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ON THE CONDITIONAL TOTAL STABILITY OF EQUILIBRIUM FOR MECHANICAL SYSTEMS

L. SALVADORI (Trento) (*)

In connection with the problem of observability, properties of total stability restricted to classes of perturbations of the governing equations are discussed for the equilibrium of holonomic mechanical systems. These systems are subject to positional conservative and dissipative forces. The particular case of a null dissipation is included. The perturbations to which the total stability is restricted are those obtained by modifying the kinetic energy, the potential of the conservative force, and the dissipative terms, without altering the Lagrangian form of the equations of motion.

1. Introduction.

The aim of this talk is to revisit some results concerning the total stability of equilibrium for holonomic mechanical systems with time independent constraints, and subject to positional conservative and dissipative forces (including the case of a null dissipation). These results have been given in a paper in cooperation with F. Visentin [7]. In contrast with the assumptions in [7], we suppose here that

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the dissipative forces are time independent, in order to restrict our analysis to autonomous Lagrangian equations and avoid inessential complications.

It is known that no periodic orbits, and then no equilibrium positions, of a positional conservative system are (uniformly) totally stable in the sense of the classic definition of Dubosin [2]. Therefore serious problems of observability arise when the conservative schema is adopted ([1], [5]), and an explanation of the observability of equilibrium is obtained by a modification of this schema. Indeed, under the assumptions in the Lagrange-Dirichlet theorem, and some regularity conditions, an isolated equilibrium position becomes (uniformly) asymptotically stable, and then totally stable [4], when small strictly dissipative forces which have been neglected in a first approximation, are taken into account.

An alternative explanation of observability may be obtained by requiring stable behaviors restricted to classes of perturbations of the equations of motion that are of prevalent importance in the problem under examination. For instance in Celestial Mechanics concepts of total stability with respect to conservative perturbations, that is perturbations which lead from a conservative system to another conservative system, are very significant.

The interest to analyze the occurrence of properties of conditional total stability, that is of total stability with respect to perturbations which satisfy appropriate conditions, then arises in the most natural way. We notice that when the perturbations of the governing equations are zero along a motion, the conditional total stability of this motion does not always coincide with its secular stability, that is with the property that the motion is stable and the stability is preserved under the above mentioned perturbations. Here, as in [7], the perturbations to which the total stability of equilibrium will be restricted, are those obtained by modifying the coefficients of the kinetic energy, the potential of the conservative force, and the dissipative terms, without altering the Lagrangian form of the equations of motion.

2 Liapunov stability and conditional total stability for ordinary differential equations.

Let $D \subseteq \mathbf{R}^s$, $s \geq 1$, be an open set which contains the origin 0 of $\mathbf{R}^s$. Consider the differential equation

$$(2.1) \qquad \dot{x} = f(x),$$

where $f \in C(D, \mathbf{R}^s)$, $f(0) = 0$. Uniqueness is not necessary and is not assumed. Let $F(x_0)$ be the set of all noncontinuable solutions $x = x(t, x_0)$ of (2.1) which satisfy the initial condition $(0, x_0)$. For $x \in F(x_0)$ let $J(x)$ be the interval of existence of x and $J^+(x) = \{t \in J(x), t \geq 0\}$. We denote by $||\cdot||$ the Euclidean norm in $\mathbf{R}^s$, by $B(\gamma)$, $\gamma > 0$, the ball $\{x \in \mathbf{R}^s : ||x|| < \gamma\}$, and by $B[\gamma]$ the closure of $B(\gamma)$. Let $\chi = \sup\{\gamma > 0 : B(\gamma) \subseteq D\}$. We recall some stability concepts in the Liapunov sense. The solution $x \equiv 0$ of (2.1) is said to be:

(i) stable if for any $\varepsilon \in (0, \chi)$ there exists $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that if $||x_0|| < \delta$ and $x \in F(x_0)$, then $||x(t)|| < \varepsilon$ for $t \geq 0$;

(ii) asymptotically stable if it is stable and there exist $\gamma \in (0, \chi)$, $\sigma \in (0, \delta(\gamma))$, such that if $||x_0|| < \sigma$, and $x \in F(x_0)$, then $x(t) \to 0$ for $t \to +\infty$ uniformly in x_0;

(iii) unstable if it is not stable, i.e. there exist $\eta > 0$, a sequence $\{x_i\}$ in $D, ||x_i|| \to 0$, such that for any $i \in \mathbf{N}$ one has $||x(t)|| \geq \eta$, for some $x \in F(x_i)$, and some $t \in J^+(x)$.

If $x \equiv 0$ is stable, then given any $\varepsilon > 0$ the supremum $r(\varepsilon)$ of the numbers $\delta(\varepsilon)$ in (i) will be called ε-radius of stability. If $x \equiv 0$ is unstable, we set

$R = \sup\{\eta > 0 : \exists \{x_i\} \subset D,\ ||x_i|| \to 0$, such that for any $i \in \mathbf{N}$ there exist $x \in F(x_i)$ and $t \in J^+(x)$ with $||x(t)|| \geq \eta\}$.

The number R (finite or infinite) will be called radius of instability.

Let $\mathcal{B}$ be a Banach space and let $|.|_{\mathcal{B}}$ be a norm in $\mathcal{B}$. Let $\Lambda \subseteq \mathcal{B}$ be a domain which contains the origin ω of $\mathcal{B}$ and such that ω is an accumulation point for Λ. Let $g \in C(D \times \Lambda, \mathbf{R}^s)$ be such that

$$(K) \qquad a(|\lambda|_{\mathcal{B}}) \leq ||g(x, \lambda) - f(x)|| \leq b(|\lambda|_{\mathcal{B}})$$

for any $(x, \lambda) \in D \times \Lambda$ and some pair of continuous functions

$a, b : \mathbf{R}^+ \to \mathbf{R}^+$, strictly increasing with $a(0) = b(0) = 0$. Let $\mathcal{U}$ be the set of all perturbations of f given by

$$\mathcal{U} = \{g(\cdot, \lambda) - f : \lambda \in \Lambda\}.$$

For any $\lambda \in \Lambda$ consider the perturbed equation

$$\dot{x} = g(x, \lambda), \tag{2.2}$$

and denote by $G(\lambda, x_0)$ the set of all noncontinuable solutions of (2.2) which satisfy the initial condition $(0, x_0)$. The solution $x \equiv 0$ of the unperturbed equation (2.1) is said to be $\mathcal{U}$-totally stable if for any $\varepsilon \in (0, \chi)$ there exist $\sigma_1 = \sigma_1(\varepsilon) \in (0, \varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon) > 0$ such that if $||x_0|| < \sigma_1$, $\lambda \in \Lambda$, $|\lambda|_B < \sigma_2$, and $x \in G(\lambda, x_0)$, then $||x(t)|| < \varepsilon$ for $t \geq 0$. In other words the solution $x \equiv 0$ of (2.1) is $\mathcal{U}$-totally stable if the solution $x \equiv 0$, $\lambda = \omega$ of the system $\dot{x} = g(x, \lambda)$, $\dot{\lambda} = 0$ is stable in the Liapunov sense.

Assume now in particular that $g(0, \lambda) \equiv 0$ so that $x \equiv 0$ is solution of (2.2) for each $\lambda \in \Lambda$. As we have pointed out in Section 1, a priori there is no relationship between the total stability of the zero solution of the unperturbed equation (2.1) and the stability of the zero solution of each perturbed equation. In particular, we may have:

(a) the origin is stable for all perturbed equations and is not $\mathcal{U}$-totally stable;

(b) the origin is unstable for all perturbed equations (with $\lambda \neq \omega$) and is $\mathcal{U}$-totally stable.

Concerning the eventualities (a) and (b) the following proposition holds:

PROPOSITION 2.1. *Let* $g(0, \lambda) = 0$ *for all* $\lambda \in \lambda$. *Then:*

(I) Assume that for any $\lambda \in \Lambda$ *the zero solution of (2.2) is stable and let* $r(\lambda, \varepsilon)$ *be the corresponding* ε*-radius of stability. Then the zero solution of (2.1) is* $\mathcal{U}$*-totally stable if and only if for any* $\varepsilon > 0$ *one has* $\lim'_{\lambda \to \omega} r(\lambda, \varepsilon) > 0$.

(II) Assume that the zero solution of (2.2) is unstable for any $\lambda \in \Lambda - \{\omega\}$, *and let* $R(\lambda)$ *be the corresponding radius of instability. Then* $\lim''_{\lambda \to \omega} R(\lambda) > 0$ *implies that the zero solution of (2.1) is not* $\mathcal{U}$*-totally stable.*

We notice that Proposition 2.1 (II) is not in general invertible. For instance consider the planar system $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1$ in a bounded open set $D \subset \mathbf{R}^2$ which contains the origin and assume for $\mathcal{U}$ a set of perturbations depending on a scalar parameter $\lambda, \mathcal{U} = \{h_\lambda : D \to D, h_\lambda(x) = \lambda x(||x||^2 - \lambda)^2, \lambda \geq 0\}$. Then, given any $\lambda > 0$, the origin is unstable for the corresponding perturbed system with a radius of instability which tends to zero as $\lambda \to 0$, while the zero solution of the unperturbed system is stable but not $\mathcal{U}$-totally stable.

3. Holonomic systems and conditional total stability of equilibrium.

Let Φ and Ψ be bounded sets in $\mathbf{R}^n$, $n \geq 1$, containing the origin. Let $T \in C^1(\Phi \times \Psi, \mathbf{R})$ and $\Pi \in C^1(\Phi, \mathbf{R})$ be such that:

(a) $T(q, v) = (2)^{-1}\langle v, A(q)v\rangle$ where $A(q)$ is an $n \times n$ symmetric positive definite matrix;

(b) $\Pi(0) = 0$, $\nabla\Pi(0) = 0$.

Consider the $2n$-dimensional system

$$\frac{d}{dt}\frac{\partial T}{\partial v}(q, v) - \frac{\partial T}{\partial q}(q, v) = -\nabla\Pi(q) + Q(q, v)$$
$$\frac{dq}{dt} = v,$$

where $Q \in C(\Phi \times \Psi, \mathbf{R}^n)$ satisfies the condition $\langle Q(q, v), v\rangle \leq 0$ for all $(q, v) \in \Phi \times \Psi$. System (3.1) may be reduced to the form (2.1) with $s = 2n$, $x = (q, v)$, $D = \Phi \times \Psi$. In the discussion of (3.1) we will adopt the notations of Section 2. Furthermore, we denote by $B'(\gamma)$ the ball $\{q \in \mathbf{R}^n : ||q|| < \gamma\}$. If $q_0 \in \Phi$ satisfies $\nabla\Pi(q_0) = 0$, then q_0 and $\Pi(q_0)$ are said to be a critical point and a critical value respectively of Π. Since Q is continuous, the condition $\langle Q(q, v), v\rangle \leq 0$ implies $Q(q, 0) \equiv 0$. Hence $q_0 \in \Phi$ is an equilibrium position if and only if q_0 is a critical point of Π. In particular $q = 0$ is one of such positions. The derivative along the solutions of (3.1) of the total energy, $H = T + \Pi$, satisfies the condition $\dot{H}(q, v) = \langle Q(q, v), v\rangle \leq 0$.

We include the case that Q is nonenergic, that is $\langle Q(q, v), v\rangle \equiv 0$, as well as the case that Q is a strictly dissipative force, that is $\langle Q(q,v),v\rangle < 0$ for $v \neq 0$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the spaces, both equipped with the norm C^1, of the bounded functions $C^1(\Phi \times \Psi, \mathbf{R})$ and $C^1(\Phi, \mathbf{R})$ respectively. Moreover let $\mathcal{B}_3$ be the space, equipped with the norm C^0 of the bounded functions $C(\Psi \times \Phi, \mathbf{R}^n)$. Consider the space $\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$ with the norm $|\cdot|_B = |\cdot|_{\mathcal{B}_1} + |\cdot|_{\mathcal{B}_2} + |\cdot|_{\mathcal{B}_3}$. Consider the subset Λ_1 of $\mathcal{B}_1$ consisting of all the functions such that $\lambda_1(q,v) = (2)^{-1}\langle v, A_{\lambda_1}(q)v\rangle$ where $A_{\lambda_1}(q)$ is an $n \times n$ symmetric positive definite matrix. Moreover, denote by Λ_3 the subset of $\mathcal{B}_3$ of the functions such that $\langle \lambda_3(q,v),v\rangle \leq 0$ for every $(q,v) \in \Phi \times \Psi$. Finally, Let $\Lambda_2 = \mathcal{B}_2$ and $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3$. For any $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ consider the differential system which has the form (3.1) with T, Π, Q replaced by the functions $T + \lambda_1, \Pi + \lambda_2, Q + \lambda_3$ respectively. This system will be denoted by $\mathcal{S}_\lambda$. It is easy to see that $\mathcal{S}_\lambda$ may assume the form (2.2) with $x = (q,v)$ and g satisfying the condition (K). For this system, and with reference to Λ, we will define the set $\mathcal{U}$ as in Section 2. From now on, we will study the conditional total stability of the solution $q \equiv 0$, $v \equiv 0$ of (3.1) with respect to the set of perturbations $\mathcal{U}$, which takes into account only the small perturbations obtained by modifying the coefficients of the kinetic energy, the potential energy and the dissipative force. The following theorem shows that under the same hypotheses of the Lagrang-Dirichlet theorem, the zero solution of system (3.1) is totally stable with respect to the above perturbations. The theorem stresses for system (3.1) a property already pointed out in [6] for perturbations depending on a finite number of parametrs, and more generally in [8].

THEOREM 3.1. *Assume the potential energy has a strict minimum at $q = 0$. Then the solution $q \equiv 0$, $v \equiv 0$ of (3.1) is $\mathcal{U}$-totally stable.*

Proof. One can proceed as for the proof of the Lagrange-Dirichlet theorem. The function H is positive definite. Then, given $\varepsilon \in (0,\chi)$ small and setting $m = min\{H(q,v), \|(q,v)\| = \varepsilon\}$, one has $m > 0$. For each $\lambda \in \Lambda$ let H_ε be the total energy relative to the perturbed system $\mathcal{S}_\lambda$. Along the solutions of $\mathcal{S}_\lambda$ we have $\dot{H}(q,v) \leq 0$. Moreover there exist $\sigma_1 = \sigma_1(\varepsilon) \in (0, \varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon) > 0$ such that

$$|H_\lambda(q,v)| < \frac{m}{2} \text{ if } ||(q_0,v_0)|| < \sigma_1 \text{ and } |\lambda|_{\mathcal{B}} < \sigma_2,$$

$$H_\lambda(q,v) \geq \frac{m}{2} \text{ if } ||(q,v)|| = \varepsilon \text{ and } |\lambda|_{\mathcal{B}} < \sigma_2.$$

Let (q_0, v_0) be such that $||(q_0,v_0)|| < \sigma_1$, and let $(q(t), v(t)) \in G(\lambda, t_0, q_0, v_0)$. This solution exists and satisfies $||(q(t), v(t))|| < \varepsilon$ for all $t \geq 0$. Indeed, the existence of $t_1 > t_0$ such that

$$||(q(t), v(t))|| < \varepsilon \text{ for each } t \in [t_0, t_1) \text{ and } ||(q(t_1), v(t_1))|| = \varepsilon,$$

implies

$$\frac{m}{2} > |H_\lambda(q_0, v_0)| \geq H_\lambda(q_0, v_0) \geq H_\lambda(q(t_1), v(t_1)) \geq \frac{m}{2},$$

that is a contradiction.

Remark 3.1.

(i) Let $\lambda \in \Lambda - \{\omega\}$ be such that $\nabla\Pi_\lambda(0) = 0$, i.e. $q = 0$ is an equilibrium position of the perturbed system $\mathcal{S}_\lambda$. We emphasize that the total stability of the zero solution of (3.1) does not imply in general that , for λ close to ω, $q \equiv 0$, $v \equiv 0$ is a stable solution of $\mathcal{S}_\lambda$. For example, assume that Π is a positive definite form of (even) degree ≥ 4. Furthermore, assume $\lambda^* = \{(\lambda_1, \lambda_2, \lambda_3) \in \Lambda : \lambda_1 = \omega$, $\lambda_2 = \mu h(q)$, $\lambda_3 = \omega$, $\mu \geq 0\}$ where h is a quadratic form whose range contains negative values. In this case, by virtue of Theorem 3.1 the zero solution of the unperturbed system is $\mathcal{U}$-totally stable, whereas for any $\lambda \in \Lambda^*$, $\lambda \neq \omega$, the zero solution of $\mathcal{S}_\lambda$ is unstable.

(ii) It is well known that the zero solution of system (3.1) is stable if the assumption that Π has a strict minimum at $q = 0$ is replaced by that of the existence of a fundamental family F of open neighborhoods of $q = 0$ such that for every $A \in F$ and for every $q \in \partial A$ one has $\Pi(q) > 0$. It is easy to prove that this is also a sufficient condition for the $\mathcal{U}$-total stability of the solution.

THEOREM 3.2. *Assume that:*

(i) the potential energy does not have a relative minimum at $q = 0$*;*

(ii) there exists $\gamma \in (0,\chi)$ such that there are no equilibrium positions in the set $P = \{q : ||q|| \leq \gamma,\ \Pi(q) < 0\}$.
Then, the zero solution of (3.1) is not $\mathcal{U}$-totally stable.

Proof. Let

$$\Lambda^* = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 = \omega,\ \lambda_2 = \omega,\ \lambda_3 = \mu\tilde{Q}(q,v),\ \mu \geq 0\},$$

where $\tilde{Q} \in C(\Phi \times \Psi, \mathbf{R}^n)$ is a strictly dissipative force. By virtue of Proposition 2.1 (II) the proof that the zero solution of (3.1) is not $\mathcal{U}$-totally stable is obtained if it is shown that for any $\mu > 0$ the zero solution of the perturbed differential system $\mathcal{S}_{\lambda(\mu)}$ corresponding to this $\mu \in \Lambda^*$ is unstable and the radius of instability $R(\mu)$ is greater than a positive number independent of μ. We have

$$\text{(3.2)} \qquad \dot{H}_\mu(q,v) \leq 0 \ \text{ and } \ \dot{H}_\mu(q,v) = 0 \ \text{ if and only if } \ v = 0,$$

where $\dot{H}_\mu$ is the derivative of H along the solutions of $\mathcal{S}_{\lambda(\mu)}$. Let $q_0 \in (0,\gamma)$ such that $\Pi(q_0) = \Pi_0 < 0$. Let $x_0 = (q_0, 0)$ and let $x(t) = (q(t), v(t))$ be a noncontinuable solution of $\mathcal{S}_{\lambda(\mu)}$ for which $x(0) = x_0$. Clearly $||x_0|| < \gamma$ and we assume that $x(t)$ exists and satisfies $||x(t)|| < \gamma$ for all $t \geq 0$. Hence the positive limit set Γ^+ of this solution is nonempty. From (3.2) it follows

$$H(q(t), v(t)) \leq H(q_0, 0) < 0 \ \text{ and } \ \Pi(q(t)) \leq \Pi(q_0) < 0, \qquad \text{for all } t \geq 0.$$

The first of these inequalities implies the existence of $a \in (0,\gamma)$ such that $||x(t)|| \geq a$ for all $t \geq 0$. Therefore we have $x(t) \in B(\gamma) - B(a)$ and $\Pi(q(t)) < 0$ for all $t \geq 0$. By using again (3.2) and the LaSalle invariance principle [3] we obtain that Γ^+ is contained in the union M of the noncontinuable orbits of $\mathcal{S}_{\lambda(\mu)}$ lying in the set $\{x = (q,v) : x \in B[\gamma] - B(a),\ \Pi(q) < 0,\ v = 0\}$. Since there are no critical points in P, it follows that M is empty. This contradicts the previous assertion that Γ^+ is nonempty and thus we have proved that for each $\mu > 0$ the zero solution of $\mathcal{S}_{\lambda(\mu)}$ is unstable and the radius of instability satisfies $R(\mu) \geq \gamma$. Then the proof is complete.

Under the assumptions (i), (ii) in Theorem 3.2, unless Q is itself strictly dissipative, the zero solution of (3.1) may be stable. It is known indeed that the Lagrange-Dirichlet theorem does not admit in general a converse, even if $Q \equiv 0$ and (ii) holds. It is also known that when Q is nonenergic, the equilibrium $q = 0$ of (3.1) may be stable even if the equilibrium is unstable for $Q \equiv 0$ (gyroscopic stabilization). We also notice that if the equilibrium position $q = 0$ is isolated, then the condition that Π has a relative minimum is necessary and sufficient in order to have $\mathcal{U}$-total stability. In conclusion, by using $\mathcal{U}$-total stability concepts, the classic theorems of Lord Kelvin concerning the secular stability of equilibrium, are reinterpreted and enriched.

Assume now that Π is a (real) analytic function. Then any compact set contains only a finite number of critical values of Π [9]. Therefore, by using the continuity of Π, there exists $\gamma \in (0, \chi)$ such that zero is the unique critical value of Π in $B'[\gamma]$. Hence condition (ii) in Theorem 3.2 is automatically satisfied and then the following theorem holds:

THEOREM 3.3. *Assume that Π is an analytic function which does not have a relative minimum at $q = 0$. Then the zero solution of (3.1) is not $\mathcal{U}$-totally stable.*

Returning to consider a C^1 potential energy Π, we wish now to inspect the case were Π has a nonstrict relative minimum at $q = 0$. We conjecture that in this case the solution $q \equiv 0$, $v \equiv 0$ of the unperturbed system (3.1) is not $\mathcal{U}$-totally stable and a general analysis concerning this question is in progress. In the present paper we restrict ourselves to consider some particular cases.

LEMMA 3.1. *Assume that Π has a nonstrict relative minimum at $q = 0$. For a given $m \geq 2$ and for $\mu \geq 0$. Let*

$$\Pi_\mu(q) = \Pi(q) - \mu ||q||^m.$$

Assume there exist $\gamma \in (0, \chi)$ and two sequences $\{a_i\}, \{\mu_i\}$, with $a_i \in (0, \gamma)$, $\mu_i > 0$, $a_i \to 0$, $\mu_i \to 0$, such that for every $i \in \mathrm{N}$ and $\mu = \mu_i$, $\Pi_\mu(q)$ does not have critical points in the set $\{q : q \in B'[\gamma] - B'(a_i)$, $\Pi_\mu(q) < 0\}$. Then the zero solution of (3.1) is not $\mathcal{U}$-totally stable.

Proof. We may assume $\Pi(q) \geq 0$ in $B'[\gamma]$. Let $\{q_n\}$ be a sequence of points in $B'(\gamma)$ with $q_n \neq 0$ and $q_n \to 0$. Let

$$\Lambda^* = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 = \omega,\ \lambda_2 = -\mu ||q||^m,\ \lambda_3 = \mu \tilde{Q}(q, v),\ \mu \geq 0\},$$

where $\tilde{Q} \in C(\Phi \times \Psi, \mathbf{R}^n)$ is a strictly dissipative force. For each $\mu > 0$ let $\mathcal{S}_{\lambda(\mu)}$ be the perturbed differential system corresponding to this μ in Λ^*. Given any n in $\mathbf{N}$ and $\varepsilon > 0$ there exist $a \in (0,\gamma)$ and $\mu \in (0,\varepsilon)$ such that $q_n \in B'(\gamma) - B'(a)$ and Π_μ does not have critical points in the set $P = \{q : q \in B'[\gamma] - B'(a),\ \Pi_\mu(q) < 0\}$. Let $x_0 = (q_n, 0)$ and let $x(t) = (q(t), v(t))$ be a noncontinuable solution of $\mathcal{S}_{\lambda(\mu)}$ such that $x(0) = x_0$. We have $||x_0|| < \gamma$ and we assume that $x(t)$ exists and satisfies $||x(t)|| < \gamma$ for all $t \geq 0$. Hence the positive limit set Γ^+ of this solution is nonempty. Denote by H_μ the total energy for $\mathcal{S}_{\lambda(\mu)}$ and by $\dot{H}_\mu$ its derivative along the solutions of $\mathcal{S}_{\lambda(\mu)}$. Since $\dot{H}_\mu(q,v) \leq 0$, we have

$$0 \leq \Pi(q(t)) \leq \mu||q(t)||^m - \mu||q_n||^m.$$

Therefore $\gamma > ||q(t)|| \geq ||q_n|| \geq a$, $x(t) \in B(\gamma) - B(a)$ and $\Pi_\mu(q(t)) < 0$ for all $t \geq 0$. Since Π_μ does not have critical points in P, and $\dot{H}_\mu(q,v) = 0$ if and only if $v = 0$, then, by using again the LaSalle invariance principle, we find that Γ^+ is empty. We get a contradiction, and, in view of the arbitrariness of n, ε, the proof is complete.

LEMMA 3.2. *Let us assume that* Π *has a nonstrict relative minimum at* $q = 0$. *Moreover suppose that for some* $\gamma \in (0,\chi)$ *and any* $a \in (0,\gamma)$ *there exists* $k > 0$ *such that*

$$k\Pi(q) - \langle \nabla\Pi(q), q\rangle \geq 0, \tag{3.3}$$

in the annulus $B'[\gamma] - B'(a)$. *Then the zero solution of (3.1) is not* $\mathcal{U}$*-totally stable.*

Proof. We may assume $\Pi(q) \geq 0$ in $B'[\gamma]$. For a given $\mu \geq max\{2,k\}$ and for $\mu \geq 0$ let

$$\Pi_\mu(q) = \Pi(q) - \mu||q||^m.$$

By virtue of Lemma 3.1, the proof will be obtained if we prove that for any $\mu > 0$ there are no critical points of Π_μ in the set $\{q : a \leq ||q|| < \gamma,\ \Pi_\mu(q) < 0\}$. For this it is sufficient to show that if $q \in B'[\gamma] - B'(a)$ satisfies $\langle \nabla\Pi\mu(q), q\rangle = 0$, then $\Pi_\mu(q) \geq 0$. Since

$$\langle \nabla\Pi_\mu(q), q\rangle = \langle \nabla\Pi(q), q\rangle - m\mu||q||^m,$$

then $\langle \nabla \Pi_\mu(q), q\rangle = 0$ implies

$$\Pi_\mu(q) = \Pi(q) - (m)^{-1}\langle \nabla \Pi(q), q\rangle,$$

and then $\Pi_\mu(q) \geq 0$ by virtue of (3.3) and taking into account that $m \geq k$ and $\Pi(q) \geq 0$. The proof is complete.

PROPOSITION 3.1. *Suppose that* Π *has at* $q = 0$ *a nonisolated zero and*

$$\Pi(q) = \sigma(q) \sum_2^h {}_j \Pi_{(j)}(q), \tag{3.4}$$

where $h \geq 2$, $\Pi_{(2)}, \ldots, \Pi_{(h)}$ *are forms of degree* $2, \ldots, h$ *for which*

$$\sum_2^s {}_j \Pi_{(j)}(q) \geq 0, \quad s = 2, \ldots, h, \tag{3.5}$$

and σ *is a* C^1 *function such that* $\sigma(q) > 0$ *for all* $q \neq 0$ *in a neighborhood of the origin. Then the zero solution of (3.1) is not* $\mathcal{U}$*-totally stable.*

Proof. Let $\gamma > 0$ be such that $\sigma(q) > 0$ for all $q \in B'(\gamma) - \{0\}$. For any $a \in (0, \gamma)$ there exists $k > h$ such that

$$(k-h)\sigma(q) - \langle \nabla\sigma(q), q\rangle \geq 0, \quad \text{for all} \quad q \in B'[\gamma] - B'(a) \tag{3.6.}$$

Then we have:

$$k\Pi(q) - \langle \nabla \Pi(q), q\rangle = \sigma(q) \sum_2^h {}_j (k-j)\Pi_{(j)}(q) - \langle \nabla\sigma(q), q\rangle \sum_2^h {}_j \Pi_{(j)}(q) =$$

$$= [(k-h)\sigma(q) - \langle \nabla\sigma(q), q\rangle] \sum_2^h {}_j \Pi_{(j)}(q) + \sigma(q) \sum_2^{h-1} {}_j (h-j)\Pi_{(j)}(q).$$

We easily verify that

$$\sum_2^{h-1} {}_j (h-j)\Pi_{(j)}(q) = \sum_2^{h-1} {}_j \Pi_{(j)}(q) + \sum_2^{h-2} {}_j \Pi_{(j)}(q) + \cdots + \sum_2^{3} {}_j \Pi_{(j)}(q) + \Pi_{(2)}.$$

Hence, by virtue of (3.5), (3.6) we get

$$k\Pi(q) - \langle \nabla \Pi(q), q\rangle \geq 0, \quad \text{for all} \quad q \in B'[\gamma] - B'(a).$$

Then the result follows from Lemma 3.2.

PROPOSITION 3.2. *Assume that* Π *depends only on* $r < n$ *coordinates, say* $q_1, \ldots, q_r$, *and is positive definite in these coordinates. Then the solution* $q \equiv 0$, $v \equiv 0$ *of (3.1) is not* $\mathcal{U}$*-totally stable.*

Proof. Let $y = (q_1, \ldots, q_r)$ and $z = (q_{r+1}, \ldots, q_n)$. For $\mu \geq 0$ let

$$\Pi_\mu(y, z) = \Pi(y) - \mu\|(y, z)\|^2. \tag{3.7}$$

Let $\gamma \in (0, \chi)$ be such that $\Pi(y) > 0$ for all $y : 0 < \|y\| \leq \gamma$, and assume $\mu > 0$. Clearly $\nabla\Pi_\mu(y, z) = 0$ if and only if

$$\nabla\Pi(y) = 2\mu y \quad \text{and} \quad z = 0.$$

For any $a \in (0, \gamma)$ the critical points $q = (y, z)$ of Π_μ lying in the annulus $B'[\gamma] - B'(a)$ satisfy $\gamma \geq \|y\| \geq a$ and $\|z\| = 0$. Hence if $\mu > 0$ is small enough, for each one of these critical points we have $\Pi_\mu(y, z) > 0$. Then the result follows from Lemma 3.1.

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Dipartimento di Matematica
Università di Trento
38050 POVO, Trento,
Italy.