LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 321-327

THERMAL CREEP PROBLEMS BY THE DISCRETE BOLTZMANN EQUATION

L. PREZIOSI (Torino)

This paper deals with an initial–boundary value problem for the discrete Boltzmann equation confined between two moving walls at different temperature. A model suitable for the quantitative analysis of the initial boundary value problem and the relative existence theorem are given.

1. Statement of the Initial-Boundary Value Problem.

One of the classical problems of the kinetic theory is the analysis of the thermal creep between walls at different temperature. Such a problem has been studied by means of linear and some nonlinear models of the continuous Boltzmann equation [1]. The analysis can be developed, as we shall see, with somewhat interesting physical results by the discrete Boltzmann equation, a nonlinear model of the discrete kinetic theory of gases [2].

The discrete Boltzmann equation is a non linear model of the kinetic theory of gases such that the particles can only attain to a finite number of velocities $\mathbf{v}_i \in \mathbb{R}^3$, $i = 1, \ldots, n$. The aim of the model is then to describe the behaviour of the gas through the time–space evolution of the number densities $N_i(t, \mathbf{x})$ of the particles travelling with velocity $\mathbf{v}_i$, i.e. to solve, with suitable initial and boundary conditions, the hyperbolic semilinear system of equations

$$\frac{\partial N_i}{\partial t} + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} N_i = J_i[\mathbf{N}] \quad , \quad i \in I = \{1, \ldots, n\} \tag{1.1}$$

where $\mathbf{N} = (N_1, \ldots, N_n)$, $N_i \geq 0$ and $J_i[\mathbf{N}]$ represents the non linear collision operator which can be split into the sum of the contributions of the binary and triple collisions

$$J_i[\mathbf{N}] = J_i^2[\mathbf{N}] + J_i^3[\mathbf{N}] \ , \tag{1.2}$$

where the two contributions may in general be expressed as

$$J_i^2[\mathbf{N}] = \sum_{jk\ell \in I} A_{ij}^{k\ell}(N_k N_\ell - N_i N_j)$$

$$J_i^3[\mathbf{N}] = \sum_{hjk\ell m \in I} A_{hij}^{k\ell m}(N_k N_\ell N_m - N_h N_i N_j) \ .$$

Let now a monoatomic gas occupy the strip

$$\Omega = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \ : \ 0 < y < \ell\}$$

between two rigid walls and assume that the wall $y = 0$ (respectively, $y = \ell$) has temperature and velocity

$$T_{wo}(t) \textit{ and } w_o(t)\mathbf{i} \ \ (\textit{respectively } T_{w1}(t) \textit{ and } w_1(t)\mathbf{i}) \ . \tag{1.3}$$

In order to solve an initial–boundary value problem, Eq.(1.1) has to be joined with the initial condition

$$\mathbf{N}(t = 0, x) = \mathbf{N}_o(x) \tag{1.4}$$

and proper boundary conditions. In order to specify these conditions we will need some notation

DEFINITION 1.1. *The set of indexes corresponding to the allowed velocities can be partitioned as* $I = I^- \cup I^+$ *where*

$$I^- = \{i \in I : \mathbf{v}_i \cdot \mathbf{j} < 0\} \quad , \quad I^+ = \{j \in I : \mathbf{v}_j \cdot \mathbf{j} \geq 0\} \ .$$

The set I *can be also partitioned in* $2\hat{n}$ *sets* $(2\hat{n} < n)$

$$I_b = \{i \in I : \mathbf{v}_i \cdot \mathbf{j} = v_b\}.$$

Thus a particle with velocity with the index in I^- is hitting the wall if it is located at the lower wall $y = 0$, but is reflecting if $y = \ell$. Trivially the opposite is true for I^+

In order to have a satisfactory reflection law, according to [4], the discrete velocity model of the Boltzmann equation should satisfy the following conditions

i) The model is characterized by at least two velocity moduli in order to describe phenomena with temperature variation;

ii) The model can be suitably oriented to avoid grazing directions;

iii) The model has sufficient velocity directions so that any particle can be reflected specularly;

iv) The number of velocities and directions is such that the space of collision invariants has the correct dimension in order to have a unique description of the Maxwellian state in terms of the macroscopic observables of the system.

Keeping this in mind, one can characterize, according to [3], the gas–surface interaction on the wall $y = 0$ as

$$\forall i \in I^+, \; v_{iy} N_i = \sum_{j \in I^-} B_{ij} |v_{jy}| N_j . \tag{1.5}$$

If one assumes that a fraction α of the particles are reflected diffusively and the fraction $1 - \alpha$ is reflected specularly, Eq.(1.5) can be re-written as

$$\forall i \in I^+, \; v_{iy} N_i = (1-\alpha) N_{i'} v_{iy} + \alpha B_i^D (T_{wo}, w_o) \sum_{j \in I^-} |v_{jy}| N_j . \tag{1.6}$$

where, from the mechanism of specular reflection, the index i' is such that $\mathbf{v}_{i'} = \mathbf{v}_i - 2 v_{iy} \mathbf{j}$ and the diffusive re-emission coefficients are

$$B_i^D(t) = \frac{v_{iy} N_i(t,0)}{\sum_{k \in I^+} v_{ky} N_k(t,0)} \tag{1.7}$$

where N_k , $k \in I^+$ are the densities in Maxwellian equilibrium with temperature T_{wo} and x–component of the drift velocity equal to w_o.

2. The Thermal Creep Problem: Solution of the Initial-Boundary Value Problem

A mathematical model which satisfies the conditions stated in the previous section is the discrete Boltzmann equation with 6 velocity directions in the plane

$$\mathbf{e}_k = \cos[(2k+1)\frac{\pi}{6}]\mathbf{i} + \sin[(2k+1)\frac{\pi}{6}]\mathbf{j}, \quad k = 1, \ldots, 6 \tag{2.1}$$

and two velocity moduli c and $2c$.

The allowed velocities $\mathbf{v}_i$, $i = 1, \ldots, 12$ are then given by

$$\mathbf{v}_i = \begin{cases} c\mathbf{e}_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ 2c\mathbf{e}_{\frac{i}{2}} & \text{if } i \text{ is even .} \end{cases} \tag{2.2}$$

After analysing the collisional scheme, one can write the following kinetic equations ($i = 1, \ldots, 12$)

$$\begin{aligned} &[\frac{\partial}{\partial t} + c\cos(\frac{i+2}{6}\pi)\frac{\partial}{\partial x} + c\sin(\frac{i+2}{6}\pi)\frac{\partial}{\partial y}]N_i = \\ &= \frac{2}{3}cS(N_{i+2}N_{i+8} + N_{i+4}N_{i+10} - 2N_iN_{i+6}) + \\ &+ \frac{\sqrt{7}}{2}cS[N_{i+6}(N_{i+3} + N_{i+11}) - N_i(N_{i+5} + N_{i+9})] + \\ &+ \frac{5}{4\sqrt{\pi}}cS^{5/2}(N_{i+2}N_{i+6}N_{i+10} - N_iN_{i+4}N_{i+8}) \end{aligned} \tag{2.3a}$$

if i is odd, and

$$\begin{aligned} &[\frac{\partial}{\partial t} + 2c\cos(\frac{i+1}{6}\pi)\frac{\partial}{\partial x} + 2c\sin(\frac{i+1}{6}\pi)\frac{\partial}{\partial y}]N_i = \\ &= \frac{4}{3}cS(N_{i+2}N_{i+8} + N_{i+4}N_{i+10} - 2N_iN_{i+6}) + \\ &+ \frac{\sqrt{7}}{2}cS[N_{i+10}N_{i+1} + N_{i+2}N_{i+9} - N_i(N_{i+3} + N_{i+7})] + \\ &+ \frac{5}{2\sqrt{\pi}}cS^{5/2}(N_{i+2}N_{i+6}N_{i+10} - N_iN_{i+4}N_{i+8}) \end{aligned} \tag{2.3b}$$

if i is even. Here S is the cross–sectional area of the gas particles and the index has to be intended modulo 12.

This problem was solved in [5] for a purely diffusive reflection law at the boundaries, i.e. $\alpha(t) = 1$ in Eq.(1.6). We will now only particularize the results of [6] to the present case, without specifying the form of the transition probability densities in (1.5).

Consider the initial–boundary value problem (1.1), (1.4), (1.5). By a (local) mild solution we mean a function $\mathbf{N}(t,x)$ such that for some $T^* > 0$, $\mathbf{N} \in \mathcal{C}([0,T^*] \times [0,\ell]; \mathrm{IR}^n)$, $\mathbf{N}(t,\cdot) \in \mathcal{X}(t)\ \ \forall\, t \le T^*$ and

$$(4.1)\qquad \mathbf{N}(t) = \mathcal{U}(t,0)\mathbf{N}_o + \int_0^t \mathcal{U}(t,r)\mathbf{J}[\mathbf{N}(r)]\,dr\ ,\quad \forall t \in [0,T^*]$$

where

$$\begin{aligned}\mathcal{X}(s) := \{\mathbf{\Phi} \in \mathcal{C}([0,\ell]; \mathrm{IR}^n)\ :\ & \\ v_{iy}\Phi_i(0) &= \sum_{j\in I^-} |v_{jy}| B_{ij}(s)\Phi_j(0)\ \ if\ \ i \in I^+ \\ |v_{iy}|\Phi_i(\ell) &= \sum_{j\in I^+} v_{jy} B_{ij}(s)\Phi_j(\ell)\ \ if\ \ i \in I^-\}\ .\end{aligned}$$

The operator $\mathcal{U}(t,s)$ in (4.1) is given by

$$\mathcal{U}(t,s)\Phi_i(y) = \Phi_i(y - v_i(t-s))\quad if\quad (t,y) \in D_i(s)$$

where

$$D_i(s) = \begin{cases} \{(t,y) \in [s, s+T] \times [0,\ell] : y \ge v_{iy}(t-s)\} & \text{if } i \in I^+ \\ \{(t,y) \in [s, s+T] \times [0,\ell] : y \le \ell + v_{iy}(t-s)\} & \text{if } i \in I^- \end{cases}$$

and $T = \dfrac{\ell}{2c}$ and by

$$\mathcal{U}(t,s)\Phi_i(y) = \begin{cases} \displaystyle\sum_{j\in I^-} \frac{|v_{jy}|}{v_{iy}} B_{ij}\left(t - \frac{y}{v_{iy}}\right)\Phi_j(y^*_{ij}) & \text{if } i \in I^+ \\ \displaystyle\sum_{j\in I^+} \frac{v_{jy}}{|v_{iy}|} B_{ij}\left(t + \frac{\ell - y}{v_{iy}}\right)\Phi_j(y^*_{ij}) & \text{if } i \in I^- \end{cases}$$

if $(t,y) \notin D_i(s)$, where

$$y^*_{ij} = \begin{cases} v_{jy}\left[\dfrac{y}{v_{iy}} - (t-s)\right] & \text{if } i \in I^+ \\ v_{jy}\left[\dfrac{y}{v_{iy}} - (t-s)\right] + \ell\left(1 - \dfrac{v_{jy}}{v_{iy}}\right) & \text{if } i \in I^-. \end{cases}$$

Furthermore we define

$$r = \max_{\substack{j,b \\ t\in[0,T^*]}} \left|\frac{v_{jy}}{v_b}\right| \sum_{i\in I_b} B_{ij}(t) \leq 4$$

where $j \in I^-$ if $v_b > 0$ and viceversa. The following theorem holds.

THEOREM. *If the transition probability densities $B_{ij}(t)$ are continuous in time and $\mathbf{N}_o \in \mathcal{X}(0)$, then there exists a unique local mild solution to the initial–boundary value problem (1.1), (1.4), (1.5). This solution is positive if $\mathbf{N}_o$ is positive. Moreover if $B_{ij}(t)$ are C^1 in t and $\mathbf{N}_o$ satisfies the compatibility conditions*

$$v_{iy} N'_{io}(0) = \sum_{j\in I^-} \frac{v_{jy}}{v_{iy}} [B'_{ij}(0) N_{jo}(0) + |v_{jy}| B_{ij}(0) N'_{jo}(0)] \text{ for } i \in I^+ ,$$

$$v_{iy} N'_{io}(0) = \sum_{j\in I^+} \frac{v_{jy}}{v_{iy}} [B'_{ij}(0) N_{jo}(\ell) + v_{jy} B_{ij}(0) N'_{jo}(\ell)] \text{ for } i \in I^- ,$$

then this solution is a classical one. Furthermore if

$$M = \int_0^\ell \sum_{i\in I} N_i(t,x)\, dx < \frac{1}{\gamma_2}$$

where $\gamma_2 = 16(\frac{8}{3} + \sqrt{7})(1+3r)S$ and

$$E_o = \max_{\substack{i\in I \\ x\in[0,\ell]}} N_{io}(x) < \frac{\sqrt{\pi}(1-M\gamma_2)^2}{3840(1+3r)rS^{5/2}M} ,$$

then this solution exists in the interval $\left[0, \frac{(L+1)\ell}{2c}\right]$ where

$$L = \left[\!\left[\frac{\log \dfrac{\sqrt{\pi}(1-M\gamma_2)^2}{3840(1+3r)rS^{5/2}ME_o}}{\log \dfrac{12r}{1-M\gamma_2}} \right]\!\right]$$

and where $[K]$ stands for the largest integer less or equal than K.

REFERENCES

[1] Kogan M.N., *Rarefied Gas Dynamics*, Plenum Press, New York, 1969.

[2] Gatignol R., *Théorie Cinétique des Gaz a Répartition Discrète de Vitesses*, Lecture Notes in Phys. No.**36**, Springer-Verlag, Berlin, New York, 1975.

[3] Gatignol R., *Kinetic theory boundary conditions for discrete velocity gases*, Phys. Fluids **20** (1977), 2022–2030.

[4] Preziosi L., Longo E., *On the decomposition of domains in non linear discrete kinetic theory*, in "Discrete Models of Fluid-Dynamics" (Advances in Mathematics for Applied Sciences), ed. A. Alves, World Scientific, London, Singapore, (1991), pp.144–155.

[5] Preziosi L., *Thermal creep problem by the discrete Boltzmann equation*, to appear in *Transport Theory Statist. Phys.*

[6] Brzezniak J., Flandoli F., Preziosi L., *On the discrete Boltzmann equation with multiple collisions*, to appear in *Stability and Nonlinear Analysis in Continuous Media*.

Dip. di Matematica - Politecnico
Corso Duca degli Abruzzi, 24
10129 - TORINO