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SOME REMARKS ON ONE-DIMENSIONAL MODELS OF WAVE MOTION IN ELASTIC RODS

PAOLO PODIO-GUIDUGLI (Roma)

An exact derivation from three-dimensional elasticity of a model equation for the longitudinal vibrations of a cylindrical elastic rod is presented, based on the results of [1]. Similarities and differences are discussed with the model of [2], whose study strongly motivated the work leading to [1] and opened the way to the present discussion. A difference is that the model of [2] is not exact, being obtained through a line of reasoning that involves truncated expansions in the radius of the cross section; a similarity is that the resulting equations share the mathematically relevant properties, and describe the same physical phenomenology (in particular, they support travelling wave solutions of the solitary type).

1. Introduction. Coleman and Newman's Model.

In the absence of body forces, the standard model equation for the axial vibrations of a rod is

$$T_Z = \rho u_{tt} \tag{1.1}$$

(*cf.* [2, 3, 4]), where

Z ... axial coordinate, t ... time;

T ... tensile force, ρ ... mass density, u ... axial displacement.

One would expect the tensile force to depend solely on the axial stretch

$$\lambda = \lambda(Z,t) := 1 + u_Z(Z,t)\,. \tag{1.2}$$

But one would not expect the functional dependence of T on λ derived by Coleman and Newman in [2], namely, for a cylindrical rod of radius a,

$$T = \tau(\lambda) + \beta(\lambda)(\lambda_Z)^2 + \gamma(\lambda)\lambda_{zz} + \delta(\lambda)(\lambda_t)^2 + \epsilon(\lambda)\lambda_{tt}\,, \tag{1.3}$$

with

$$8\gamma(\lambda) = -[\lambda(\lambda^3 - 1)]^{-1}\tau(\lambda)a^2\,,\, 2\beta(\lambda) = \gamma'(\lambda)\,; \tag{1.4}$$

$$8\epsilon(\lambda) = \lambda^{-3}\rho a^2\,,\; 2\delta(\lambda) = \epsilon'(\lambda)\,. \tag{1.5}$$

As remarked by Coleman and Newman, the function $\lambda \mapsto \tau(\lambda)$ completely determines the tensile force T; starting from a simple([1]) three-dimensional material body having the shape of a shaft, and employing an approximation procedure based on truncated expansions in the cross-sectional radius, they obtain a one-dimensional material body whose mechanical response is not simple because the tensile force depends strongly not only on the axial stretch λ but also on the first and second spatial and temporal derivatives of λ; moreover, as the radial stretch ν is assumed to be determined by the axial one *via* the incompressibility condition

$$\nu = \lambda^{-1/2}\,, \tag{1.6}$$

their one-dimensional model predicts also the deformed shape of a stretched rod. A test case is when the material of which the three-dimensional shaft is made is *neohookean*; as

$$\tau(\lambda) = \mu(\lambda - \lambda^{-2}),\; \mu > 0\,, \tag{1.7}$$

in this case,([2]) it follows from $(1.4)_1$ and $(1.5)_1$ that

$$\gamma(\lambda) = -c^{-2}\epsilon(\lambda)\,,\; c^2 := \mu\rho^{-1}\,, \tag{1.8}$$

([1]) In the sense of Noll (*cf.* [5], Section 28).

([2]) It would appear that (1.7) is a material function, but it is not: indeed, by definition, $\tau(\lambda) = \mu\lambda - \pi\lambda^{-1}$, with π the pressure that arises as a consequence of the incompressibility constraint; (1.7) obtains when one chooses $\pi = \mu\lambda^{-1}$.

so that (1.3)-(1.5) reduce to

$$(1.9)_1 \qquad T = \mu(\lambda - \lambda^{-2}) + \beta(\lambda)[(\lambda_Z)^2 - c^{-2}(\lambda_t)^2] + \gamma(\lambda)(\lambda_{ZZ} - c^{-2}\lambda_{tt}),$$

$$(1.9)_2 \qquad 2\beta(\lambda) = \gamma'(\lambda).$$

Even in its simplest instance (1.9) the unusual "constitutive" dependence (1.3)-(1.5) allows for a striking variety of statical and dynamical solutions to (1.1), (1.2); the most interesting ones are those for which the axial stretch is *inhomogeneous*, *i.e.*, at least one of the spatial derivatives of λ is not zero for all (Z,t). We here list two significant classes of motions with inhomogeneous stretch.

a. Quasi-Static Motions

These obtain when inertia is negligible, and system (1.1)-(1.5) reduces to

$$(1.10) \qquad T^\circ = \tau(\lambda) + \beta(\lambda_Z)(\lambda)^2 + \gamma(\lambda)\lambda_{ZZ}.$$

This equation was proposed and studied by Coleman in [6]; a derivation was given by Coleman and Newman [7].

b. Travelling Waves

Here we look for waves travelling along the Z-axis with velocity V, *i.e.*, for solutions having the form

$$(1.11) \qquad u = u(\xi), \; \xi := Z - Vt$$

of the equation to which system (1.1)-(1,5) reduces, namely,

$$(1.12) \qquad T^\circ = \bar{\tau}(\lambda) + \bar{\beta}(\lambda)(\dot{\lambda})^2 + \bar{\gamma}(\lambda)\ddot{\lambda},$$

where

$$(1.13) \qquad \bar{\tau}(\lambda) = \tau(\lambda) - \rho V^2\lambda, \; \bar{\gamma}(\lambda) = \gamma(\lambda) + V^2\epsilon(\lambda), \; 2\bar{\beta}(\lambda) = \bar{\gamma}'(\lambda),$$

and where a superscript dot denotes differentiation with respect to ξ.

Under resonable assumptions on the problem's parameters — that is, the function $\lambda \mapsto \tau(\lambda)$, the radius a and, when appropriate, the density ρ and the wave velocity V — solutions are possibile that, in case a., are very much reminiscent of the necking and drawing phenomena observed when certain polymeric fibers are taught (*vid.* [7, 6, 2], and the literature quoted therein); and that, in case b., are interpreted as *solitary waves of contraction* which, with the use of (1.6), are found to have the form of a travelling bulge [2]. All in all, Coleman and Newman's model has primary interest as a model for longitudinal vibrations of elastic rods, but it seems also apt to capture certain features of cold-drawing processes, presumably those features that are not particularly influenced by temperature and/or viscosity.

2. An Exact Model.

We now furnish, in an abridged manner, a rational and exact deduction of a one-dimensional model for the longitudinal vibrations of an elastic rod; greater detail may be found in [1].

As has been done before to motivate and somewhat justify various one-dimensional models [*cf.*, *e.g.*, 2, 3, 4], we consider a special class of motions of a cylindrical rod, namely, those motions with cylindrical symmetry that leave both cross sections planar and volume unchanged, and therefore have the following representation:

$$r = \nu(Z,t)R\,,\ \theta = \Theta\,,\ z = Z + u(Z,t)\,,\ \lambda\nu^2 = 1\,. \tag{2.1}$$

The interest of this class of motions is in its peculiar simplicity: (i) the axial stretch mapping $(Z,t) \mapsto \lambda(Z,t)$ completely determines the motion; (ii) irrespectively of constitutive choices to come, the axial component of the momentum balance equation, when integrated over the (undeformed) cross section $\mathcal{S}$, yields

$$d + T_Z = \rho_{tt} \tag{2.2}$$

provided the "body force" d and the "resultant force" T are defined to be, respectively,

$$d := \frac{1}{|\mathcal{S}|}\Big(\int_{\partial\mathcal{S}} \mathrm{SM}\Big) \cdot \mathrm{e}_Z\,, \tag{2.3}$$

where

$|\mathcal{S}|$ ... area of $\mathcal{S}$, $\partial\mathcal{S}$ ... boundary curve of $\mathcal{S}$,

S ... Piola stress, **M** ... outward normal to $\partial\mathcal{S}$,
$\mathbf{e}_Z$ ... unit vector of Z-axis, **N** ... normal to $\mathcal{S}$,
n ... normal to deformed cross section;(3)

$$T := \fint T_N\,, \quad T_N := \mathbf{SN}\cdot\mathbf{n}\,, \quad \fint f := \frac{2}{a^2}\int_0^a f(R)RdR\,. \tag{2.4}$$

We now pick the stored energy mapping

$$2\sigma(\mathbf{F}) = \mu\mathbf{FP}\cdot\mathbf{F} + \alpha\mathbf{F}(\mathbf{I}-\mathbf{P})\cdot\mathbf{F}\,, \quad \mathbf{P} := \mathbf{N}\otimes\mathbf{N}\,, \quad \det\mathbf{F} = 1\,, \tag{2.5}$$

which induces the following dependence of the Piola stress **S** on the deformation gradient **F**:

$$\mathbf{S} = \partial_{\mathbf{F}}\sigma(\mathbf{F}) = \mu\mathbf{FP} + \alpha\mathbf{F}(\mathbf{I}-\mathbf{P}) - \pi\mathbf{F}^{-T}\,. \tag{2.6}$$

Relations (2.5) and (2.6) describe an incompressible, nonlinearly elastic material class that has *transversely isotropic* response with respect to the cross-sectional normal N; for $\alpha = \mu$, the isotropic neohookean material class results. With (2.1), (2.6) yields

$$(\mathbf{S}) = \mu\begin{pmatrix} 0 & 0 & \nu_Z X \\ 0 & 0 & \nu_Z Y \\ 0 & 0 & \lambda \end{pmatrix} + \alpha\begin{pmatrix} \nu & & \\ & \nu & \\ & & 0 \end{pmatrix} - \pi\begin{pmatrix} \nu^{-1} & & \\ & \nu^{-1} & \\ \dfrac{\lambda_Z X}{2\lambda^2} & \dfrac{\lambda_Z X}{2\lambda^2} & \lambda^{-1} \end{pmatrix}, \tag{2.7}$$

where $R^2 = X^2 + Y^2$.

(2.7) has manifold consequences. First, with (2.4), it gives

$$T = \mu\lambda - (\fint \pi)\lambda^{-1}\,. \tag{2.8}$$

(3) It follows from (2.1) that $\mathbf{N} = \mathbf{n} = \mathbf{e}_Z$.

In order to determine the average pressure we look at the radial component of the momentum balance equation, which reads

$$-R^{-1}\pi_R + \nu\nu_{ZZ} = c^{-2}\nu\nu_{tt}\,; \tag{2.9}$$

we note that, due to $(2.1)_4$,

$$\begin{aligned} &4\nu(\nu_{ZZ} - c^{-2}\nu_{tt}) = \\ &\qquad 3\lambda^{-3}[(\lambda_Z)^2 - c^{-2}(\lambda_t)^2] - 2\lambda^{-2}(\lambda_{ZZ} - c^{-2}\lambda_{tt})\,; \end{aligned} \tag{2.10}$$

we integrate (2.9) over $[0, R]$ to get $\pi(R, Z, t)$, and then average the latter over the cross section to obtain, with the use of (2.10) and setting $\pi(0) := \pi(0, Z, t)$,

$$\begin{aligned} &⨍\pi = \pi(0) + \\ &\qquad \lambda\beta(\lambda)[(\lambda_Z)^2 - c^{-2}(\lambda_t)^2] + \lambda\gamma(\lambda)(\lambda_{ZZ} - c^{-2}\lambda_{tt})\,. \end{aligned} \tag{2.11}$$

Finally, combining (2.8) and (2.11) we arrive at

$$\begin{aligned} T =& \mu\lambda - \pi(0)\lambda^{-1} + \\ &\qquad \beta(\lambda)[(\lambda_Z)^2 - c^{-2}(\lambda_t)^2] + \gamma(\lambda)(\lambda_{ZZ} - c^{-2}\lambda_{tt}) \end{aligned} \tag{2.12}$$

(*cf.* (1.9)). This part of our derivation makes clear that, in a formula for the tensile force such as (1.3), (1.9) or (2.12), spatial and temporal derivatives of the axial stretch have no real constitutive character, but rather reflect the influence of the transverse motion that accompanies longitudinal vibrations of type (2.1) in a material of type (2.6).

Secondly, (2.7) implies that

$$(\mathbf{SM})_{(a,Z,t)} = -\begin{pmatrix} [\pi(a,Z,t) - \alpha\lambda^{-1}(Z,t)]\nu^{-1}\dfrac{X}{a} \\ [\pi(a,Z,t) - \alpha\lambda^{-1}(Z,t)]\nu^{-1}\dfrac{Y}{a} \\ \pi(a,Z,t)\dfrac{\lambda_Z a}{2\lambda^2} \end{pmatrix}; \tag{2.13}$$

therefore, in view of definition (2.3) and for $\pi(a) := \pi(a, Z, t)$,

$$d = -\pi(a)\lambda^{-2}\lambda_Z\,. \tag{2.14}$$

We see from (2.13) that, at a point of $\partial\mathcal{S}$, the traction vector SM has null radial components if and only if

$$\pi(a) = \alpha\lambda^{-1}\,; \tag{2.15}$$

but, if (2.15) holds, then both $\pi(a)$ and d are of the order of $\frac{\alpha}{\mu}$.

3. Concluding Remarks.

(i) In specifying the material parameters of our exact model the drastic choice $\alpha = 0$ is admissible, and actually is the only one that allows us to recover the standard model equation (1.1).

(ii) All the results obtained in the preceding section are valid for the neohookean material considered as an example in [2], provided that, as indicated, one sets $\frac{\alpha}{\mu} = 1$ throughout; but then $\pi(a) = \mu\lambda^{-1}$, and thus $\pi(a)$ cannot vanish for any (Z, t); since d is proportional to λ_Z as (2.14) shows, either $\lambda_Z(Z, t) \equiv 0$ (a situation compatible only with equilibrium) or the term d should appear in the one-dimensional equation for longitudinal vibrations.

(iii) When the constitutive choice (2.5) is made, travelling wave solutions in the form of solitary waves of contraction are possible for a wide range of values of α, $\alpha = 1$ included [1].

(iv) When one strives to generate from a three-dimensional theory an exact one-dimensional model equation, the idea of choosing a transversely isotropic three-dimensional material body may prove useful not only for elastic constitutive laws more general than (2.5), as indicated in [1], but also for not necessarily elastic types of material response, among which the viscoelastic response has obvious interest in modelling cold-drawing processes.

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Dipartimento di Ingegneria Civile
Università di Roma - Tor Vergata
00173 Roma, Italia