LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 393-402

ON THE HYPERBOLICITY CONDITION IN LINEAR ELASTICITY

REMIGIO RUSSO (Napoli)

This talk, which is mainly expository and based on [2–5], discusses the *hyperbolicity condition* in linear elastodynamics. Particular emphasis is devoted to the key role it plays in the uniqueness questions associated with the *mixed boundary–initial value problem* in unbounded domains.

Notation – Light face letters indicate scalars; bold face lower-case letters, different from o and x, denote vectors (on $\mathbb{R}^n$, $n = 2, 3$), while o and x stand respectively for the origin of the reference frame $\{o; e_i\}$ $(i = 1, .., n)$ and the generic point of $\mathbb{R}^n$; bold face upper-case letters denote second-order tensors (linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$); *Lin* denotes the set of all second-order tensors and $Sym = \{\mathbf{A} : \mathbf{A} = \mathbf{A}^T\}$; $\nabla \mathbf{u}$ is the second-order tensor with components $(\nabla \mathbf{u})_{ij} = \mathbf{u}_{i,j}$ $({}_{,j} = \partial / \partial \mathbf{x}_j)$; div**S** is the vector with components $S_{ij,j}$.

1. The system of linear elastodynamics.

Let $\mathcal{B}$ an *linearly elastic body* we identify with the regular open connected set B of $\mathbb{R}^n$ $(n = 2, 3)$ it occupies in an assigned reference configuration. As is well-known, the regular motions of $\mathcal{B}$ are the *one parameter families* of regions of $\mathbb{R}^n$ $\{\mathbf{x} + \mathbf{u}(\mathbf{x}, t), \ (\mathbf{x}, t) \in Q = B \times (0, +\infty)\}$, where the *displacements* $\mathbf{u}$ are the classical solutions (1) to the system [1]

$$\rho\,\ddot{\mathbf{u}} = \operatorname{div}\mathbf{C}(\nabla\mathbf{u}) + \mathbf{b}, \tag{1}$$

where

ρ *(positive) mass density*;

b *body force per unit volume*;

$\mathbf{C}$ *elasticity tensor* : $\overline{B} \times Lin \to Sym$, regular on $\overline{B}$, linear on Lin and : $\mathbf{C}(\mathbf{W}) = \mathbf{0}$, $\forall$ skew $\mathbf{W}$.

Assume that B is *unbounded.* Moreover, assume that $\mathbf{C}$ is *symmetric, i.e.*,

$$\mathbf{L}\cdot\mathbf{C}(\mathbf{M}) = \mathbf{M}\cdot\mathbf{C}(\mathbf{L}), \quad \forall\,\mathbf{L},\mathbf{M},$$

and, setting $\pi(\mathbf{L}) = \mathbf{L}\cdot\mathbf{C}(\mathbf{L})$, satisfies at least one of the following *definiteness conditions*:

$$\begin{aligned} \textit{positive semi-definiteness} &\iff \pi(\mathbf{L}) \geq 0, \quad \forall\,\mathbf{L};\\ \textit{semi-strong ellipticity} &\iff \pi(\mathbf{L}) \geq 0, \quad \forall\,\mathbf{L} = \mathbf{a}\otimes\mathbf{b};\\ \textit{strong ellipticity} &\iff \pi(\mathbf{L}) > 0, \quad \forall\,\mathbf{L} = \mathbf{a}\otimes\mathbf{b} \neq \mathbf{0}. \end{aligned}$$

Let $\partial_1 B$ and $\partial_2 B$ be two disjoint subsets of ∂B such that $\overline{\partial_1 B}\cup\partial_2 B = \partial B$. Let us assign:

a) two smooth fields $\hat{\mathbf{u}}$ (*surface displacement*) and $\hat{\mathbf{s}}$ (*surface traction*) on $\overline{\partial_1 B}\times[0,+\infty)$ and $\partial_2 B\times[0,+\infty)$;

aa) two smooth fields $\mathbf{u}^*$ (*initial displacement*) and $\dot{\mathbf{u}}^*$ (*initial distribution of velocities*) on $\overline{B}$.

The *mixed boundary–initial value problem of elastodynamics* consists in finding a classical solution $\mathbf{u}$ to System (1) which satisfies the *boundary conditions*

$$\begin{aligned} \mathbf{u} &= \hat{\mathbf{u}} && \text{on } \overline{\partial_1 B}\times[0,+\infty)\\ \mathbf{C}(\nabla\mathbf{u})\mathbf{n} &= \hat{\mathbf{s}} && \text{on } \partial_2 B\times[0,+\infty) \end{aligned}$$

and the *initial conditions*

$$\mathbf{u} = \mathbf{u}^*, \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}^* \qquad \text{on } B\times\{0\},$$

(1) By a classical solution to System (1) we mean a vector field $\mathbf{u}$ on $\overline{Q}$ which satisfies pointwise (1) on Q and is twice continuously differentiable on $\overline{Q}$.

where $\mathbf{n}$ denotes the outward unit normal to ∂B.

When $\partial_2 B = \emptyset$ [resp. $\partial_1 B = \emptyset$] the above boundary-initial value problem is known as *displacement problem* [resp. *traction problem*] of linear elastodynamics.

We denote by $\mathfrak{C}$ the set of all classical solutions to System (1).

An interesting problem related to elastic solutions is concerned with the research of the best assumptions at infinity on the *material data* ρ and $\mathbf{C}$ assuring the boundedness of the support (at each instant) of an elastic solution which corresponds to data $\mathbf{b}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{s}}$, $\mathbf{u}^*$ and $\dot{\mathbf{u}}^*$ having compact supports. To this end we give the following definition.

DEFINITION 1. *System (1) is said to be hyperbolic in the class $\mathfrak{I}$, if any $\mathbf{u} \in \mathfrak{I} \cap \mathfrak{C}$ corresponding to data vanishing outside bounded regions, has a compact support at each instant.* □

2. The hyperbolicity condition. Uniqueness theorems.

Let $\mathbf{A}(\mathbf{m})$ ($\mathbf{m} : |\mathbf{m}| = 1$) be the *acoustic tensor for the direction* $\mathbf{m}$, defined by [1]

$$\mathbf{A}(\mathbf{m})\mathbf{a} = \rho^{-1}\mathbf{C}(\mathbf{a} \otimes \mathbf{m})\mathbf{a}, \quad \forall \mathbf{a}.$$

Let

$$\varphi^2(\mathbf{x}_o, \xi) = \max\{|\mathbf{A}(\mathbf{m})|,\ \mathbf{x} \in \overline{B} \cap \overline{S}_\xi(\mathbf{x}_o),\ \mathbf{m} : |\mathbf{m}| = 1\},$$
$$p(\mathbf{x}_o, r) = \int_{r_o}^{r} [\varphi(\mathbf{x}_o, \xi)]^{-1}\, d\xi;\ \varphi^2(\mathbf{o}, \xi) \equiv \varphi^2(\xi),\ p(\mathbf{o}, r) \equiv p(r),$$

where $r = |\mathbf{x} - \mathbf{o}|$, $\mathbf{x}_o \in \overline{B}$, r_o is a fixed (but arbitrarily chosen) constant and $S_\xi(\mathbf{x}_o)$ is the ball of radius ξ centered at $\mathbf{x}_o$.

It is evident that $p(\mathbf{x}_o, r)$ is a smooth, positive and nondecreasing function of r, so that

$$\lim_{r \to +\infty} p(\mathbf{x}_o, r) = l(\mathbf{x}_o) \in (0, +\infty].$$

Of course, the inverse function $p^{-1}(\mathbf{x}_o, r)$ exists only in $[0, l(\mathbf{x}_o))$.

It is worth observing that $\varphi(\mathbf{x}_o, \xi)$ represents the maximum speed of propagation in $B \cap S_\xi(\mathbf{x}_o)$ corresponding to $\mathbf{C}$ and ρ, and $p(\mathbf{x}_o, r)$ gives un upper bound for the time employed by a signal travelling with velocity $\varphi(\mathbf{x}_o, \xi)$ to reach $B \cap \partial S_r(\mathbf{x}_o)$ by starting from $B \cap \partial S_{r_o}(\mathbf{x}_o)$.

Taking into account the above observation, we realize that a necessary condition in order that System (1) is hyperbolic is

$$l(\mathbf{x}_o) = +\infty\,, \quad \forall\, \mathbf{x}_o \in B\,. \tag{2}$$

Since

$$p(\mathbf{x}_o, r) = \int_{r_o}^{r} [\varphi(\mathbf{x}_o, \xi)]^{-1}\,\mathrm{d}\xi \geq \int_{r_o}^{r} [\varphi(|\mathbf{x}_o - \mathrm{o}| + \xi)]^{-1}\,\mathrm{d}\xi$$
$$= p(r + |\mathbf{x} - \mathrm{o}|) \geq p(r)\,,$$

we see that

$$l(\mathbf{x}_o) = +\infty \;\; \forall\, \mathbf{x}_o \in B \Leftrightarrow l(\mathrm{o}) = +\infty\,,$$

so that (2) is equivalent to

$$l(\mathrm{o}) = +\infty\,. \tag{3}$$

Since, as we shall see, (3) is necessary and sufficient to the uniqueness of solutions to the boundary-initial value problems of elastodynamics and it is sufficient to guarantee, among other important properties, the hyperbolicity of System (1), we are led to give the following [2]

DEFINITION 2. *The acoustic tensor is said to satisfy the hyperbolicity condition iff (3) holds.* □

Observe that, if $\mathbf{A}$ satisfies the hyperbolicity condition, then $p(\mathbf{x}_o, r)$ is *one-to one*, $\forall \mathbf{x}_o \in B$. Moreover, $\mathbf{A}$ certainly satisfies the hyperbolicity condition if there exists a smooth, increasing and unbounded function q such that

$$[q'(r)]^2 |\mathbf{A}(\mathbf{m})| \leq \mathrm{C} \qquad \text{on } B, \tag{4}$$

for some positive constants C. If we choose $q = \log r$, then (4) reads

$$|\mathbf{A}(\mathbf{m})| \leq \mathrm{C} r^2 \qquad \text{on } B.$$

The following theorems are proved in [2, 3] and [4] respectively(2).

(2) For previous results about uniqueness and the hyperbolicity of System (1) (in the sense of Definition 1) cf. [1] and the references here quoted.

THEOREM 1. *Let* $\mathbf{C}$ *be positive semi-definite and let* $\mathbf{A}$ *satisfy the hyperbolicity condition. Then, System (1) is hyperbolic in* $\mathfrak{C}$ *and the mixed boundary–initial value problem of elastodynamics has at most one classical solution.* □

THEOREM 2. *Let* Σ *be a finite regular subset of* ∂B*, Let* $\mathbf{A}$ *satisfy the hyperbolicity condition and let* $\mathbf{C}$ *be either strongly elliptic or constant and semi–strongly elliptic. Then, System (1) is hyperbolic in the class*

$$\{\mathbf{u} \in \mathfrak{C} : \mathbf{u} = \mathbf{0} \text{ on } (\partial B \backslash \Sigma) \times [0, +\infty)\}.$$

Moreover, if $\partial_2 B$ *is bounded, then the mixed boundary–initial value problem of elastodynamics has at most one classical solution.* □

3. Counter-examples to uniqueness

We aim at showing now that, if the acoustic tensor does not satisfy the hyperbolicity condition, then the solutions to the boundary–initial value problems of linear elastodynamics in an unbounded domain are not uniquely determined by the boundary and initial data and the body forces. Of course, since System (1) is linear, in order to show nonuniqueness, it is sufficient to prove existence of nontrivial solutions to the associated homogeneous system.

Let p be a smooth and increasing function on $\mathbb{R}$ such that $\lim_{x \to +\infty} p = l$. Let $B = [1, +\infty)$ and let

$$\rho = p'(x), \quad \mathbf{C}(x) = [p'(x)]^{-1}, \quad \forall x \in [0, +\infty), \tag{5}$$

with $p(1) = 0$.

It is a simple matter to verify that in such a case the acoustic tensor (which is now the scalar $\rho^{-1}\mathbf{C}$) does not satisfy the hyperbolicity condition.

Set

$$\mathcal{I}_1 = \{(x,t) : x \in [1,+\infty),\ 0 \le t \le p(x)\};$$

$$\mathcal{I}_2 = \{(x,t) : x \in [1,+\infty),\ t \ge p(x)\};$$

$$\mathcal{I}_3 = \{(x,t) : x \in [1,+\infty),\ 0 \le t \le l - p(x)\};$$

$$\mathcal{I}_4 = \{(x,t) : x \in [1,+\infty),\ t \ge l - p(x)\};$$

$$\mathcal{I}_5 = \{(x,t) : x \in \mathbb{R},\ 0 \le t \le p(x) + l\};$$

$$\mathcal{I}_6 = \{(x,t) : x \in \mathbb{R},\ t \ge p(x) + l\}.$$

a) Counter-examples to the displacement problem.

Given the above data, the homogeneous boundary-initial value problem of linear elastodynamics takes the form

$$\text{(6)}\qquad \begin{aligned} &p'\,\ddot{u} = \partial_x[(p')^{-1}\partial_x u] && \text{on } [1,+\infty)\times[0,+\infty)\,,\\ &u = 0\,, \quad \dot{u} = 0 && \text{on } [1,+\infty)\times\{0\}\,,\\ &u = 0 && \text{on } \{1\}\times[0,+\infty)\,. \end{aligned}$$

Since equation $(6)_1$ can be written as

$$(p'\partial_t + \partial_x)(\partial_t - (p')^{-1}\partial_x)u = 0\,,$$

its general solution is expressed by the *D'Alembert integral*

$$u(x,t) = u_1(t - p(x)) + u_2(t + p(x))\,.$$

We are able to show that System (6) admits infinitely many non trivial solutions depending on the *additional condition*

$$\text{(7)}\qquad \lim_{x\to+\infty} u(x,t) = u_\infty(t)\,,$$

provided the *additional datum* $u_\infty(t)$ is suitably chosen. To this aim, consider a regular function $\tilde{u}_\infty$ on $[0,+\infty)$ such that

$$\begin{aligned} \tilde{u}_\infty(0) &= \tilde{u}_\infty(l) = \tilde{u}_\infty(2l) = \tilde{u}'(l)\\ &= \tilde{u}'(2l) = \tilde{u}''(l) = \tilde{u}''(2l) = 0 \end{aligned}$$

and let

$$u_\infty(t) = \tilde{u}_\infty(t) - \tilde{u}_\infty(t+2l), \quad \forall t \ge 0\,.$$

In order to show that System (6)–(7) admits nonzero solutions, we first look for a solution $\tilde{u}(x,t) = u_1(t-p(x))$ to System $(6)_{1,2}$ such that.

$$\lim_{x\to+\infty} \tilde{u}(x,t) = \tilde{u}_\infty(t)$$

and easily get

$$\tilde{u}(x,t) = \begin{cases} 0 & \text{on } \mathcal{I}_1, \\ \tilde{u}_\infty(t-p(x)+l) & \text{on } \mathcal{I}_2. \end{cases} \tag{8}$$

Then, setting

$$\hat{u}_\infty(t) = -\tilde{u}_\infty(t+2l)\,,$$

we look for a solution $\hat{u}(x,t) = u_2(t+p(x))$ to System $(6)_{1,2}$ such that

$$\lim_{x\to+\infty} \hat{u}(x,t) = \hat{u}_\infty(t)$$

and find

$$\hat{u}(x,t) = \begin{cases} 0 & \text{on } \mathcal{I}_3, \\ \hat{u}_\infty(t+p(x)-l) & \text{on } \mathcal{I}_4. \end{cases} \tag{9}$$

It is obvious that the function

$$u(x,t) = \tilde{u}(x,t) + \hat{u}(x,t) \tag{10}$$

satisfies $(6)_{1,2}$–(7). Finally, since

$$u(1,t) = \tilde{u}(1,t) + \hat{u}(1,t) = \tilde{u}_\infty(t+l) + \hat{u}_\infty(t-l) = 0, \ \ \forall t \geq 0\,,$$

(10) satisfies $(6)_3$ too. Thus, it results a nonzero solution to System (6). □

b) Counter-examples to the traction problem

Let

$$u_\infty(t) = \tilde{u}_\infty(t) + \tilde{u}_\infty(t+2l)\,.$$

Then, the function

$$u_\infty(x,t) = \tilde{u}(x,t) - \hat{u}(x,t)\,,$$

with $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ given by (8) and (9) respectively, satisfies $(6)_{1,2} - (7)$. Moreover, since

$$(\mathbf{C}\partial_x u)(1,t) = -\tilde{u}'_\infty(t+l) - \hat{u}'_\infty(t-l) = 0\,, \ \ \forall t \geq 0\,,$$

the corresponding tractions vanish on the boundary. Then, it furnishes a nontrivial solution to the traction problem of elastodynamics corresponding to vanishing data.

It is not difficult, by reproducing the above technique, to obtain some counter-examples to the mixed problem in the case where $B = (-\infty, -1] \cup [1, +\infty)$. □

c) Counter-examples to the Cauchy problem

As far as the Cauchy problem is concerned, let $B = \mathbb{R}$ and let ρ and $\mathbf{C}$ be given by (5), where p is an *odd* function. Then, System (6) becomes

$$
\begin{aligned}
&p'\,\ddot{u} = \partial_x[(p')^{-1}\partial_x u] && \text{on } \mathbb{R}\times[0,+\infty)\,,\\
&u = 0\,, \quad \dot{u} = 0 && \text{on } \mathbb{R}\times\{0\}\,.
\end{aligned} \tag{11}
$$

By using the previous arguments, we show that (11) admits nonzero solutions. Indeed, let $u_\infty(t)$ be a regular function on $\mathbb{R}$, periodic with period $2l$ and such that

$$u_\infty(0) = u'_\infty(0) = u''(0) = 0\,.$$

Then, it is readily seen that the function

$$u(x,t) = \begin{cases} 0 & \text{on } \mathcal{I}_5,\\ u_\infty(t - p(x) - l) & \text{on } \mathcal{I}_6, \end{cases}$$

is a nonzero solution to System (11) which satisfies (7). □

From the above results we derive now the desired counter-examples in the n-dimensional case.

A) Counter-examples in exterior domains

Let $B = S_1^c$ and assume that $\mathcal{B}$ is isotropic with Lamé moduli $\lambda = \lambda(r)$ and $\mu = \mu(r)$ such that $\lambda + \mu = 0$. In the case $n = 3$ [resp. $n = 2$] this assumption assures that the elasticity tensor is strongly elliptic [resp. positive definite].

For the sake of simplicity, we look for solutions of the type $\mathbf{u}(\mathbf{x}, t) = \mathrm{u}(r, t)\mathrm{e}_1$. Then, the system of linear elastodynamics with vanishing body force is reduced to the equation

$$\rho\ddot{u} = \partial_i(\mu\partial_i u)\,, \tag{12}$$

A strightforward calculation shows that (12) is equivalent to

$$r^{n-1}\rho\,\ddot{u} = \partial_r(r^{n-1}\mu\partial_r u)\,. \tag{13}$$

If we assume that

$$r^{n-1}\rho = p'(r)\,, \quad r^{n-1}\mu(r) = [p'(r)]^{-1}\,,$$

we see that the acoustic tensor $\mu\rho^{-1}$ does not satisfy the hyperbolicity condition and (13) becomes

$$p'(r)\,\ddot{u} = \partial_r[(p')^{-1}\partial_r u]\,. \tag{14}$$

Since (14) is formally equivalent to $(6)_1$, nontrivial solutions to the displacement, traction and Cauchy problems are found by replacing x by r in all the relations concerning the one-dimensional counter-examples a), b) and c). □

B) Counter-examples in domains with unbounded boundaries

Let B be the exterior of the infinite cylindrical region having axis x_3 and radius 1, and let $\mathcal{B}$ be isotropic with Lamé moduli λ and μ such that $\lambda = 0$, so that the elasticity tensor is positive definite.

Let (r, θ, x_3) be a cylindrical coordinate system. An elastic solution of the tipe $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(r,t)\mathbf{e}_3$ corresponding to zero body force satisfies the equation

$$r\rho\,\ddot{u} = \partial_r(r\mu\partial_r u)\,. \tag{15}$$

If we assume that λ and μ are given by

$$r\rho = p'(r)\,, \quad r\mu(r) = [p'(r)]^{-1}\,,$$

we see that the acoustic tensor does not satisfy the hyperbolicity condition and (15) becomes

$$p'\ddot{u} = \partial_r[(p')^{-1}\partial_r u]\,,$$

so that the conclusion of b) holds in this case too. □

The counter–examples given in *a)* and *b)* have been established in [2, 5].

The results of this section show that, when the hyperbolicity condition is violated, then the system of linear elastodynamics may admit solutions corresponding to zero data that are different from zero on the whole of B in the time interval $[l(o), +\infty)$. Hence it follows that System (1) is not hyperbolic in the whole class of elastic solutions.

REFERENCES

[1] Gurtin M.E, *The linear theory of elasticity*, in Handbuch der Physik (ed. C. Truesdell) vol. VI/a, Springer–Verlag, 1972.

[2] Carbonaro B., Russo R., Singularity problems in linear elasticity, *Ann. Scuola Norm. Sup. Pisa* (VI) **18**, 1991.

[3] Knops R.J., Rionero S., Russo R., *Stability and related results in linear elastodynamics on unbounded domains*, in press on Arch. Rational Mech. Anal.

[4] Maremonti P., Russo R., *On the wave propagation property in finite elasticity,* in press on Proc. Roy. Soc. Edinburgh.

[5] Carbonaro B., Russo R., *Sharp uniqueness classes in linear elastodynamics*, in press on J. Elasticity.

Dipartimento di Matematica e Applicazioni
Università "Federico II"
via Mezzocannone, 8 – Napoli, ITALY