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ASYMPTOTIC CONVERGENCE TO TRAVELLING WAVES FOR THE STEFAN PROBLEM AND RELATED PROBLEMS

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We investigate the stability of travelling wave solutions of the one-dimensional supercooled Stefan problem and other related problem. A complete characterization of the set of initial data under which the free boundary is asymptotic to a travelling wave front is found. The method applies also to other types of solutions like similarity solutions.

1. Introduction.

The Stefan problem is an old and intensively studied problem. It is the simplest mathematical model for heat exchange driven phase transition. We give a brief description of the problem which will be usefull to fix notation. We limit from the start to the one-dimensional problem, i.e. the case of planar symmetry, since our analysis will be confined within this problem. We take a frame of reference with the (y, z) axis on the plane of symmetry so that all relevant quantities only depend on the x space variable and the time variable t. We assume that the whole space is filled with a material in the liquid state on the right of the plane $x = s(t)$, in the solid state on its left.

The main assumptions of the Stefan description of the solid-liquid transition is that the transition takes place at a fixed (melting) temperature and it is confined on the (moving) interface, $x = s(t)$, between the solid and the liquid. Then the material is solidifing iff the interface is moving to the right ($\dot{s}(t) > 0$), the converse for melting. Moreover the phenomenon is totally driven by heat exchange, governed by Fourier's law in the two phases. The melting of a volume of solid demands an extra quantity of energy (in addition to that required to rise the temperature) proportional to the volume itself, the constant of proportionality is called the latent heat. Symmetrically the solidification release the same amount of energy. The density difference of the two phases is negletted. The mathematical setting of the problem is then

$$u_t - u_{xx} = 0, \qquad x \neq s(t), \quad t > 0, \tag{1.1}$$

$$s(0) = 0, \tag{1.2}$$

$$u(s^+(t), t) = u(s^-(t), t) = 0, \qquad t > 0, \tag{1.3}$$

$$u_x(s^+(t), t) - u_x(s^-(t), t) = -\lambda \dot{s}(t), \qquad t > 0, \tag{1.4}$$

$$u(x, 0) = \varphi(x), \qquad x \in (-\infty, \infty), \tag{1.5}$$

where u represents the temperature, eq. (1.3) is the assumption of "local equilibrium" at the interface, the melting temperature being scaled to zero. Eq. (1.4) is the so called Stefan condition which gives the energy balance through the interface. In the following we concetrate on a simpler problem, coming from the previous one assuming that one of the phase is initially at the melting temperature everywhere. For instance assume that $\varphi(x) = 0$ for negative x. Then the maximum principle for the heat equation, [12] ensure that $u(x,t)$ is identically zero in all the region $x < s(t)$. We have to concetrate only on what happens on the right of $x = s(t)$, i.e. we have to consider only the set of equations

$$u_t - u_{xx} = 0, \qquad s(t) < x < \infty, \quad t > 0, \tag{1.6}$$

$$s(0) = 0, \tag{1.7}$$

$$u(s(t),t) = 0, \qquad t > 0, \tag{1.8}$$

$$u_x(s(t),t) = -\lambda \dot{s}(t), \qquad t > 0, \tag{1.9}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,\infty). \tag{1.10}$$

The problem above is called one-phase Stefan problem. The minus sign in (1.9) reminds that we are assuming that the phase on the right of $x = s(t)$ is liquid, i.e. has higher energy than the phase on the left. Accordingly the standard problem will have $u(x,t) > 0$, which is the case if $\varphi(x) > 0$. Consequently the Vyborny-Friedman boundary point principle (the parabolic version of the Hopf lemma, [12]) ensures that the interface moves toward the left, i.e. the material melts.

However in some experiments the liquid can be undercooled below the melting temperature without solidification. This makes reasonable to investigate problem (1.6)-(1.10) even in the case of $\varphi(x) < 0$ (and consequently $u(x,t) < 0$). This is the so called supercooled Stefan problem.

The supercooled problem is much more delicate from the mathematical point of view than the standard problem. In fact in the case φ positive the problem can be mathematically solved under very unrestrictive conditions on the initial data and the solution always exists for any positive time, i.e. is a global solution. In the supercooled problem strong restrictions on the initial data must be assumed in order to be sure of the existence of the solution, expecially if we are interested in the global existence.

The main reason for that is that the free boundary $x = s(t)$ "blows up" if the temperature becomes too low. The precise meaning of the above sentence can be found in [10], and [6], where it is proved that $\dot{s}$ becomes infinite iff the level line $u(x,t) = -\lambda$ hits the free boundary. In particular this immediatly implies that the supercooled problem has no solution for initial data $\varphi < -\lambda$. On the contrary if $\varphi > -\lambda$ everywhere, then local existence of the solution implies global existence. What happens for initial data oscillating about the value $-\lambda$ is an open problem. In [11] examples of very complicated behaviours are given: for instance the free boundary may blow up at a given t_0, i.e. $\dot{s}(t_0) = +\infty$, but with the function $u(x,t_0) > -\lambda$ for any $x > s(t_0)$ and such that it can be used as an initial datum of the

Stefan problem for $t > t_0$. Then the solution can be extended for $t > t_0$ matching the solution for $t < t_0$ with that corresponding to the initial datum $u(x, t_0)$ for $t > t_0$. In this case the blow up is said "inessential".

The typical solutions of problem (1.6)-(1.10) are the similarity solutions, i.e. solution depending on x and t only through the ratio $\frac{x}{\sqrt{t}}$, with free boundary of the form $x = \beta\sqrt{t}$. These solutions correspond to constant in space initial data φ_0, β and φ_0 being related by a transcendental equation which is uniquely solvable iff $\varphi_0 > -\lambda$, [19].

There exist also travelling wave solutions of the problem, corresponding to initial data which decay exponentially to $-\lambda$ as $x \to \infty$. This solutions are of less importance from the physical point of view because their initial data are almost indiscernible.

However the stability analysis of these solutions is of importance in a different contest. In fact, as we shall see, problem (1.6)-(1.10) is closely related to a problem in the NEF (near equilibrium flame) theory [2] where travelling waves are the typical solutions.

Our aim is to characterize the asymptotic behaviour of the solutions of (1.6)-(1.10) in terms of the initial data. This problem as been the object of previous investigations, see [4], [21], [8]. In particular, in the letter reference, an asympotic analysis based on a Laplace transform approach to the problem was performed, giving some insite of the behaviour of the solution in the travelling waves case.

In [4] the case of similarity solution was considered. The authors were able to prove the following global existence result and asympotic characterization of the free boundary behaviour.

THEOREM [4]. *Suppose* $\varphi \in C^1[0, +\infty)$, $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = \varphi_\infty$ *and* $\varphi(x) - \varphi_\infty \in L^1(0, +\infty)$.

a) *Problem* (1.6)-(1.10) *has a unique global solution if* $\varphi_0(x) > -\frac{\lambda}{8}$.

b) *If a global solution exists, then there exist* n, $N > 0$ *such that, for* $t > 0$, *one has* $(1 - \epsilon)\beta\sqrt{t} - n < s(t) < (1 + \epsilon)\beta\sqrt{t} + N$ *where* β *is the coefficient of the parabola corresponding to the initial datum* $u(x, 0) = \varphi_\infty$ *and* $\epsilon \to 0$ *as the Stefan number* $\frac{\varphi_\infty}{\lambda} \to 0$.

Both this two resuts leave a quite large gap. For what the global solution is concerned we have noticed above that the condition $\varphi(x) > -\lambda$ is sufficient to guarantee the global existence once some reasonable conditions for local existence are satisfied. Another sufficient condition for global existence will be given later. In the following we also prove that a sharper asymptotic characterization can be proved even if one drops the L^1 assumption on the datum.

Our approach mainly consists in a tranformation of the original problem into a more regular one, for which a global comparison principle can be applied. An intermediate step in the regularization transforms the Stefan problem into the NEF model of combustion theory. For this problem global existence and asymptotic behaviour were discussed in papers by Brauner and Schmidt-Lainé and by Hilhorst and Hulshof. We discuss the relations among those papers and our approch in the following.

2. Related problems.

For sake of simplicity we take $\lambda = 1$ in the (1.9) throughout the rest of the paper. If $u(x,t)$ is a solution of problem (1.6)-(1.10) we define a new function $c(x,t)$ by means of the Baiocchi-type transform introduced in [10]:

$$c(x,t) = \int_{s(t)}^{x} dy \int_{s(t)}^{y} (u(\xi,t)+1)\, d\xi. \tag{2.1}$$

The function $c(x,t)$ is defined by (2.1) for any $x > s(t)$. It is a simple computation to verify that $c(x,t)$ and the free boundary $x = s(t)$ solves a new free boundary problem.

$$c_t - c_{xx} + 1 = 0, \qquad s(t) < x < \infty, \quad t > 0, \tag{2.2}$$

$$c(s(t),t) = 0, \qquad t > 0, \tag{2.3}$$

$$c_x(s(t),t) = 0, \qquad t > 0, \tag{2.4}$$

$$c(x,0) = c_0(x), \qquad s(0) = 0 \le x < \infty, \tag{2.5}$$

where

$$c_0(x) = \int_0^x dy \int_0^y (\varphi(\xi) + 1)\, d\xi. \tag{2.6}$$

Problem (2.2)-(2.5) is known as the *oxygen consumption problem* since it was first introduced in [7] to describe the sorption of oxigen into a living tissue. A constant sorption term is present in eq. (2.2) and it can cause the function $c(x,t)$ to become negative somewhere in $x > s(t)$. If this does not happen, than the free boundary problem (2.2)-(2.5) is equivalent to a variational inequality, [13], [9], namelly to find the non negative solution of the equation

$$c_t - c_{xx} + H(c) = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{2.7}$$

with initial datum $c(x,0) = c_0(x), x > 0$ and $c \equiv 0, x < 0$, where the H is the Heaviside function

$$H(c) = \begin{cases} 1, & c > 0 \\ 0, & c \le 0. \end{cases} \tag{2.8}$$

Conversely, if problem (2.2)-(2.5) has a sufficiently regular solution, positive in the domain $x > s(t)$ and vanishing for $x \le s(t)$ for some C^1 function $s(t)$, then its time derivative $u(x,t) = c_t(x,t)$, and the free boundary $x = s(t)$ are the solution of the Stefan problem (1.6)-(1.10).

In the case we are considering the positivity of the function $c(x,t)$, defined by (2.1) starting from the solution of the Stefan problem, is granted if, for instance, $u(x,t) > -1$ (remember that this is the case if $u(x,0) > -1$).

In the present contest solutions of (2.7) have little physical meaning because, with our choice of the data, they are unbounded. However the main interest of problem (2.7) is that a comparison principle holds for its solutions. This makes the tranformation from the temperature u to the oxygen concentration c a useful tool in proving stability results.

Suppose now that the initial datum φ is such that $\varphi(x) + 1 \in L^1(0,+\infty)$ and let

$$0 < f_s = \int_0^\infty (\varphi(x) + 1)\, dx < \infty. \tag{2.9}$$

Then we define a function $v(x,t)$ by

$$v(x,t) = f_s - \int_{s(t)}^{x} (u(\xi,t)+1)\,d\xi, \tag{2.10}$$

where $u(x,t)$ and $s(t)$ give the solution to (1.6)-(1.10).

Now the couple (v,s) solves the free boundary problem

$$v_t - v_{xx} = 0, \qquad s(t) < x < \infty, \quad t > 0, \tag{2.11}$$

$$v(s(t),t) = f_s, \qquad t > 0, \tag{2.12}$$

$$v_x(s(t),t) = -1, \qquad t > 0, \tag{2.13}$$

$$v(x,0) = v_0(x), \qquad s(0) = 0 \le x < \infty, \tag{2.14}$$

where v_0 is defined according to (2.10), $v_0 \to 0$ as $x \to \infty$.

These equations were introduced by Buckmaster and Ludford, [2] to describe the temperature evolution for the equidiffusion case in the NEF (Near-Equilibrium Flames) theory. The value f_s represents the flame temperature and $x = s(t)$ is the location of the flame.

It is worth noting that the tranformation (2.10) can be defined also when the sum $\varphi + 1$ is not in L^1, simply by omitting the term f_s. The resulting $v(x,t)$ is no more bounded but the existence and uniqueness results kwown for the case v bounded can easily been proved in this case if $v(x,0)$ is negative for any $x > 0$.

For problem (2.11)-(2.14) travelling waves become significative, since the wave speed is now controlled by the flame temperature f_s, which is a typical parameter of the problem.

The stability of these solutions has been considered by Brauner and coworkers, see [1], and with a small modification in the equation, which does not change the quality of the results, by Hilhorst and Hulshof [14].

The technique used in [1] is an adapted form of a general tecnique developped by Sattinger, [20] for nonlinear reaction diffusion equation. This tecnique makes use of weighted space, introducing a norm with exponentially growing weight. What is proved in [1] is that, if the initial datum v_0 is close to a travelling wave profile in this norm then there exists a phase shift, such that the difference, in the same norm, between the solution and shifted wave decreases exponentially in time. However the norm involved requires that both the difference between v_0 and the travelling wave profile, and that of their derivatives, decay exponentially fast in space at infinity, limiting this stability result to a very narrow class of initial perturbation of travelling waves.

The approach by Hilhorst and Hulshof, [14], consists in imbedding the free boundary problem into a fixed domain problem for an elliptic-parabolic equation. The solution of (2.11)-(2.14) is prolongated with constant slope behind the free boundary in order to construct a solution of the equation

$$h(v)_t - v_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{2.15}$$

where $h(v) = \max\{v, f_s\}$. (Actually in their paper also a convecting nonlinear term vv_x is present in the equation, but their resuts hold also in the linear case). Assuming for the initial data that $v_0 \in C^0(0,+\infty) \cap L^1(0,+\infty)$, and $0 \leq v_0 \leq f_s$, they were able to prove asymptotic convergence to a travelling waves for any solution of the problem.

3. Some properties of the solution of the oxigen problem.

Our results on the asymptotic behavior are deeply dependent on some properties of the solutions of the oxigen problem (2.5)-(2.6). In particular the comparison principle will be the most important tool in the following. It states that equation (2.5) is order preserving, i.e. if the initial data are ordered, so are the corresponding solutions. A more general form, including a convection term in the equation is the following

LEMMA 3.1. *Let Ω be a subdomain of R and let $c_i(x,t), i = 1,2,$ be continuous non-negative functions satisfing*

$$c_{1\,t} + a(x,t)c_{1\,x} - c_{1\,xx} + H(c_1) \geq c_{2\,t} + a(x,t)c_{2\,x} - c_{2\,xx} + H(c_2),$$

$$\textit{in } D_T = \Omega \times (0,T)$$

where $a(x,t)$ is a bounded function and H is the Heaviside function defined by (2.8). *Assume that*

$$c_1 \geq c_2, \qquad \text{on } \partial_P D_T = \Omega \times \{0\} \cup \partial\Omega \times (0,T).$$

Then

$$c_1(x,t) \geq c_2(x,t), \qquad \text{in } D_T.$$

Proof.. Denote by $\delta(x,t) = c_1(x,t) - c_2(x,t)$. Then δ is solves the following equation

$$\delta_t - \delta_{xx} \geq H(c_2) - H(c_1) := f(x,t), \qquad -\infty < x < \infty, \quad t > 0,$$
$$\delta = c_1 - c_2 \geq 0, \qquad \text{on } \partial_P D_T = \Omega \times \{0\} \cup \partial\Omega \times (0,T).$$

Notice that the non-homogeneous term $f(x,t)$ is different from zero only in the set

$$N = \{(x,t) : |\delta| > c_1 c_2 = 0\},$$

i.e. only when one and only one of the c_i is zero. Moreover in N we have $f = -1$ where $\delta > 0$, $f = 1$ where $\delta < 0$. Then the maximum principle for the heat equation implies $\delta \geq 0$.

There is a main difference with the heat equation which is worth noting and it is that the comparison principle has only the previous weak form and the strong maximum principle does not hold (the simplest example is the case of $c(x,0) \equiv 1$ on R, which has solution $c = 1 - t$ for $t < 1$ and $c \equiv 0$ for $t > 1$)

The following lemma deals with the asympotic behaviour of the difference between two solutions.

LEMMA 3.2. *Let $c_i(x,t), i = 1,2$, be two solutions of (2.7) and suppose that $c_1 - c_2$ is a bounded function satisfing*

$$\lim_{|x|\to\infty} (c_1(x,0) - c_2(x,0)) = 0,$$

then

$$\lim_{t\to\infty} \sup_{x\in R} |c_1(x,t) - c_2(x,t)| = 0. \tag{3.1}$$

Proof. Denoting by δ the difference of the two solutions the maximum principle again implies that $|\delta|$ is dominated by the solution $\varepsilon(x,t)$ of the heat equation with initial datum $\varepsilon(x,0) = |c_1(x,0) - c_2(x,0)|$. The Lemma is a simple corollary of a general results on the asymptotic behaviour of the solution of the heat equation proved by Mihaĭlov [16], which imply that $\varepsilon(x,t)$ tends to zero uniformly in R. This can be proved directly looking carefully at the Poisson integral representation of the solution of the heat equation [18].

Notice that it is not required that the two solutions are bounded but only that their difference is, and this, in turn, is true if the difference is bounded for some $t_0 \geq 0$. In case that the solutions themselves are bounded the result is trivial since both of them will vanish in finite time.

A more delicate analysis is needed in order to establish some estimates on the asymptotic behaviour of the free boundaries of solution of (2.7).

The case in which we are interested in is that in which the solutions of (2.7) are such that their support at any time t is of the form $x > s(t)$ for some function $s(t)$. Suppose that we have two solution of this kind, and that the assumptions of Lemma 3.2 hold. Then are the free boundaries $s_1(t)$ and $s_2(t)$ asymptotically the same as $t \to \infty$?

The answer is yes. Let us first justify euristically why. The solutions of (2.7) are continuous together with their first derivative so $c_x(s(t),t) = 0$. However the second derivative c_{xx} has a fixed jump from zero to 1 at the free boundary. Since the c's vanish on the left of $x = s_i(t)$, this implies that, if the difference $c_1 - c_2$ decay to zero like, say, $\epsilon(t)$, the diffence between the free boundaries must decay like $\sqrt{\epsilon(t)}$. To transform this argument into a rigorous proof will involve a very delicate estimate from below of c_{xx} which in turn will demand extra assumptions on the data. However it is possible to prove the above statement using supersolutions which are reminiscent of those used by Bundle and Stakgold for the dead core problem [3].

We give a sketch of the proof, se also [18]. In a moving reference frame in which one of the free boundary is at rest, for instance with the origin in $x = s_1(t)$, we construct a supersolution for the second solution c_2 of (2.7) as follows. We start from the solution of the equation

$$K\,v_y + v_{yy} = \lambda H(v), \qquad y < 0, \tag{3.2}$$

$$v(0) = \delta, \tag{3.3}$$

where y is the coordinate in the moving frame, K is a constant greater than $|\dot{s}_1(t)|$, and λ is a positive parameter less than 1.

Then we add v to $w(t) = \delta_0 - (1-\lambda)(t-t_1)$. Carefully choosing the parameter δ_0 and t_1 the function $v(y) + w(t)$ is a supersolution for c_2, vanishing after finite time for any $y < y_\delta < 0$ where y_δ is the infimum of the support of the solution of (3.2)-(3.3). This implies that the free boundary of c_2 is eventually on the right of the point of coordinate $s_1(t)+y_\delta$. Now we can choose δ to be $\sup_x |c_1(x,t)-c_2(x,t)|$, which tends zero as t increases. Finally it remains to notice that y_δ is of order $\sqrt{\delta}$ as $\delta \to 0$ to obtain that $(s_1 - s_2)_+ \to 0$. Reversing c_1 with c_2 we finally have that $|s_1(t) - s_2(t)| \to 0$ as $t \to \infty$.

We can now summarize the previous discussion.

PROPOSITION 3.3. *Suppose $c_i, i = 1,2$ are two solutions of* (2.7) *as in Lemma* 3.2 *and suppose that*

$$supp\, c_i(\cdot,t) := \{(x,t),\, c_i(x,t) > 0\} = \{(x,t),\, x > s_i(t)\},$$

$$s_i(0) = b_i > +\infty,$$

$$\dot{s}_i(t) < K, \quad t > t_o \geq 0. \tag{3.7}$$

Then

$$\lim_{t\to\infty} |s_1(t) - s_2(t)| = 0.$$

4. Stability results for the Stefan problem.

We are now in a position to prove the stabilty results for the Stefan problem. The idea to tackle the problem is relatively simple. We transform the initial data of the Stefan problem into initial data for the oxigen problem: if the diffence between the data of the oxigen problem tends to zero as $x \to \infty$, then the free boundaries will be asympotically the same. In this way, for any fixed initial, datum we can characterize a whole set of initial data whose corresponding free boundaries are asympotic to the previous one. In particular this will be done for the travelling waves case and for the similarity solutions.

Let us start with the travelling waves. Because of the invariance of the problem under space and time translation, for any fixed speed, there exists a one parameter family of travelling waves with the same speed, differing only for a space shift (which for a travelling wave is equivalent to a time shift). Then for the set of initial data whose free boundary is asymptotic to a travelling wave we must expect one condition to select the speed and one more condition to select the initial shift.

Let us assume the following hypotheses on the initial datum φ

(H.1) φ bounded, continuous, $\varphi(x) > -1, \quad x > 0;$

(H.2) $\varphi(x) \to -1 \quad \text{as} \quad x \to +\infty;$

(H.3) $(\varphi(x)+1) \in L^1(R^+).$

We can now state the stability result for travelling waves.

THEOREM 4.1. *Let φ satisfy hypotheses (H.1), (H.2) and (H.3) and let*

$$\frac{1}{V} = \int_0^{+\infty} (\varphi(x)+1)\,dx, \tag{4.1}$$

Then the following two conditions are equivalent:

(*A*) There exists $b \in R$ such that $\lim_{t\to+\infty}(s(t) - Vt - b) = 0;$

(*B*) $\lim_{x\to+\infty}\left[\frac{x}{V} - c_0(x)\right] = L < +\infty;$

where $c_0(x)$ is given by (2.6). *The value of b is given by*

$$b = VL - \frac{1}{V}. \tag{4.2}$$

The proof of $(B) \Rightarrow (A)$ is an immediate corollary of the stability result of Proposition 3.3. Let us first recall that the travelling wave solution to problem of (1.6)-(1.10) is given by

$$u_{b,v}(x,t) = e^{-v(x-b-vt)} - 1, \qquad b + vt = s(t) < x, \quad t > 0, \tag{4.3}$$

where v is a positive number and b is any number.

In particular take $v = V$ and consider the oxigen concetration $c_{b,V}$ associated to the travelling wave solution defined by (4.3). Then

$$\begin{aligned} c_{b,V}(x,t) &= \int_b^{x-Vt} dy \int_b^y e^{-V(\xi-b)}\, d\xi \\ &= \frac{1}{V}(x - Vt - b) + \frac{1}{V^2}\left(e^{-V(x-Vt-b)} - 1\right). \end{aligned} \tag{4.4}$$

Then the choice of b according to (4.2) ensure that

$$\lim_{x \to +\infty} (c_0(x) - c_{b,V}(x,0)) = 0,$$

where c_0 is the initial datum for the oxigen problem corresponding to φ by (2.6). Then proposition 3.3 implies (A).

The converse is also true. In fact, if we assume that (A) holds together with $L = +\infty$ in (B), we end with a contraddiction. We refer to [18] for details.

Observe that condition (B) can be written in equivalent way, if our hypotheses (H) are satisfied, as

$$x(\varphi(x) + 1) \in L^1(R^+). \tag{B'}$$

It worth noting that assumption (H.1) is a quite natural assumption for the Stefan problem since it says that the local energy density of the liquid phase is higher that the energy density of the solid at the melting temperature.

The mathematical role of this condition in our analysis is twofold: for one side it is a sufficient condition for the global existence of the solution of the Stefan problem, as we already noticed. From the other side condition (H.1) trivially implies the existence of the limit (finite or infinite)

$$\lim_{x\to+\infty}\int_0^x (\varphi(\xi)+1)d\xi,$$

which is used in proving the equivalence of conditions (B) and (B'). Then we can reformulate our results substituting (H.1) with the explicit requirement that the solution of the Stefan problem globally exists and that the above integral as a limit.

In particular (H.1) is nomore a natural condition when the combustion problem (2.11)-(2.14) is concerned, since in this case it would implies that the initial datum is a monotone function, which is a unnaturally restrictive condition. However in this case the global existence of the solution can be proved independently from (H.1). In fact the global existence proof of [14] can be rephrased with minor changes to be adapted to the general case of problem (1.6)-(1.10), even if the condition $\varphi+1\in L^1$ is not satifisfied. In this case we define the function v by

$$v(x,t) = -\int_{s(t)}^{x} (u(\xi,t)+1)\,d\xi,$$

instead of (2.10) (v can be unbounded). Now it turns out that the solution globally exists if $v(x,0)<0$ for any $x>0$. This is a natural condition for the combustion problem since it means that the initial temperature is everywhere less than the flame temperature (here the flame temperature is taken to be zero). This condition can also be rephrased in the Stefan problem language saying, at the initial time, the total energy of the portion of the melt that is between the front and x is larger than the energy of the same quantity of solid at the melting temperature, for any $x>0$. This is a weaker assumption with respect to $\varphi>0$, since it does not forbid that somewhere "sufficiently far from the melting front" the energy density may drop below that of the solid at melting temperature.

As a corollary of Thoerem 4.1 and the comparison principle for the oxigen problem we can also prove a weaker stability result in the case that condition (B) does not hold.

THEOREM 4.2. *Let φ satisfy hypotheses $(H.1)$, and $(H.2)$ Then the* *following condition is equivalent to* $(H.3)$

(C) There exists $V > 0$ such that $\lim_{t\to+\infty} \frac{s(t) - Vt}{t} = 0.$

The proof of this theorem simply follows comparing the initial datum with initial data for travelling waves with speeds $V_1 < V < V_2$.

This result can be rephrased for the combustion problem saying that any solution to problem (4.11)-(4.14) has a free boundary which satisfies (C).

A similar asymptotic analysis can be done for the case of similarity solutions. We just state here the main result.

THEOREM 6.1. *Suppose* $\varphi > -1$ *and* $\lim_{x\to+\infty}\varphi(x) = \varphi_\infty > -1$, *then*

$$\lim_{t\to+\infty} \frac{s(t) - \beta(\varphi_\infty)\sqrt{t}}{\sqrt{t}} = 0.$$

where $x = \beta(\varphi_\infty)\sqrt{t}$ *is the free boundary corresponding to the initial datum* $u(x,0) = \varphi_\infty$.

Moreover, if $\varphi(x) - \varphi_\infty$ *and* $x(\varphi(x) - \varphi_\infty)$ *belong to* $L^1(R^+)$, *then*

$$\lim_{t\to+\infty} \left(s(t) - \beta(\varphi_\infty)\sqrt{t - t_0} - x_0\right) = 0,$$

where t_0 *and* x_0 *are solution of the system*

$$\int_0^{+\infty} (\varphi - \varphi_\infty)\, dx = \beta \exp\left(\frac{\beta^2}{4}\right)$$

$$\int_{s_0}^{+\infty} \left(\int_{\frac{x-x_0}{2\sqrt{t}}}^{+\infty} \exp\left(-u^2\right) du\right) dx - (\varphi_\infty + 1)\, s_0$$

$$\int_0^{+\infty} x(\varphi - \varphi_\infty)\, dx = \beta \exp\left(\frac{\beta^2}{4}\right)$$

$$\int_{s_0}^{+\infty} x \left(\int_{\frac{x-x_0}{2\sqrt{t}}}^{+\infty} \exp\left(-u^2\right) du\right) dx - \frac{1}{2}(\varphi_\infty + 1)\, s_0^2$$

where $s_0 = \beta\sqrt{t_0} + x_0$.

Notice that a slight difference exists from this case and that of the travelling waves. In this case the existence of the limit at $+\infty$ is enough to select the asymptotic behaviour and to ensure the weak stability result. Two extra conditions are now needed in order to have the strong asymptotic result, This is because time and space shifts are nomore equivalent for the parabola shaped free boundaries of the similarity solutions and they must be fixed solving a system of two coupled equations.

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