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INHOMOGENEOUS DEFORMATIONS AND MOTIONS OF ELASTIC MATERIALS

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Unsteady motions are discussed within the context of the linearized theory of elasticity, the neo-Hookean theory of elasticity and the theory of interacting continua. In the last case, we discuss unsteady motions in an isotropic solid infused with a fluid, and a transversely isotropic solid infused with a fluid. We are able to show that the theory reduces to one that has a structure similar to that introduced by Biot.

1. Introduction.

We shall discuss several unsteady motions within the context of linearized elasticity, non-linear elasticity, and non-linearly elastic solids infused with fluids. The unsteady motions we shall consider within the realm of linearized elasticity theory are extensions to elasticity of unsteady solutions to the Navier-Stokes equations in which the non-linear terms, while individually non-vanishing, are self cancelling, thereby reducing the Navier-Stokes equations to a set of linear equations. We extend solutions due to Kelvin (cf. Thompson [17]), Taylor [16] and others in the Navier-Stokes theory to that for a linearized elastic material (cf. Rajagopal [12]). Next, we discuss within the framework of the neo-Hookean and Mooney-Rivlin theories some classes of exact solutions (cf. Hayes and Rajagopal [10]). Included in this class are the elliptically polarized wave solutions found by Boulanger and Hayes [6] for neo-Hookean materials. Finally, we briefly discuss the recent work of Rajagopal and Tao [13] on the propagation of waves in solids infused with a fluid, undergoing large deformations, where it is shown that a Biot like theory is imbedded in a more general setting.

All the unsteady motions that are considered are inhomogeneous, and thus not possible in all isotropic elastic materials (cf. Ericksen [8], [9]), and it is for this reason we consider such motions within the context of special constitutive theories, say the neo-Hookean or Mooney-Rivlin theory.

2. New unsteady motions in linearized elasticity.

The equations of motion for isotropic linearized elastic materials take the form

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}) = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

where λ and μ are the Lame constants. Equation (1) admits solutions of the form (cf. Rajagopal [3])

$$u = \mathrm{Re}\Sigma A_n \exp(\alpha_{n^x})\exp(\beta_n y)\exp(\gamma t), \tag{2}$$

$$v = \mathrm{Re}\Sigma B_n \exp(\alpha_{n^x})\exp(\beta_n y)\exp(\gamma t), \tag{3}$$

where A_n, B_n, α_n and β_n are complex constants.

a. Standing waves in an array of square or rectangular cells:

We notice that (1) admits solutions of the form

$$u = \sin mx \cos ny e^{-\alpha t}, \tag{4}$$

$$v = \cos mx \sin ny e^{-\alpha t}, \tag{5}$$

provided

(6) $$-\{\mu(m^2+n^2)+(\lambda+\mu)(m+n)m\}=\rho\alpha^2,$$

(7) $$-\{\mu(m^2+n^2)+(\lambda+\mu)n(m+n)\}=\rho\alpha^2,$$

If $\lambda+\mu\neq 0$, then (6) and (7) imply that $m=n$ or $m=-n$. If $m=n$, then a simple computation gives

(8) $$\alpha^2=\frac{-2m^2(\lambda+2\mu)}{\rho}.$$

If $\rho>0$ and the stored energy is positive definite, which implies that, $\mu>0$ and $3\lambda+2\mu>0$, it follows that α is imaginary. Thus, only oscillating solutions are possible. This in marked contrast to the result established by Taylor [16] within the context of the Navier-Stokes theory, where only decaying of the array of vortices is possible. We also notice that det $\mathbf{F}\neq 1$, and thus the motion is not isochoric. If $m=-n$, then

(9) $$\alpha^2=\frac{-\mu}{\rho}(2m^2),$$

and once again only oscillatory solutions are possible. In this case it is easy to verify that div$\mathbf{u}=0$.

Rajagopal [12] has shown that these results also hold in the case of a transversely isotropic linearized elastic solid, as also in the case of an infinitesimal deformation superposed on a non-linear elastic solid subject to biaxial extension. We shall not discuss these results here.

Changing the displacement field slightly to

(10) $$u=A\sin mx\cos nye^{-\alpha t},$$

(11) $$v=B\cos mx\sin nye^{-\alpha t},$$

leads to similar results, the vortex cells being rectangular instead of square.

b. An analogue of Kelvin's «cats eye» vortex.

The displacement field

(12) $$u = \cos hax \sin bye^{\alpha t}$$

(13) $$v = B \sin hax \cos bye^{\alpha t}$$

satisfies equation (1), provided

(14) $$\mu(a^2 - b^2)A + (\lambda + \mu)a[Aa - Bb] = \rho\alpha^2 A,$$

(15) $$\mu(a^2 - b^2)B + (\lambda + \mu)b[Aa - Bb] = \rho\alpha^2 B.$$

Thus, if $\lambda + \mu \neq 0$, then $Aa = Bb$ or $aB = bA$. if $aA = bB$, then

(16) $$\alpha^2 = \frac{\mu(a^2 - b^2)}{\rho}.$$

A simple computation also verifies that $\det \mathbf{F} \neq 1$. If $aB = bA$, then

(17) $$\alpha^2 = \frac{(\lambda + 2\mu)(a^2 - b^2)}{\rho},$$

and once again the solution can oscillate, blow-up or decay with time.

There are several other exact solutions that can be established for (1) and we refer the reader to [12] for the same.

3. Unsteady motions in finite elasticity.

We shall now discuss some unsteady exact solutions which have been established recently by Hayes and Rajagopal [10] to the equations of motion of a neo-Hookean material. The Cauchy stress in a neo-Hookean material is given by

(18) $$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{B},$$

whre $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{F}$ is the deformation gradient.

Let us consider inhomogeneous motions of the form (cf. Hayes and Rajagopal [10])

$$x = f_1(t)X + f_2(t)Y, \tag{19}$$

$$y = g_1(t)X + g_2(t)Y, \tag{20}$$

$$z = Z + \alpha\frac{X^2}{2} + \beta\frac{Y^2}{2} + \gamma XY, \tag{21}$$

where (X, Y, Z) and (x, y, z) denote the reference and current coordinates of a particle, respectively.

Since the material is incopressible, it can undergo only isochoric motion, and thus

$$\det \mathbf{F} = 1. \tag{22}$$

For the motion under consideration $\mathbf{F}$ has the matrix representation

$$\mathbf{F} = \begin{pmatrix} f_1 & f_2 & 0 \\ g_1 & g_2 & 0 \\ \alpha x + \gamma y & \beta Y + \gamma X & 1 \end{pmatrix}, \tag{23}$$

and thus (22) implies that

$$f_1 g_2 - f_2 g_1 = 1. \tag{24}$$

It can be shown that (cf. Hayes and Rajagopal [10]) the motion is an unsteady isochoric plane motion superposed on a finite isochoric anti-plane strain.

A simple calculation shows that the balance of linear momentum reduces to

$$-\frac{\partial p}{\partial X}g_2 + \frac{\partial p}{\partial Y}g_1 - \frac{\partial p}{\partial Z}[-(\alpha X + \gamma Y)g_2 + (\gamma X + \beta Y)g_1] = \rho[Xf_1'' + Yf_2''], \tag{25}$$

$$\frac{\partial p}{\partial X}f_2 + \frac{\partial p}{\partial Y}f_1 - \frac{\partial p}{\partial Z}[-(\alpha X + \gamma Y)f_2 - (\beta Y + \gamma Y)f_1] = \rho[Xg_1'' + Yg_2''], \tag{26}$$

$$\mu(f_1g_2 - f_2g_1)(\alpha + \beta) - \frac{\partial p}{\partial Z} = 0. \tag{27}$$

If follows that we need to satisfy

$$f_1''f_2 + g_1''g_2 - f_2''f_1 - g_2''g_1 = 0. \tag{28}$$

Thus, all we need to ensure is that the functions f_1, f_2, g_1, g_2 satisfy equations (24) and (28), which gives us a great deal of flexibility in choosing the class of allowable motions.

a. Finite amplitude elliptically polarized waves superposed on static inhomogeneous extension:

Let

$$f_1(t) = \cos\theta(t),\ f_2(t) = \left(\frac{C}{D}\right)\sin\theta(t) \tag{29}$$

$$g_1(t) = -\left(\frac{D}{C}\right)\sin\theta(t),\ g_2(t) = \cos\theta(t), \tag{30}$$

where $\theta(t) = \omega t + \varepsilon$, ω and ε being constants. It is easy to verify that both (24) and (28) are met and hence (29), (30) is an allowable motion. Thus

$$x = X\cos\theta(t) + \frac{C}{D}Y\sin\theta(t), \tag{31}$$

$$y = -\frac{D}{C}X\sin\theta(t) + Y\cos\theta(t), \tag{32}$$

$$z = Z + \frac{\alpha X^2}{2} + \frac{\beta Y^2}{2} + \gamma XY. \tag{33}$$

This immediately implies that

$$\frac{x^2}{C^2} + \frac{y^2}{D^2} = \frac{x^2}{C^2} + \frac{y^2}{D^2}, \tag{34}$$

and

$$\ddot{x} = -\omega^2 c, \tag{35}$$

$$\ddot{y} = -\omega^2 y. \tag{36}$$

Equation (34) implies that particles which lie on the elliptic cylinders $x^2/C^2 + y^2/D^2$ =constant remain on these elliptic cylinders. Moreover, we see from (35) and (36) that the particles oscillate harmonically along these ellipses. The motion (29), (30) is a generalization of the motion studied by Boulanger and Hayes [6] in that the body is subjected to a static inhomogeneous deformation along the z-direction.

b. Unsteady motion along hyperbolas superposed on inhomogeneous extension:

Consider the motion

$$x = X \cos h\theta(t) + \left(\frac{C}{D}\right) Y \sin h\theta(t), \tag{37}$$

$$y = \frac{D}{C} X \sin h\theta(t) + Y \cos h\theta(t), \tag{38}$$

$$z = Z + \frac{\alpha X^2}{2} + \frac{\beta Y^2}{2} + \gamma XY, \tag{39}$$

where $\theta = \omega t + \varepsilon$ once again C, D, ω and ε are constants. The motion (37)-(39) satisfies (24) and (28), in the absence of body forces, and is hence possible in a neo-Hookean material. It is easy to verify that

$$\frac{X^2}{C^2} - \frac{Y^2}{D^2} = \frac{x^2}{C^2} - \frac{y^2}{D^2}, \tag{40}$$

and

$$\ddot{x} = \omega^2 x, \; \ddot{y} = \omega^2 y. \tag{41}$$

Thus, by virtue of (39) particles on hyperbolic cylinders $x^2/C^2 - y^2/D^2$ =constant remain on these cylinders.

c. Arbitrary motion along rectangular hyperbolas:

Consider the case

$$f_1(t) = g_2(t) = (1 + f^2)^{1/2}, \; f_2(t) = g_1(t) = f(t), \tag{42}$$

where $f(t)$ is any arbitrary twice differentiable function of t. We find that the choice (42) automatically satisfies (24) and (28) and hence the motion corresponding to the choice (42) is possible, in the absence of body forces, in a neo Hookean material. It is easy to verify that

$$X^2 - Y^2 = x^2 - y^2, \tag{43}$$

and thus particles on hyperbolic cylinders $X^2 - Y^2$ =constant remain on these cylinders. The interesting feature is that these particles can move in a totally arbitrary manner as the choice of the function $f(t)$ is left to us.

Several other unsteady motions of the form (19)-(21) are discussed by Hayes and Rajagopal [10].

4. Wave propagation in solids infused with fluids.

Recently, there has been considerable interest in the study of solids infused with fluids that are undergoing large deformations. An up-to-date discussion of these efforts can be found in the review article by Rajagopal and Wineman [14]. However, much if not all this effort has been directed towards the study of steady problems. Even there, problems present daunting obstacles with regard to proper choices for the constitutive expressions and boundary conditions. The issues are all the more complicated when we consider unsteady motions of solids infused with fluids. Tao, Rajagopal and Wineman [14] have recently presented a theory, within the context of the theory of interacting continua, for the study of diffusing singular surfaces in a solid infused with a fluid. During the mid-nineteen hundreds, Biot [2], [3], [4], [5] developed a theory for the study of elastic solids infused with fluids. Rajagopal and Tao [13] have used the theory of interacting continua to develop a method for studying unsteady motions of solid infused with fluids, which can be reduced to the theory proposed by Biot by making appropriate assumptions. Here, we shall discuss wave propagations within the context of the theory proposed by Rajagopal and Tao [13].

The theory of interacting continua assumes that each point in space in the mixture, a particle belonging to each constituent that makes up the mixture is present. The kinematical variables associated with the motion of each constituent, the basic balance laws for each constituent and constitutive assumptions which are necessary for each constituent are postulated (cf. Atkins and Craine [1], Bowen [7], Truesdell [18], [19]). Here we shall not discuss any of these details, but we shall only document the final equations which govern unsteady motions in solids infused with fluids. Details of the same can be found in the paper by Rajagopal and Tao [13].

Let $\mathbf{V}$ and $\mathbf{U}$ denote the displacement of the fluid and solid, respectively. Let us also suppose that the specific Helmhotz free energy for the mixture, which we shall denote hy A is given by

$$A = A(\rho_f, I, II, III, \theta, \beta), \tag{44}$$

where ρ_f is the density of the fluid, θ is the temperature, β is the porosity of the solid, and

$$\begin{aligned} I &= tr\mathbf{B}, \\ II &= \frac{1}{2}[(tr\mathbf{B})^2 - tr\mathbf{B}^2], \\ III &= \det \mathbf{B}, \end{aligned} \tag{45}$$

and $\mathbf{B}$ is the Cauchy-Green tensor defined through $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, $\mathbf{F}$ being the deformation gradient. Then, using standard arguments in continuum mechanics, we can obtain the constitutive representations for the partial stress for the solid $\mathbf{T}_s$ the partial stress for the fluid $\mathbf{T}_f$ and the interaction terms $\mathbf{I}$ (cf. Rajagopal and Tao [13]). If we make the assumptions that the solid in the reference state has constant porosity β, the fluid is compressible, the velocities and accelerations of both the solid and fluid are «small» in some sense, the virtual mass tensor is isotropic and the partial stress tensors for the solid and fluid have a special form (cf. Rajagopal and Tao [13]), we can show that the balance of linear momentum for the two constituents reduce to

$$
\begin{aligned}
&[\rho_{s0} + (\rho_{s0} + \rho_{f0})H_0]\frac{\partial^2 U_i}{\partial t^2} - (\rho_{s0} + \rho_{f_0})H_0\frac{\partial^2 V_i}{\partial t^2} \\
&+ b\left(\frac{\partial U_i}{\partial t} - \frac{\partial V_i}{\partial t}\right) = \frac{1}{2}\beta_2 U_{i,kk} + \\
&\left(\beta_1 - \alpha_1 + \frac{1}{2}\beta_2\right) e_{kk,i} + (\alpha_1 + \gamma_1)V_{k,ki},
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&[\rho_{f0} + (\rho_{s0} + \rho_{f0})H_0]\frac{\partial^2 V_i}{\partial t^2} - (\rho_{f0} + \rho_{f_0})H_0\frac{\partial^2 V_i}{\partial t^2} \\
&- b\left(\frac{\partial U_i}{\partial t} - \frac{\partial V_i}{\partial t}\right) = (\alpha_1 + \gamma_1)e_{kk,i} + (\alpha_2 + \gamma_2)V_{k,ki}.
\end{aligned} \tag{47}
$$

Here β_1, β_2, α_1, γ_1 are constants, e is the symmetric part of the displacement gradient $\nabla \mathbf{U}$, b is the drag coefficient and $\mathbf{H}_0 = H_0\mathbf{1}$ is the virtual mass tensor, ρ_{s0} and ρ_{f0} are the density of the solid and the fluids in their reference configurations, respectively. We notice that the equations have the same structure as Biot's equation (cf. Biot [4]) and we can have two distinct dilational wave speeds.

If the constituents of the mixture are incompressible, then we have to satisfy a volume additivity constraint (cf. Mills [11]). In this case Rajagopal and Tao [13] find that only one dilatational wave speed is possible, unlike two dilatational wave speeds in the case of a mixture of compressible constituents.

These results can also be extended to the case of an anisotropic solid infused with a fluid. Rajagopal and Tao [13] study unsteady motions of a transversely isotropic solid infused with a fluid. In this case, the specific Helmholtz free energy has the form

$$A = A(\rho_f, I, II, III, E_{33}, E_{13}^2 + E_{23}^2, \theta, \beta), \tag{48}$$

where

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}). \tag{49}$$

Under assumptions similar to those discussed earlier in the case of unsteady motions of an isotropic solid infused with a fluid, we can derive the appropriate approximate form of the balance of linear momentum. We shall not discuss the details here but refer the reader to [13] for the same.

Rajagopal and Tao [13] study the propogation of transverse plane waves, longitudinal waves and spherical waves which are considered as disturbances on the state of finite triaxial extension. This then allows the possibility to study the effect of finite deformation on the propogation of these unsteady motions. Rajagopal and Tao [13] consider motions of the form

(50)
$$X_1 = \lambda_1 x_1 + U_1(x_k, t),$$
$$X_2 = \lambda_2 x_2 + U_2(x_k, t),$$
$$X_3 = \lambda_3 x_3 + U_3(x_k, t),$$

where λ_i are constants and U_i denote the perturbances to the state of triaxial extension. While it is customary to express the coordinates of the particle in the deformed configuration x_i in terms of the coordinates of the particle in the reference configuration X_i, Rajagopal and Tao [6] choose to express the coordinate X_i in terms of x_i for ease of computation.

In the case of transverse plane waves Rajagopal and Tao [6] assume

(51)
$$U = U_1 \mathrm{e}_1 + U_2 \mathrm{e}_2, \ \mathbf{v} = V_1 \mathrm{e}_1 + V_2 \mathrm{e}_2,$$

where

(52)
$$U_k(x_k, t) = U_{k0} e^{i(l_k x_3 - \omega_k t)}, \qquad \text{no sum on } k,$$

(53)
$$V_k(x_k, t) = V_{k0} e^{i(l_k x_3 - \omega_k t)}, \qquad \text{no sum on } k,$$

while in the case of longitudinal waves

(54)
$$\mathbf{U} = U \mathrm{e}_3, \ \mathbf{v} = V \mathrm{e}_3,$$

with

(55)
$$U = U_0 e^{i(l x_3 - \omega t)},$$

(56) $$V = V_0 e^{i(lx_3-\omega t)}.$$

In the case of spherical waves

(57) $$\mathbf{U} = U\mathbf{e}_r, \ \mathbf{v} = v\mathbf{e}_r,$$

where

(58) $$U = U_0 \frac{\partial}{\partial r}\left(\frac{1}{r}e^{i(lr-\omega w t)}\right)$$

(55) $$V = V_0 \frac{\partial}{\partial r}\left(\frac{1}{r}e^{i(lr-\omega w t)}\right).$$

In all the above cases they show that such waves of the assumed form are possible.

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