, $(0; 0^J; 0^j; D^j(\varphi_k))$ where $D^j(\varphi_k)$ is the independent solution of $\sum_{j=1}^{3} B_{ij}(\varphi_k)D^j(\varphi_k) = 0$.

III) Other solutions of $\det(A_{ij}) = 0$.

With long calculations we can obtain that the roots of $\det(A_{ij}) = 0$ are the above considered $\varphi = 0$ and those of

$$\alpha\varphi^4 + \beta\varphi^2(\phi^1)^2 + \gamma(\phi^1)^4 = 0 \tag{2.6}$$

where

$$\alpha = \frac{28}{3}\varepsilon^{10} + \varepsilon^{11}\bar{\alpha};\ \beta = -\frac{28}{9}\varepsilon^{10} + \varepsilon^{11}\bar{\beta};\ \gamma = \frac{28}{15}\varepsilon^{12} + \varepsilon^{13}\bar{\gamma} \tag{2.7}$$

with $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ continuous functions also in $\varepsilon = 0$.

Now the two roots of (2.6) can be expressed in the form

$$\varphi^2 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}(\phi^1)^2 \text{ and } \varphi^2 = \frac{2\gamma}{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}(\phi^1)^2.$$

By using the expressions (2.7) their dominant terms in ε are $\varphi^2 = \frac{1}{3}(\phi^1)^2$ and $\varphi^2 = \frac{3}{5}\varepsilon^2(\phi^1)^2$ respectively. To each of these values φ_1, φ_2, φ_3, φ_4 of φ it corresponds the eigenvector $(0; E^J(\varphi_k); 0^j; 0^j)$ where $E^J(\varphi_K)$ is the independent solution of $\sum_{j=1}^{3} A_{ij}(\varphi_k)\ E^J(\varphi_k) = 0$.

Summarizing the results , *I* have found from the system (2.4)

6 independent eigenvectors corresponding to $\varphi = 0$

2 independent eigenvectors corresponding to $\varphi = \sqrt{\frac{7}{15}}\ \phi^1\varepsilon$

2 independent eigenvectors corresponding to $\varphi = -\sqrt{\frac{7}{15}}\ \phi^1\varepsilon$

4 independent eigenvectors corresponding to $\varphi = \pm\frac{\sqrt{3}}{3}\ \phi^1\varepsilon$,

$$\pm\sqrt{\frac{3}{5}}\phi^1\varepsilon.$$

It is obvious that $\varphi = 0$ has multiplicity 6, $\varphi = \pm\sqrt{\frac{7}{15}}\ \phi^1\varepsilon$ have multiplicity 2, while the remaining values of φ have multiplicity 1. All these values of φ are such that

$$\varphi^2 = k(\phi^\mu\phi_\mu) \text{ with } 0 \le k < 1 \tag{2.8}$$

and precisely $k = 0, \frac{7}{15}\varepsilon^2, \frac{1}{3}, \frac{3}{5}\varepsilon^2$. In the next section we shall see that the corresponding eigenvalues are all real.

I want to conclude this section noticing that from $\det(\varphi_\alpha A_B^{\alpha A}) = 0$ it has followed the condition (2.8) for $\varphi = -\varphi_\alpha u^\alpha$ and $\phi_\alpha = h_{\alpha\beta}\varphi^\beta$; Therefore if $\det(\xi_\alpha A_B^{\alpha A}) = 0$ we would have (ξ_α instead of φ_α) $(\xi_\alpha u^\alpha)^2 = kh^{\alpha\beta}\xi_\alpha\xi_\beta$, i.e. in the reference frame Σ where $u^\alpha \equiv (1,0,0,0)$; $\xi_\alpha \equiv (\xi_0, \xi_1, 0, 0)$ we would have $(\xi_0)^2 = k(\xi_1)^2$; but $\xi_\alpha\xi^\alpha = -1 = -(\xi_0)^2 + (\xi_1)^2$ from which $(\xi_0)^2 = 1 + (\xi_1)^2$ that substituted above gives $1 = (k-1)(\xi_1)^2$ against the fact that $k - 1 < 0$. Consequently the requirement $\det(\xi_\alpha A_B^{\alpha A}) \neq 0$ holds.

3. The speeds of propagation of discontinuity waves.

Inserting the expression $(2.3)_1$ in the definitions $\varphi = -\varphi_\alpha u^\alpha$; $\phi_\alpha = h_{\alpha\beta}\varphi^\beta$ we obtain

$$\varphi = \phi(-\zeta_\alpha u^\alpha + \lambda\xi_\alpha u^\alpha);\ \phi^\mu\phi_\mu = \phi^2[1 - \lambda^2 + (\zeta_\alpha u^\alpha - \lambda\xi_\alpha u^\alpha)^2]$$

that substituted in (2.8) give

$$f(\lambda) = [k + (1-k)a^2]\lambda^2 + 2ab(k-1)\lambda - k(1-k)b^2 = 0 \tag{3.1}$$

where

$$a = u^\alpha\xi_\alpha;\ b = u^\alpha\zeta_\alpha;\ u^\alpha = a\xi^\alpha + b\zeta^\alpha + V^\alpha$$

from which $u^\alpha u_\alpha = -1$ gives $a^2 - b^2 = 1 + V^\alpha V_\alpha > 0$.

Now from (3.1) we have that $\frac{\Delta}{4} = k^2 + k(1-k)\ (a^2 - b^2) \geq 0$ (because $0 \leq k < 1$), which assures that the roots of (3.1) are real.

Moreover we have

$$f(1) = (1-k)(a-b)^2 \text{ because } a^2 - b^2 > 0 \text{ implies } a \neq b;$$

it implies also $a^4 - a^2b^2 > 0$ from which

$$-a^2 < ab < a^2 \text{ and therefore } f'(1) = 2k + 2(1-k)(a^2 - ab) > 0. \tag{3.2}$$

Consequently the roots of (3.1) are smaller than 1 (the coefficient of λ^2 is positive). Similarly we have

$$f(-1) = (1-k)(a+b)^2 > 0 \text{ because } a^2 - b^2 > 0 \text{ implies } a \neq -b;$$

$$f'(-1) = 2[k + (1-k)(a^2 + ab)] < 0 \quad \text{because of (3.2);}$$

consequently the roots of (3.1) are greater than -1. With the above result this says us that every root $\bar{\lambda}$ of (3.1) is such that $|\bar{\lambda}| < 1$, i.e. the speeds of propagation of discontinuity waves do not exceed that of light.

The roots of (3.1) become simplier in the case $\xi_\alpha = u_\alpha$ because for

$$k = 0, \ \frac{7}{15}\varepsilon^2, \ \frac{1}{3}, \ \frac{3}{5}\varepsilon^2$$

they are

$$\lambda = 0; \pm\sqrt{\frac{7}{15}}\varepsilon; \pm\sqrt{\frac{1}{3}}; \pm\sqrt{\frac{3}{5}}\varepsilon.$$

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