LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 361-365

# COVARIANT FLUX-LIMITED DIFFUSION THEORY

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Explicit flux-limited expression are obtained for relativistic radiation energy flux and stress tensor in the case of small shear.

## 1. Introduction.

In this article, following an earlier approach by M.A. Anile and M. Sammartino [2] , we present a general relativistic flux-limited diffusion theory, which holds for arbitrary inhomogeneous and nonstationary media, provided the shear is sufficiently small.

The latter limitation is essential if we want ot obtain explicit expressions for the radiation energy-flux and stress tensor because, otherwise, a host of highly non-linear terms describing the coupling of the radiation energy-density gradient and shear would appear. In section 2 we start from the covariant radiative transfer equation and set up the basic approximation method and in section 3 we perform a small shear analysis.

## 2. Covariant radiative transfer equation and flux-limit diffusion approximation.

For the sake of simplicity we treat the case of gray medium and neglet polarization, dispersion and coherence of the radiation field. Then the integrated form of the radiative trasfer equation, explicitly covariant (see [1]), is:

$$(u^\mu + l^\mu)\Big[\nabla_\mu I + 4Il^\sigma\nabla_\mu u_\sigma + (\nabla_\mu u_\sigma)u^\rho l^\sigma \frac{\partial I}{\partial l^\rho} + + (\nabla_\mu u_\sigma)l^\rho l^\sigma \frac{\partial I}{\partial l^\rho} - (\nabla_\mu u^\rho)\frac{\partial I}{\partial l^\rho}\Big] = \frac{1}{\tau}(S - I) \tag{1}$$

where $I = I(u^\mu, l^\mu)$ is the integrated intensity, $I = \int_0^\infty f\nu_0^3 d\nu_0$ (f being the invariant distribution function), $\nu_0$ the rest-frame frequency as measured by the observer whose 4-velocity is $u^\mu$, $l^\mu$ is the unit vector in the observer's rest frame, $l^\mu l_\mu$=1, $l^\mu u_\mu$=0, S is the isotropic source function and $\tau$ the photon mean free path.

Let $J, H^\mu, K^{\mu\nu}$ denote the radiation energy density, energy-flux and stress tensor as measured in the observer's rest frame,

$$J = \frac{1}{4\pi}\int_\Omega I d\Omega \qquad H^\mu = \frac{1}{4\pi}\int_\Omega l^\mu I d\Omega \qquad K^{\mu\nu} = \frac{1}{4\pi}\int_\Omega l^\mu l^\nu I d\Omega \tag{2}$$

where $d\Omega$ is the element of solid angle $d\Omega = \delta(u_\mu l^\mu)\delta(l^\alpha l_\alpha - 1)d^4 l$.

In terms of normalized intensity $\psi = I/J$, using an expansion in spherical harmonics of $\psi$, introducing the standard kinematic decomposition $\nabla_\mu u_\sigma = \frac{1}{3}\Theta h_{\mu\sigma} + \sigma_{\mu\sigma} + \omega_{\mu\sigma} - a_\sigma u_\mu$ where $\Theta, \sigma_{\mu\sigma}, \omega_{\mu\sigma}$ are the expansion , shear and rotation and $h_{\mu\sigma}$ is the projection tensor, eq. (1) can be rewritten

$$\psi(u^\mu + l^\mu)\nabla_\mu J + J(u^\mu + l^\mu)\nabla_\mu\psi + 4J\psi\left[l^\sigma a_\sigma + \frac{\Theta}{3} + \sigma_{\mu\sigma}l^\mu l^\sigma\right] + + J\left[a_\sigma l^\sigma l^\rho - a_\rho + \sigma_{\mu\sigma}l^\mu l^\sigma l^\rho - l_\mu\sigma_{\mu\rho} - l^\mu\omega^{\mu\rho}\right]\frac{\partial\psi}{\partial l^\rho} = \frac{1}{\tau}(S - J\psi) \tag{3}$$

As discussed by Anile and Sammartino [3], we generalize the Levermore and Pomraning ansatz [4] by supposing that $\psi$ is slowly varing along the bicharacteristics of (3), i.e.

$$(u^\mu + l^\mu)\nabla_\mu\psi + (a_\sigma l^\sigma l^\rho - a_\rho + \sigma_{\mu\sigma}l^\mu l^\sigma l^\rho - l^\mu\sigma^{\mu\rho} - l^\mu\omega^{\mu\rho})\frac{\partial\psi}{\partial l^\rho} = 0 \tag{4}$$

Let $f^\mu$ denote the normalized flux ($H^\mu = Jf^\mu$), with $f^\mu = \frac{1}{4\pi}\int_\Omega l^\mu \psi d\Omega$. By using the sperical harmonics decomposition and the energy equation for the radiation we obtain

$$\psi = \frac{S}{\tau\left[S/\tau + (l^\mu - f^\mu)(\nabla_\mu J + 4Ja_\mu) + 4J\sigma_{\mu\nu}\left(l^\mu l^\nu - \frac{K^{\mu\nu}}{J}\right)\right]} \tag{5}$$

## 3. Small shear analysis.

From the definitions (2) and the explicit expression (4) for $\psi$, it follows that both $f^\mu$ and $K^{\mu\nu}$ are functions of $\sigma^{\alpha\beta}$ and of $\tilde{R}^\alpha \equiv (\tau/S)h^\beta_\alpha(\nabla_\beta J + 4Ja_\beta)$. From exact representation theorem [5] for small shear, keeping only linear terms in $\sigma_{\mu\nu}$, we obtain

$$f^\mu = (\lambda_1 + \lambda_2\sigma^{\tau\rho}\tilde{R}_\tau\tilde{R}_\rho)\tilde{R}^\mu + \sigma\sigma^{\mu\nu}\tilde{R}_\nu \tag{6}$$

$$\begin{aligned} K^{\mu\nu} = & \frac{1}{3}(\alpha_1 + \alpha_2\sigma^{\tau\rho}\tilde{R}_\tau\tilde{R}_\rho)Jh^{\mu\nu} + \beta\sigma^{\mu\nu} + \\ & + \gamma\sigma^{\mu\rho}\tilde{R}_\rho\tilde{R}_\nu + (\delta_1 + \delta_2\sigma^{\tau\rho}\tilde{R}_\tau\tilde{R}_\rho)\tilde{R}^\mu\tilde{R}^\nu \end{aligned} \tag{7}$$

with $\lambda_1, \lambda_2, \sigma, \alpha_1, \alpha_2, \beta, \gamma, \delta_1, \delta_2$ functions of J , $\tilde{R}^\alpha\tilde{R}_\alpha$.

In this paper we consider only the case $\sigma^{\mu\nu}\tilde{R}_\mu\tilde{R}_\nu = 0$ (for the general case we refer to an article to appear [3]). Then, by using eq. (2) and (5), we obtain, for small shear:

$$4\pi = \int_\Omega \frac{d\Omega}{1 + l^\mu\tilde{R}_\mu - \lambda_1\tilde{R}^\mu\tilde{R}_\mu} + O(\sigma^2_{\mu\nu}) \tag{8}$$

$$\begin{aligned} 4\pi f^\mu = & \int_\Omega \frac{l^\mu d\Omega}{1 + l^\mu\tilde{R}_\mu - \lambda_1\tilde{R}^\mu\tilde{R}_\mu} - \\ & - 4\tilde{J}\sigma_{\tau\rho}\int_\Omega \frac{l^\tau l^\rho l^\mu d\Omega}{\left[1 + l^\mu\tilde{R}_\mu - \lambda_1\tilde{R}^\mu\tilde{R}_\mu\right]^2} + O(\sigma^2_{\mu\nu}) \end{aligned} \tag{9}$$

$$(10)\qquad \begin{aligned}\frac{4\pi}{J}K^{\mu\nu} = \int_\Omega \frac{l^\mu l^\nu d\Omega}{1 + l^\mu \tilde{R}_\mu - \lambda_1 \tilde{R}^\mu \tilde{R}_\mu} - \\ - 4\tilde{J}\sigma_{\tau\rho}\int_\Omega \frac{l^\tau l^\rho l^\mu l^\nu d\Omega}{\left[1 + l^\mu \tilde{R}_\mu - \lambda_1 \tilde{R}^\mu \tilde{R}_\mu\right]^2} + O(\sigma^2_{\mu\nu})\end{aligned}$$

$$\sigma_{\tau\rho}K^{\tau\rho} = \sigma_{\tau\rho}\left[\frac{1}{3}\alpha_1 J h^{\tau\rho} + \beta\sigma^{\tau\rho} + \gamma\sigma^{\tau\rho}\tilde{R}_\nu \tilde{R}^\rho + \delta_1 \tilde{R}^\tau \tilde{R}^\rho\right] = O(\sigma^2_{\mu\nu})$$

where $\tilde{J} = \tau J/S$. Then, by choosing an orthonormal tetrad ($u^\mu$, $e^\mu_{(i)}$ $i = 1, 2, 3$), in which $\underline{\tilde{R}} = e_{(3)}\tilde{R}, \sigma^{33} = 0$, eq. (8, 9, 10) can be inverted and yield

$$(11)\qquad \lambda_1 = \frac{1}{\tilde{R}}\left[\frac{1}{\tilde{R}} - \coth \tilde{R}\right]$$

which coincides with the shear free case [2].

Now, from the representation (6) we have $f^\mu f_\mu = (\lambda_1 \tilde{R})^2 + 2\lambda_1 \sigma_{\nu\mu}\tilde{R}_\mu \tilde{R}_\nu(\lambda_2 \tilde{R}^2 + \sigma) + O(\sigma^2_{\mu\nu})$ and therefore, for $\sigma^{\mu\nu}\tilde{R}_\mu \tilde{R}_\nu = 0$, we obtain

$$f^2 = (\lambda_1 \tilde{R})^2$$

which gives a flux-limited theory. The remainig coefficients are then given by

$$\alpha_1 = \frac{3}{2}\left[1 + \coth \tilde{R}\left(\frac{1}{\tilde{R}} - \coth \tilde{R}\right)\right]$$

$$\delta_1 = \frac{1}{2\tilde{R}^2}\left[3\coth \tilde{R}\left(\coth \tilde{R} - \frac{1}{\tilde{R}}\right) - 1\right]$$

$$\beta = -J\tilde{J}\left[\frac{1 - 4\coth^2 \tilde{R} + \coth^4 \tilde{R}}{\tilde{R}^2(\coth^2 \tilde{R} - 1)} + \frac{9\coth^2 \tilde{R} - 5}{3\tilde{R}^2} + \right.$$
$$\left. + \frac{4\coth^2 \tilde{R}(1 - \coth^4 \tilde{R})}{\tilde{R}}\right]$$

$$\gamma = \frac{J\tilde{J}}{\tilde{R}^2}\left[\frac{1 - 12\coth^2 \tilde{R} + 5\coth^4 \tilde{R}}{\tilde{R}^2(\coth^2 \tilde{R} - 1)} - \frac{13}{3\tilde{R}^2} + \frac{15\coth^2 \tilde{R}}{\tilde{R}^2} - \right.$$
$$\left. - \frac{12\coth \tilde{R} - 20\coth^3 \tilde{R}}{\tilde{R}}\right]$$

$$\sigma = -\frac{\tilde{J}}{\tilde{R}}\left[\frac{2}{\tilde{R}} + \frac{6\coth \tilde{R}}{\tilde{R}^2} - \frac{6\coth^2 \tilde{R}}{\tilde{R}}\right].$$

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