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A TRANSPORT EQUATION FOR THE EVOLUTION OF SHOCK AMPLITUDES ALONG RAYS

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A new asymptotic method is derived for the study of the evolution of weak shocs in several dimension. The method is based on the *Generalized Wavefront Expansion* derived in [1]. In that paper the propagation of a shock into a known background was studied under the assumption that shock is weak, i.e. Mach Number $= 1 + O(\epsilon)$, $\epsilon \ll 1$, and that the perturbation of the field varies over a length scale $O(\epsilon)$. To the lowest order, the shock surface evolves along the rays associated with the unperturbed state.

An infinite system of compatibility relations was derived for the jump in the field and its normal derivatives along the shock, but no valid criterion was found for a truncation of the system.

Here we show that the infinite hierarchy is equivalent to a single equation that describes the evolution of the shock along the rays. We show that this method gives equivalent results to those obtained by Weakly Nonlinear Geometrical Optics [2].

The infinite hierarchy.

Let us consider a hyperbolic quasi-linear system of conservation

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laws:

$$\partial_t \mathbf{f}^0(\mathbf{U}) + \partial_i \mathbf{f}^i(\mathbf{U}) = \mathbf{g} \tag{1}$$

where $\mathbf{U} \in \mathbb{R}^N$ is a vector field and $\mathbf{f}^\alpha(\mathbf{U}) \in \mathbb{R}^N$, $\alpha = 0, \dots, 3$ are smooth vector functions of $\mathbf{U}$ in an open domain $\Omega \subset \mathbb{R}^N$, and $\mathbf{g} = \mathbf{g}(\mathbf{U}, x^\alpha)$ is smooth on $\Omega \times \mathbb{R}^4$. Summation over repeated indices from 1 to 3 is assumed. We define the $N \times N$

$$\mathcal{A}^\alpha(\mathbf{U}) = \nabla_{\mathbf{U}} \mathbf{f}^\alpha(\mathbf{U}),$$

for $\alpha = 0, \dots, 3$. We suppose that $\mathbf{U}^{(0)}$ is a known smooth unperturbed solution of system (1) and consider the propagation of a weak shock into the unperturbed state $\mathbf{U}^{(0)}$. Let $\Sigma(t)$ denote a moving surface of discontinuity for system (1), V_Σ the normal speed of propagation of Σ and (n_1, n_2, n_3) its unit normal. The jump conditions across Σ are [3]

$$-V_\Sigma [\![\mathbf{f}^0]\!] + n_i [\![\mathbf{f}^i]\!] = 0$$

where, for any quantity $h(\mathbf{U})$ we denote $[\![h]\!] = h(\mathbf{U}_-) - h(\mathbf{U}_+)$, with $\mathbf{U}_-$ and $\mathbf{U}_+$ the states just ahead and behind $\Sigma(t)$.

We denote by $\mathbf{L}$ and $\mathbf{R}$ respectively the left and right null vectors of the matrix

$$\mathcal{A}^i n_i - \lambda^{(k)} \mathcal{A}^0$$

corresponding to the eigenvalue $\lambda^{(k)}(n_i)$ (which we assume to be simple), computed in the unperturbed state.

Let us define the normal jump of the space derivatives of the field:

$$\mathbf{Y}^0 = [\![\mathbf{U}]\!], \mathbf{Y}^1 = [\![n^i \partial_i \mathbf{U}]\!], \dots, \mathbf{Y}^k = [\![n^{i_1} \dots, n^{i_k} \partial^k_{i_1 \dots i_k} \mathbf{U}]\!], \dots$$

The basic asymptotic assumption on the field is [1]

$$\mathbf{Y}^0 = O(\epsilon), \mathbf{Y}^1 = O(1), \dots, \mathbf{Y}^k = O(\epsilon^{-k+1}), \dots$$

We make use of the formal expansion

$$\mathbf{Y}^0 = \epsilon \mathbf{Y}_1^0 + \epsilon^2 \mathbf{Y}_2^0 + \dots$$

$$\mathbf{Y}^1 = \mathbf{Y}_0^1 + \epsilon \mathbf{Y}_0^1 + \dots$$

$$\dots$$

$$\mathbf{Y}^k = \epsilon^{-k+1} \mathbf{Y}_0^k + \epsilon^{-k+2} \mathbf{Y}_1^k + \dots$$

$$\dots$$

and apply it to the kinematic and geometric compatibility conditions across the shock front Σ to system (1) and to its space derivatives. To the lowest order in ϵ one obtains the following infinite system of compatibility relations for the jump in the field and its normal derivatives [1]:

$$\text{(2)} \qquad \frac{d\pi^k}{dt} + \alpha \sum_{p=1}^{k} \binom{k}{p} \pi^p \pi^{k-p+1} + (\beta + k\gamma)\pi^k + \frac{\alpha}{2}\pi^0 \pi^{k+1} = 0,$$

where $k \geq 0$,

$$\pi_0 = \mathbf{L}\mathbf{Y}^0, \dots, \pi_k = \mathbf{L}\mathbf{Y}^k, \dots$$

and α, β, γ are given by:

$$\text{(3)} \qquad \alpha \equiv \mathbf{L} \nabla_{\mathbf{U}} \mathcal{A}^i n_i \mathbf{R}\mathbf{R}$$

$$\text{(4)} \qquad \beta \equiv \mathbf{L}(\partial_t + \lambda n^i \partial_i)\mathbf{R} + \mathbf{L}\mathcal{A}^i \nabla_{\mathbf{U}} \mathbf{R} \tilde{\partial}_i \mathbf{U}^{(0)} + \mathbf{L}\mathcal{A}^i \frac{\partial \mathbf{R}}{\partial n^j} \chi_{ij} +$$

$$\mathbf{L} \nabla_{\mathbf{U}} \mathcal{A}^i \mathbf{R} \partial_i \mathbf{U}^{(0)} - \mathbf{L} \nabla_{\mathbf{U}} \mathbf{g} \mathbf{R}$$

$$\text{(5)} \qquad \gamma \equiv \mathbf{L} \nabla_{\mathbf{U}} \mathcal{A}^i n^j \partial_j \mathbf{U}^{(0)} n_i \mathbf{R}$$

Here $\tilde{\partial}_i = (\delta_{ij} - n_i n_j)\partial_i$ is the tangential derivative on Σ, $\chi_{ij} \equiv \tilde{\partial}_i n_j$ is the *second fundamental form* of Σ, and d/dt represents the derivative along the rays [1]. These transport coefficients are evaluated in the unperturbed state just ahead of the shock and depend on the field $\mathbf{U}^{(0)}$ and on the geometry of the surface. System (2) is supplemented with equations that describe the evolution of the geometry, i.e. with transport equations for the position x^i on the ray and χ_{ij} [4].

The generating function.

The infinite system (2) can be interpreted as a power series expansion of a single partial differential equation for a *generating function*. Let

$$\pi(\theta,t)=\sum_{k=0}^{\infty}\pi^k(t)\frac{\theta^k}{k!}.$$

Then the Cauchy product of the series corresponding to π and $\partial\pi/\partial\theta$ is:

(6) $$\pi\frac{\partial\pi}{\partial\theta}=\sum_{k=0}^{\infty}\left(\sum_{p=0}^{k}\binom{k}{p}\pi^p\pi^{k-p+1}\right)\frac{\theta^k}{k}$$

Let us muliply the k-th equation of (2) by $\theta^k/k!$, sum over k and make use of (6). We obtain the following equation for $\pi(\theta,t)$:

(7) $$\frac{\partial\pi}{\partial t}+\left[\alpha\left(\pi-\frac{1}{2}\pi^0\right)+\gamma\sigma\right]\frac{\partial\pi}{\partial\theta}+\beta\pi=0$$

where $\pi=\pi(\theta,t)$ is the unknown function, $\pi^0(t)=\pi(0,t)$, α, β and γ are functions of t that can be evaluated from the known background. The function $\pi^0(t)$ is most important because it is related to the shock amplitude, which is the principal quantity of interest. This equation must be supplemented with transport equations for the rays and the curvature of the shock front. At the lowest order in the perturbation expansion these equations are not coupled to the shock. Therefore we can solve these equations to compute α,β and γ as functions of time.

We shall consider the initial value problem for Eq. (7), with

$$\pi(\xi,0)=\begin{cases}0 & \xi>0\\ f(\xi) & \xi<0\end{cases}$$

The quantity $f(\xi)$ could be defined via the power series:

$$f(\xi)=\sum_{k=0}^{\infty}\pi^k(0)\frac{\xi^k}{k!}.$$

or, better, as

(8) $$f(\xi)=\mathbf{L}(\xi)(\mathbf{U}(\xi,0)-\mathbf{U}^{(0)}(\xi))$$

where ξ is the arc length along the rays and we interpret the left eigenvector and the initial value of the field as functions of ξ.

Equation (7) can be written in characteristic form,

$$\frac{d\pi}{dt} + \beta(t)\pi = 0 \tag{9}$$

$$\frac{d\theta}{dt} = \alpha(t)\left[\pi(\theta,t) - \frac{1}{2}\pi(0,t)\right] + \gamma(t)\theta. \tag{10}$$

Integrating the first equation gives

$$\pi = f(\xi)\exp\left(-\int_0^t \beta(\tau)d\tau\right), \tag{11}$$

and substituting in the second yields

$$\frac{d\theta}{dt} = \alpha(t)\left[f(\xi)\exp\left(-\int_0^t \beta(\tau)d\tau\right) - \frac{1}{2}\pi^0(t)\right] + \gamma(t)\theta. \tag{12}$$

Integrating this equation gives

$$\theta(\xi,t) = \left\{\xi + \int_0^t \left[f(\xi)\alpha(\tau)E(\tau) - \frac{1}{2}\pi^0(\tau)\alpha(\tau)\right]\rho(\tau)d\tau\right\}/\rho(t), \tag{13}$$

where

$$\rho(t) \equiv \exp\left(-\int_0^t \gamma(\tau)d\tau\right), \quad E(t) \equiv \exp\left(-\int_0^t \beta(\tau)d\tau\right). \tag{14}$$

By setting $\theta(\xi,t) = 0$ and using (11) evaluated at $\xi = \xi(t)$ to eliminate π^0, we obtain the parameter $\xi(t)$ of the characteristic which terminates on the shock at time t. The relation has the form of an integral equation for $\xi(t)$:

$$\xi(t) + f(\xi(t))\int_0^t F(\tau)d\tau - \frac{1}{2}\int_0^t f(\xi(\tau))F(\tau)d\tau = 0 \tag{15}$$

where

$$F(t) \equiv \alpha(t)E(t)\rho(t). \tag{16}$$

Differentiating equation (15) with respect to time we obtain an ordinary differential equation for $\xi(t)$:

(17)
$$\frac{d\xi}{dt} = -\frac{1}{2}\frac{f(\xi)F(t)}{1 + f'(\xi)\int_0^t F(\tau)d\tau}.$$

Solution of the equation.

With the change of variable

$$y = \int_0^t F(\tau)d\tau$$

Eq. (17) can be written in the form

$$\frac{dy}{d\xi} = -\frac{2}{f(\xi)} - 2\frac{f'(\xi)}{f(\xi)}y$$

Solving the linear equation one obtains an integral relation between $F(t)$ and $f(\xi)$:

(18)
$$f(\xi)^2\int_0^t F(\tau)d\tau + 2\int_0^\xi f(\xi)d\xi = 0$$

This relation, together with the relation (11) gives the time evolution of the shock.

When applied to the propagation of weak shocks in gas dynamics in a stratified atmosphere, one finds a good agreement with numerical results.

Propagation into a constant state.

In this case α is constant and $\beta = \frac{1}{2}(d/dt)\log J$, where J is the expansion of the rays. In the case of $1-D$ propagation in plane, cylindrical and spherical geometry, Eq. (18) can be solved for t:

$$\text{(19)} \qquad \begin{aligned} &1D \qquad t = -\frac{2\int_0^{\xi} f(\xi)d\xi}{\alpha f^2(\xi)} \\ &2D \qquad t = t_0 \left(1 - \frac{\int_{\xi_0}^{\xi} f(\xi)d\xi}{\alpha t_0 f^2(\xi)} \right)^2 \\ &3D \qquad t = t_0 \exp \left(\frac{f^2(\xi_0)}{f(\xi)} - \frac{2\int_{\xi_0}^{\xi} f(\xi)d\xi}{\alpha t_0} \right) \end{aligned}$$

In the case of triangular initial profile these relations can be solved for $\xi(t)$.

The resulting expression give the well-known shock decay for the three geometries [5].

Comparison with Weakly Nonlinear Geometrical Optics.

The theory of *weakly nonlinear geoemtrical optics* (WNGO), introduced by Choquet-Bruhat [6] to study the evolution of a high frequency wave in quasilinear systems without shocks, has been extended in [2] to the case of several waves in systems of conservation laws in presence of shocks.

It is assumed that the vector field is a superposition of an unperturbed field and m small-amplitude, high-frequency waves. Here we shall consider the case of a single high-frequency wave (the one corresponding to the eigenvalue $\lambda^{(k)}$). One looks for solutions of system (1) of the form

$$\text{(20)} \qquad \mathbf{U}(x^\alpha, \epsilon) = \mathbf{U}^{(0)}(x^\alpha) + \epsilon \mathbf{V}(x^\alpha, \sigma) + O(\epsilon^2)$$

where $\sigma = \phi/\epsilon$, and $\phi(x^\alpha)$ is a phase variable to be determined. Performing a detailed asymptotic analisys, one obtains an expression for the asymptotic solution (20) in terms of the phase ϕ and known functions of the unperturbed field [2]. The asymptotic solution takes the form

$$\mathbf{U}(x^\alpha,\epsilon) = \mathbf{U}^{(0)}(x^\alpha) + \epsilon a(x^\alpha, \phi(x^\alpha)/\epsilon)\mathbf{R}(x^\alpha) + O(\epsilon^2)$$

where the amplitude satisfies a scalar equation

$$\frac{\partial a}{\partial t} + \alpha|\nabla\phi| a \frac{\partial a}{\partial \sigma} + \beta a = 0. \tag{21}$$

Here $\partial/\partial t$ is the derivative with respect to time along the rays. In this framework, shocks can be treated by applying *shock-fitting* to Eq. (21). We consider (21) with intial data $a(\sigma,0) = f(\sigma)$, with $f(\sigma) = 0$ for $\sigma > 0$. Let $z(t)$ be the initial value of σ on the characteristic hitting the shock from behind at time t. Shock-fitting [2, 5] shows that z and the shock strength $a(z(t),t)$ are given by,

$$[\![\mathbf{U}]\!] = a\mathbf{R}, \quad a = f(z)E(t), \quad 2\int_0^z f(\zeta)d\zeta + f^2(z)I(t) = 0 \tag{22}$$

with

$$E(t) \equiv \exp\left\{-\int_0^t \beta(\tau)d\tau\right\}, \quad I(t) \equiv \int_0^t \alpha|\nabla\phi|E d\tau, \tag{23}$$

Here $f(\zeta)$ is given by (8), β is defined in Eq. (4), and $\mathbf{R}$ is the same as the one used in the previous sections. The phase $\phi(x^i,t)$ satisfies the eikonal equation

$$\phi_t + \lambda|\nabla\phi| = 0,$$

and the intial value along the phase is the arc length of the ray: $\phi(0,x_i) = \xi$.

We shall prove that in this case GWE and WNGO give the same evolution law for the shock amplitude. Comparing Eq. (22) with Eq. (18), and using (14) and (16), it follows that the two methods give the same evolution law for the shock amplitude, provided that

$$I(t) = \int_0^t \alpha(\tau)E(\tau)\exp\left[-\int_0^\tau \gamma(t')dt'\right]d\tau, \tag{24}$$

where all the terms appearing in the integral are evaluated along the rays.

From (23), Eq. (24) is satisfied if

$$|\nabla\phi| = \exp\left[-\int_0^T \gamma(t')dt'\right]. \tag{25}$$

In order to prove Eq. (25) we obtain a differential equation for $|\nabla\phi|$. The derivative of $|\nabla\phi|$ along the ray is

$$\frac{d|\nabla\phi|}{dt} = |\nabla\phi|\frac{\partial\phi}{\partial x_i}\frac{d}{dt}\frac{\partial\phi}{\partial x_i}, \tag{26}$$

where the derivative along the ray, d/dt, is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Lambda^i\frac{\partial}{\partial x^i}.$$

Here Λ^i is the ray velocity [1, 2] defined by

$$\Lambda^i = \mathbf{L}\mathcal{A}^i\mathbf{R} = \frac{\partial\lambda}{\partial n_i}.$$

We assume, without loss of generality, that $\mathcal{A}^0 = I$ and $\mathbf{LR} = 1$. Using this expression for d/dt in (26) gives

$$\begin{aligned}\frac{d}{dt}\frac{\partial\phi}{\partial x_i} &= \left(\frac{\partial}{\partial t} + \Lambda^j\frac{\partial}{\partial x^j}\right)\frac{\partial\phi}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial t} + \Lambda^j\phi_{ij}\\ &= \frac{\partial}{\partial x_i}(-\lambda|\nabla\phi|) + \Lambda^j\phi_{ij}\\ &= -\frac{\partial\lambda}{\partial x_i}|\nabla\phi| + \phi_{ij}(\Lambda^j - \lambda n^j)\end{aligned}$$

where the eikonal equation has been used and $\phi_{ij} \equiv (\partial^2\phi/\partial x_i\partial x_j)$. Substituting this equation into (26) gives

$$\frac{d|\nabla\phi|}{dt} = -\frac{\partial\lambda}{\partial x_i}\frac{\partial\phi}{\partial x^i} + n^j\phi_{ij}(\Lambda^i - \lambda n^i) \tag{27}$$

From the definition of n_i, we have

$$\phi_{ij} = \frac{\partial}{\partial x^j}(|\nabla\phi| n_i) = n_i n^k \phi_{kj} + |\nabla\phi|\chi_{ij}.$$

Using this equation in (27) gives

$$n^j \phi_{ij}(\Lambda^i - \lambda n^i) = |\nabla\phi| n^j \chi_{ij}(\Lambda^i - \lambda n^i) + n^k \phi_{jk} n^j (\Lambda^i n_i - \lambda) = 0,$$

because $\chi_{ij} n^j = 0$ and, from the eigenvalue equation for λ, $\Lambda^i n_i - \lambda = 0$. It now follows from (27) that

$$\frac{d|\nabla\phi|}{dt} = -\frac{\partial\phi}{\partial x_i}\frac{\partial\lambda}{\partial x^i} = -|\nabla\phi| n_j \nabla_{\mathbf{U}}\lambda \frac{\partial \mathbf{U}^{(0)}}{\partial x_j}.$$

To prove (25), we observe that γ can be written as

$$\gamma = n_i \nabla_{\mathbf{U}}\lambda \frac{\partial \mathbf{U}^{(0)}}{\partial x_i}. \tag{28}$$

In fact, differentiating the eigenvalue equation for λ, and left multiplying it by $\mathbf{L}$, it follows that

$$\mathbf{L}\nabla_{\mathbf{U}}\mathcal{A}^i n_i \mathbf{R} = \nabla_{\mathbf{U}}\lambda.$$

Substituting this expression in Eq. (5), Eq. (28) follows.

$$\frac{d|\nabla\phi|}{dt} = -|\nabla\phi|\gamma.$$

Integrating this ODE gives (25), which proves the result.

REFERENCES

[1] Anile A.M., Russo G., *Generalized Wavefront Expansion I. Higher Order Corrections for the Propagation of Weak Shock Waves,* Wave Motion **8,** (1986) 243-258.

[2] Hunter J.K., Keller J.B., *Weakly Nonlinear High Frequency Waves,* Comm. Pure Appl. Math., **XXXVI**, (1983) 547-569.

[3] Jeffrey A., *Quasilinear Hyperbolic Systems and Waves,* Pitman. London (1976).

[4] Russo G., *On the evolution of ordinary discontinuities and characteristic shocks,* Le Matematiche, University of Catania, Italy, **XLI**, (1986) 123-141.

[5] Whitham G.B., *Linear and Nonlinear Waves,* Wiley, New York (1974).

[6] Choquet-Bruhat Y., *Ondes asymptotique et approchées pour systèmes d'équations aux dériv'ees partielles nonlinéaires,* J. Math. Pures et Appl., **48**, (1969) 117-158.

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