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ON AN EVOLUTION PROBLEM OF THERMOCAPILLARY CONVECTION

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We consider a free boundary problem of incompressible viscous flow governing the motion of an isolated liquid mass. The liquid is subjected to capillary forces at the boundary, and the coefficient of the surface tension depends on the temperature satisfying the heat equation with convection and dissipation terms.

It is shown that this problem is solvable in a certain finite time interval, however, if the data are close to the rest state, the solution can be extended to the interval $t > 0$. In the case when the temperature satisfies the heat equation without dissipation term a local existence theorem was proved by M.V. Lagunova and V.A. Solonnikov [2], and global result was obtained by V.A. Solonnikov [7].

1. Introduction.

Consider the following free boundary problem: find a bounded domain $\Omega_t \subset \mathbb{R}^3$, $t > 0$, velocity vector filed $\tilde{v}(x,t) = (v_1, v_2, v_3)$ and scalar pressure $p(x,t)$ and temperature $\theta(x,t)$ satisfying in Ω_t the following equations, initial and boundary conditions:

$$
(1.1)\qquad
\begin{aligned}
&\rho(\tilde{v}_t + (\tilde{v}\cdot\nabla)\tilde{v}) - \nabla\mathbb{T} = \rho\tilde{f},\ \nabla\cdot\tilde{v} = 0,\\
&\rho c_p(\theta_t + (\tilde{v}\cdot\nabla)\theta) - \nabla\cdot\kappa\nabla\theta = \frac{\mu}{2}|\,\mathbb{S}(\tilde{v})|^2\ (x\in\Omega_t, t>0),\\
&\tilde{v}|_{t=0} = \tilde{v}_0(x), \theta|_{t=0} = \theta_0(x)\ (x\in\Omega_0),\\
&\frac{\partial\theta}{\partial n} + \beta\theta = 0,\\
&\mathbb{T}_{\tilde{n}} - \delta(\theta)H\tilde{n} = \nabla_\tau\sigma(\theta)\ (x\in\Gamma_t = \partial\Omega_t, t>0).
\end{aligned}
$$

Here $\mu, \kappa, \rho, c_p, \beta$ are positive constants, $\sigma(\theta)$ (the coefficient of the surface tension) is a smooth strictly positive function of the temperature: $\sigma(\theta) \geq \sigma_0 > 0$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, $\nabla p = \mathrm{grad} p$, $\nabla\cdot\tilde{v} = \mathrm{div}\tilde{v}$, $\nabla\mathbb{T} = \left(\sum_{i=1}^{3}\frac{\partial}{\partial x_i}T_{ik}\right)_{k=1,2,3}$, $\nabla_\tau\sigma = \nabla\sigma - \tilde{n}\frac{\partial\sigma}{\partial n}$ is the gradient of σ at the surface Γ_t, $\tilde{n}$ is a unit exterior normal to Γ_t, $H(x,t)$ is the twice mean curvature of Γ_t at the point x which is negative if Ω_t is convex in the neighbourhood of x, $\mathbb{T} = -p\mathbb{I} + \mu\,\mathbb{S}(\tilde{v})$ is the stress tensor, and $\mathbb{S}(\tilde{v})$ is the strain tensor with the elements $S_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$.

The domain Ω_t is given at the initial moment $t = 0$; for $t > 0$ Ω_t is the set of points $x = x(\xi, t)$ such that the corresponding radii-vectors $\tilde{x}(\xi, t)$ are solutions to the Cauchy problem

$$
(1.2)\qquad
\begin{aligned}
&\frac{\partial\tilde{x}}{\partial\tau} = \tilde{v}(x(\xi,\tau),\tau), \quad 0\leq\tau\leq t,\\
&\tilde{x}(\xi,0) = \tilde{\xi}, \quad \forall\xi\in\Omega_0.
\end{aligned}
$$

If Ω_t is defined by the equation

$$
(1.3)\qquad |x| = R\left(\frac{x}{|x|}, t\right)
$$

the kinematic boundary condition (1.2) can be written in an equivalent form

$$
(1.4)\qquad RR_t = Rv_r - \nabla_\omega R\cdot\tilde{v}_\omega
$$

where $\nabla_\omega R(\omega,t)$, is the gradient of R on the unit sphere $S_1 : |\omega| = 1$, v_r and $\tilde{v}_\omega = \tilde{v} - \frac{\tilde{x}}{|x|} v_r$ are radial and angular components of $\tilde{v}$, respectively.

Condition (1.2) makes it possible to rewrite our free-boundary problem as a nonlinear initial-boundary value problem in a given domain Ω_0, using the Lagrangean coordinates $\xi \in \Omega_0$.

Let $\tilde{u}(\xi,t) = \tilde{v}(x(\xi,t),t)$, $q(\xi,t) = p(x(\xi,t),t)$, $\psi(\xi,t) = \theta(x(\xi,t),t)$. The Cauchy problem (1.2) is easily integrated:

$$\tilde{x} = \tilde{\xi} + \int_0^t \tilde{u}(\xi,\tau)d\tau \equiv X_u(\xi,t)$$

and (1.1) take the form

$$\begin{aligned}
&\tilde{u}_t - \mu\nabla_u^2\tilde{u} + \nabla_u q = \tilde{f}(X_u,t),\ \nabla_u\cdot\tilde{u} = 0,\\
&\psi_t - \chi\nabla_u^2\psi - \lambda|\mathbb{S}_u(\tilde{u})|^2 = 0\ (\xi\in\Omega_0, t>0),\\
&\tilde{u}|_{t=0} = \tilde{v}_0(\xi),\ \psi|_{t=0} = \theta_0(\xi)\ (\xi\in\Omega_0),\\
&\mathbb{T}_u\tilde{n} - \sigma(\psi)\Delta(t)X_u = (\nabla_u - \tilde{n}(\tilde{n}\cdot\nabla_u))\sigma(\psi),\\
&\tilde{n}\cdot\nabla_u\psi + \beta\psi = 0\ (\xi\in\Gamma_0)
\end{aligned}\tag{1.5}$$

where $\nabla_u = \mathcal{A}\nabla$, $\mathcal{A}$ is a matrix with the elements $A_{ij} = \frac{\partial\xi_j}{\partial x_i}$ which can be computed as cofactors of $a_{ij} = \frac{\partial x_i}{\partial\xi_j} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial\xi_j}d\tau$,

$\mathbb{T}_u = -p\mathbb{I} + \mu\,\mathbb{S}_u(\tilde{u})$, $(\mathbb{S}_u(\tilde{w}))_{ij} = \sum_{k=1}^{3}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right)$.

We have put $\rho = 1$, $\chi = \frac{\kappa}{\rho c_p}$, $\lambda = \frac{\mu}{2\rho c_p}$ and used the formula $H\tilde{n} = \Delta(t)X_u$ where $\Delta(t)$ is the Laplace-Beltrami operator on the surface $\Gamma_t = X_u(\Gamma_0)$. The exterior normals to Γ_t and to Γ_0 at the points $X_u(\xi,t)$ and ξ are related to each other by the formula $\tilde{n} = \frac{\mathcal{A}\tilde{n}_0}{|\mathcal{A}\tilde{n}_0|}$.

Problem (1.5) is considered in anisotropic Hölder spaces $C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$, $Q_T = \Omega_0\times(0,T)$ whose definition can be found in particular in Lagunova and Solonnikov [2]. The initial data $\tilde{v}_0$ and θ_0 are taken from the isotropic Hölder spaces $C^{2+\alpha}(\tilde{\Omega})$. Norms in $C^{l,\frac{l}{2}}(Q_T)$ and in $C^l(\Omega)$ are denoted by $|\cdot|_{Q_T}^{(l,\frac{l}{2})}$ and $|\cdot|_{\Omega}^{(l)}$.

THEOREM 1. *Let Ω_0 be a bounded domain with the boundary $\Gamma_0 \in C^{3+\alpha}$, $\alpha \in (0,1)$, and let $\tilde{f}$, $\tilde{f}_{x_k} \in C^{\alpha,\frac{\alpha+\varepsilon}{2}}(\mathbb{R}^3 \times (0,T))$ where $\varepsilon \in (0,1-\alpha)$. For arbitrary $\theta_0 \in C^{2+\alpha}(\Omega_0)$, $\tilde{v}_0 \in C^{2+\alpha}(\Omega_0)$ satisfying the compatibility conditions*

$$\nabla \cdot \tilde{v} = 0,\ \mu(\mathbb{S}(\tilde{v}_0)\tilde{n}_0 - \tilde{n}_0(\tilde{n}_0 \cdot \mathbb{S}(\tilde{v}_0)\tilde{n}_0))|_{\Gamma_0} = \left(\nabla - \tilde{n}\frac{\partial}{\partial n_0}\right)\sigma(\theta_0)|_{\Gamma_0},$$

$$\frac{\partial \theta_0}{\partial n_0} + \beta\theta_0|_{\Gamma_0} = 0$$

problem (1.5) has a unique solution in the interval $(0,T')$, $T' \leq T$, with the following differentiability properties: $\theta \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T'})$, $\tilde{u} \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T'})$, $\nabla q \in C^{\alpha,\frac{\alpha}{2}}(Q_{T'})$, $q \in C^{1+\alpha\frac{1+\alpha}{2}}(S_{T'})$ $(S_{T'} = \Gamma_0 \times (0,T'))$. The magnitude of T' depends on the norms of the data $\tilde{v}_0$, θ_0 and on Γ_0.

THEOREM 2. *Let the hypotheses of theorem 1 hold and let Ω_0 be defined by the equation (1.3) $|x| = R\left(\frac{x}{|x|},0\right)$, $R \in C^{3+\alpha}(S_1)$.*

If

$$|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)} + |\theta_0|_{\Omega_0}^{(2+\alpha)} + |R - R_0|_{S_1}^{(3+\alpha)} \leq \varepsilon_1$$

where $R_0 = \left(\frac{3|\Omega_0|}{4\pi}\right)^{\frac{1}{3}}$ and ε_1 is a sufficiently small number, then the solution exists for all $t > 0$, and Ω_t is defined by (1.3) with $R(\cdot,t) \in C^{3+\alpha}(S_1)$ satisfying (1.4). As $t \to \infty$, it tends to a quasistationary solution of problem (1.1)-(1.2), which corresponds to a uniform motion of a liquid mass rotating as a rigid body about a certain axis, and having the temperature $\theta_\infty = 0$.

In the case $\lambda = 0$ these theorems are proved by M.V. Lagunova and V.A. Solonnikov [2] and by V.A. Solonnikov [7]. We refer the reader to the review paper of V.A. Pukhnachov [6] for many other results conserning problems of thermocapillary convection.

The proof of theorems 1 and 2 is based on estimates of solutions of the linear initial-boundary value problems

$$
\begin{aligned}
&\psi_t - \chi\nabla_u^2\psi = f(\xi,t) \quad (\xi\in\Omega_0, t\in(0,T)),\\
&\psi|_{t=0} = \psi_0(\xi), \qquad\qquad (1.6)\\
&\tilde{n}\cdot\nabla_u\psi + \beta\psi = g \ (\xi\in\Gamma_0)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{w}_t - \mu\nabla_u^2\tilde{w} + \nabla_u s = \tilde{f},\ \nabla_u\cdot\tilde{w} = \rho,\\
&\tilde{w}|_{t=0} = \tilde{w}_0, \qquad\qquad (1.7)\\
&\mathbb{T}_u(\tilde{w},s)\tilde{n} - \sigma(\xi)\tilde{n}(\tilde{n}\cdot\Delta(t)\int_0^t \tilde{w}d\tau) = \tilde{b} \quad (\xi\in\Gamma_0)
\end{aligned}
$$

with a given vector field $\tilde{u}\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$ satisfying the condition

$$
(T+T^{\frac{1}{2}})|\tilde{u}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \le \delta, \tag{1.8}
$$

$\delta \ll 1$. Problem (1.7) is treated in the papers by I. Sh. Mogilevskii and V.A. Solonnikov [4, 5]. It is proved that this problem is uniquely solvable in Hölder spaces of functions, and in the case $\rho=0$ the solution satisfies the estimate

$$
\begin{aligned}
&|\tilde{w}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} + |\nabla s|^{(\alpha,\frac{\alpha}{2})} + |s|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})} \le C_1(T)(|\tilde{f}|_{Q_T}^{(\alpha,\frac{\alpha+\varepsilon}{2})}+\\
&+|\tilde{w}_0|_{\Omega_0}^{(2+\alpha)} + |\tilde{b}|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})})+ \qquad\qquad (1.9)\\
&+C_2(T)(T^{\frac{1-\alpha}{2}}|\nabla\tilde{u}|_{Q_T} + |\nabla\tilde{u}|_{Q_T}^{(\alpha,\frac{\alpha+\varepsilon}{2})})(|\tilde{w}_0|_{\Omega_0}+\\
&+|\nabla\tilde{w}_0|_{\Omega_0} + |\tilde{b}(\cdot,0)\cdot\tilde{n}_0|_{\Gamma_0})
\end{aligned}
$$

where $\varepsilon\in(0,1-\alpha)$, $\nabla\tilde{u} = \left\{\dfrac{\partial u_i}{\partial\xi_j}\right\}_{i,j=1,2,3}$, $|w|_\Omega = \sup\limits_{\xi\in\Omega}|w(\xi)|$.

The case $\rho\ne 0$ is also considered. Problem (1.6) is a usual parabolic problem and the estimate for the Hölder norm of its solution follows from results of the book O.A. Ladyzhenskaya, V.A. Solonnikov, N.M. Uraltzeva [1] (see also M.V. Lagunova, V.A. Solonnikov [2]).

This estimate has a form

$$|\psi|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \le C_3(|f|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + |\psi_0|_{\Omega_0}^{(2+\alpha)} + |g|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})})+ \tag{1.10}$$
$$+ C_4 T^{\frac{1-\alpha}{2}} |\nabla \tilde{u}|_{Q_T} |\nabla \psi_0|_{\Omega_0}.$$

2. Proof of theorem 1,

Let us present main ideas of the proof of theorem 1. Following the arguments of M.V. Lagunova and V.A. Solonnikov [2], we construct the solution of (1.5) by the method of successive approximations. We put $\tilde{u}_0^{(0)} = 0$, $q^{(0)} = 0$ and we define first $\psi^{(m+1)}$, then $(\tilde{u}^{(m+1)}, q^{(m+1)})$, $m \ge 1$ as solutions of linear problems

$$\begin{aligned} &\psi_t^{(m+1)} - \chi \nabla_m^2 \psi^{(m+1)} = \lambda |\,\mathbb{S}_m(\tilde{u}^{(m)})|^2, \\ &\psi^{(m+1)}|_{t=0} = \theta_0(\xi), \\ &\tilde{n}_m \cdot \nabla_m \psi^{(m+1)} + \beta \psi^{(m+1)}|_{\xi \in \Gamma_0} = 0, \end{aligned} \tag{2.1}$$

$$\tilde{u}_t^{(m+1)} - \mu \nabla_m^2 \tilde{u}^{(m+1)} + \nabla_m q^{(m+1)} = \tilde{f}(X_m, t),$$

$$\nabla_m \cdot \tilde{u}^{(m+1)} = 0,$$

$$\tilde{u}^{(m+1)}|_{t=0} = \tilde{v}(\xi),$$

$$\mu \prod\nolimits_0 \prod\nolimits_m \mathbb{S}_m(\tilde{u}^{(m+1)}) \tilde{n}_m = \prod\nolimits_0 \prod\nolimits_m \nabla_m \sigma(\psi^{(mH)}), \tag{2.2}$$

$$\tilde{n}_0 \cdot \mathbb{T}_m(\tilde{u}^{(m+1)}, q^{(m+1)}) \tilde{n}_m - \sigma(\psi^{(m+1)}) \tilde{n}_0 \cdot \sigma_m(t) X_{m+1}|_{\xi \in \Gamma_0} =$$
$$= \tilde{n}_0 \cdot \prod\nolimits_m \nabla_m \sigma(\psi^{(m+1)})$$

where $\nabla_m = \nabla_{u^{(m)}}$, $X_m = X_{u^{(m)}}(\xi, t) = \tilde{\xi} + \int_0^t \tilde{u}^{(m)}(\xi, \tau) d\tau$, $\mathbb{S}_m(\tilde{w}) = \mathbb{S}_{u^{(m)}}(\tilde{w})$, $\mathbb{T}_m = \mathbb{T}_{u^{(m)}}$, $\prod_m \tilde{h} = \tilde{h} - \tilde{n}_m(\tilde{n}_m \cdot \tilde{h})$, $\tilde{n}_m = \mathcal{A}_m \tilde{n}_0 |\mathcal{A}_m \tilde{n}_0|^{-1}$ is the exterior normal to the surface $\Gamma_m = X_m(\Gamma_0)$, $\Delta_m(t)$ is the Laplace-Beltrami operator on Γ_m.

In particular, the function $\psi^{(1)}$ is defined for all $t \geq 0$; the estimate (1.10) implies

$$|\psi^{(1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \leq C_1 |\theta_0|_{\Omega_0}^{(2+\alpha)}.$$

The initial-boundary value problem for $(\tilde{u}^{(1)}, q^{(1)})$ can be written in the form

$$\tilde{u}_t^{(1)} - \mu\nabla^2\tilde{u}^{(1)} + \nabla q^{(1)} = \tilde{f}(\xi,t),\ \nabla\cdot\tilde{u}^{(1)} = 0,\ \tilde{u}^{(1)}|_{t=0} = \tilde{v}_0(\xi),$$

$$\mathbb{T}(\tilde{u}^{(1)}, q^{(1)})\tilde{n}_0 - \sigma(\psi^{(1)})\tilde{n}_0\left(\tilde{n}_0\cdot\Delta_0\int_0^t \tilde{u}^{(1)}d\tau\right)|_{\xi\in\Gamma_0} =$$
$$= \left(\nabla - \tilde{n}_0\frac{\partial}{\partial n_0}\right)\sigma(\phi^{(1)}) + \tilde{n}_0\sigma(\psi^{(1)})H_0(\xi)$$

where $H_0 = \tilde{n}_0\cdot\Delta\tilde{\xi}$ is the mean curvature of Γ_0. This problem is solvable in the interval $(0,T)$ and in virtue of (1.9)

$$|\tilde{u}^{(1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} + |\nabla q^{(1)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + |q^{(1)}|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})} \leq$$
$$\leq C_2(|\tilde{f}|_{Q_T}^{(\alpha,\frac{\alpha+\epsilon}{2})} + |\tilde{v}_0|_{\Omega_0}^{(2+\alpha)} + |\nabla_\tau\sigma(\psi^{(1)})|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})} +$$
$$+ |\tilde{n}_0\sigma(\psi^{(1)})H_0|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})}).$$

Suppose that $\tilde{u}^{(j)}$, $q^{(j)}$, $\theta^{(j)}$, $j = 1,\ldots,m+1$, are defined on the interval $(0,T_m)$ and that $\tilde{u}^{(j)}$ satisfy (1.8) for $T \leq T_m$. Then for $T \leq T_m$

$$|\psi^{(j+1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \leq C_3(|\theta_0|_{\Omega_0}^{(2+\alpha)} + \lambda(|\mathbb{S}_j(\tilde{u}^{(j)})|_{Q_T}^{(\alpha,\frac{\alpha}{2})})^2 +$$
$$+ T^{\frac{1-\alpha}{2}}|\nabla\tilde{u}^{(j)}|_{Q_T}|\nabla\theta_0|_{\Omega_0} \leq C_4(|\theta_0|_{\Omega_0}^{(2+\alpha)} + (|\tilde{v}_0|_{\Omega_0}^{(1+\alpha)})^2 +$$
$$+ \delta^2 + T^{\frac{1-\alpha}{2}}|\nabla\tilde{v}_0|_{\Omega_0}|\nabla\theta_0|_{\Omega_0}) \equiv \Theta_1(T),$$

since

$$|\mathbb{S}_j(\tilde{u}^{(j)})|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq |\mathbb{S}_j(\tilde{v}_0)|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + |\mathbb{S}_j(\tilde{u}^{(j)} - \tilde{v}_0)|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq$$
$$\leq C_5(|\tilde{v}_0|_{\Omega_0}^{(1+\alpha)} + T^{\frac{1}{2}}|\tilde{u}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})}) \leq C_5(|\tilde{v}_0|_{\Omega_0}^{(1+\alpha)} + \delta),$$

$$|\nabla\tilde{u}^{(j)}|_{Q_T} \leq |\nabla|\tilde{u}^{(j)} - \tilde{v}_0)|_{Q_T} + |\nabla\tilde{v}_0|_{\Omega_0} \leq |\nabla\tilde{v}_0|_{\Omega_0} + \delta$$

The differences $\omega^{(j+1)} = \psi^{(j+1)} - \psi^{(j)}$, $j \geq 1$, are solutions to the problems

$$\omega_t^{(j+1)} - \chi\nabla_j^2\omega^{(j+1)} = l^{(j)}(\psi^{(j)}) - l^{(j-1)}(\psi^{(j)}) +$$

$$+ \lambda(|\,\mathbf{S}_j(\tilde{u}^{(j)})|^2 - |\,\mathbf{S}_{j-1}(\tilde{u}^{(j-1)}|^2),$$

$$\omega^{(j+1)}|_{t=0} = 0,$$

$$\tilde{n}_j \cdot \nabla_j\omega^{(j+1)} + \beta\omega^{(j+1)}|_{\xi\in\Gamma_0} = \lambda^{(j)}(\psi^{(j)}) - \lambda^{(j-1)}(\psi^{(j)})$$

where $l^{(j)}(\psi) = \chi(\nabla_j^2 - \nabla^2)\psi$, $\lambda^{(j)}(\psi) = (\tilde{n}_0 \cdot \nabla - \tilde{n}_j \cdot \nabla_j)\psi$.

These problems differ from problems (3.7) in [2] by the term $\lambda(|\,\mathbf{S}_j(\tilde{u}^{(j)})|^2 - |\,\mathbf{S}_{j-1}(\tilde{u}^{(j-1)}|^2)$ in the heat equation for $\omega^{(j+1)}$, hence $\omega^{(j+1)}$ satisfis the estimate analogous to (3.10) from [2], namely.

$$|\omega^{(j+1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \leq C_6\Theta_1(T)(M_T[\tilde{w}^{(j)}] + T^{\frac{1-\alpha}{2}}|\nabla\tilde{w}^{(j)}|_{Q_T} +$$

$$+ C_7\lambda\big||\,\mathbf{S}_j(\tilde{u}^{(j)})|^2 - |\,\mathbf{S}_{j-1}(\tilde{u}^{(j-1)})|^2\big|_{Q_T}^{(\alpha,\frac{\alpha}{2})}$$

with $M_T[\tilde{w}] = \int_0^T |\tilde{w}|^{(2+\alpha)}dt + \sup\limits_{0<h<t<T} h^{-\frac{\alpha}{2}}\int_{t-h}^t (|\nabla\tilde{w}|_{\Omega_0} + |\mathcal{D}_\xi^2\tilde{w}|_{\Omega_0})d\tau$, $\tilde{w}^{(j+1)} = \tilde{u}^{(j+1)} - \tilde{u}^{(j)}$. As

$$|\,\mathbf{S}_j(\tilde{u}^{(j)})|^2 - |\,\mathbf{S}_{j-1}(\tilde{u}^{(j-1)})|^2 = [\,\mathbf{S}_j(\tilde{u}^{(j)}) - \mathbf{S}_{j-1}(\tilde{u}^{(j)}) + \mathbf{S}_{j-1}(\tilde{w}^{(j)})] :$$

$$: [\,\mathbf{S}_j(\tilde{u}^{(j)}) + \mathbf{S}_{j-1}(\tilde{u}^{(j-1)})]$$

and

$$|\,\mathbf{S}_j(\tilde{u}^{(j)}) - \mathbf{S}_{j-1}(\tilde{u}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq C_8(\int_0^T |\nabla\tilde{w}^{(j)}|_{\Omega}^{(\alpha)}dt +$$

$$+ \sup_{0<h<t<T} h^{-\frac{\alpha}{2}}\int_{t-h}^t |\nabla\tilde{w}^{(j)}|_{\Omega_0}d\tau)|\nabla\tilde{u}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq C_9\delta|\nabla\tilde{w}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})},$$

$$|\nabla\tilde{w}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq \delta_{j1}|\nabla\tilde{v}_0|_{\Omega_0}^{(\alpha)} + C_{10}T^{\frac{1}{2}}|\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})},$$

we have in virtue of (1.8)

$$||\,\mathbf{S}_j(\tilde{u}^{(j)})|^2 - |\,\mathbf{S}_{j-1}(\tilde{u}^{(j-1)})|^2|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq C_{11}(|\nabla\tilde{w}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + \delta_{j1}|\nabla\tilde{v}_0|_{\Omega_0}^{(\alpha)})$$

$$\times (|\nabla\tilde{u}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + |\nabla\tilde{u}^{(j-1)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})}) \leq C_{12}(\delta|\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} +$$

$$\delta_{j1}|\nabla\tilde{v}_0|_{\Omega_0}^{(\alpha)}|\nabla\tilde{u}^{(1)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})})$$

and consequently

$$(2.3)\quad \begin{aligned} &|\omega^{(j+1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} \leq C_{13}\Theta_1(T)\{M_T[\tilde{w}^{(j)}] + T^{\frac{1-\alpha}{2}}|\nabla\tilde{w}^{(j)}|_{Q_T} + \\ &+ \delta|\tilde{w}^{(j)}|_{Q_T}^{2+(\alpha,1+\frac{\alpha}{2})} + \delta_{j1}|\nabla\tilde{v}_0|_{\Omega_0}^{(\alpha)}|\nabla\tilde{u}^{(1)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})}) \end{aligned}$$

The differences $\tilde{w}^{(j+1)} = \tilde{u}^{(j+1)} - \tilde{u}^{(j)}$, $S(^{(j+1)} = q^{(j+1)} - q^{(j)}$ satisfy the same equations and boundary conditions as in M.V. Lagunova and V.A. Solonnikov [2] (see problem (3.8)), hence the inequality (3.16) holds, i.e.

$$(2.4)\quad \begin{aligned} &N_T[\tilde{w}^{(j+1)}, S^{(j+1)}] \equiv |\tilde{w}^{(j+1)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} + |\nabla S^{(j+1)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + \\ &+ |S^{(j+1)}|_{S_T}^{(1+\alpha,\frac{1+\alpha}{2})}] \leq C_{14}\{\delta N_T[\tilde{w}^{(j)}, S^{(j)}] + M_T[\tilde{w}^{(j)}] + \\ &+ |\nabla\tilde{w}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} + \delta_{j1}(|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)})^2\}(1+\Theta_1(T))^{3+\alpha} \end{aligned}$$

$(T \leq T_m)$, provided that $(T + T^{\frac{1}{2}})\Theta_1(T) \leq \delta$.

Norms in the right-hand side of (2.4) satisfy the estimates

$$|\nabla\tilde{w}^{(j)}|_{Q_T}^{(\alpha,\frac{\alpha}{2})} \leq \varepsilon|\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} + C_{15}(\varepsilon)|\tilde{w}^{(j)}|_{Q_T} \leq$$

$$\leq \varepsilon|\tilde{w}^{(j)}|_{Q_T}^{2+(\alpha,1+\frac{\alpha}{2})} + C_{15}\delta_{j1}|\tilde{v}_0|_{\Omega_0} + C_{15}\int_0^T |\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})}dt,$$

$$\sup_{0<ht<T} h^{-\frac{\alpha}{2}}\int_{t-h}^{t}(|\nabla\tilde{w}^{(j)}|_{\Omega_0} + |\mathcal{D}_\xi^2\tilde{w}^{(j)}|_{\Omega_0})d\tau \leq \varepsilon|\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} +$$

$$+ C_{16}(\varepsilon)[|\tilde{w}^{(j)}|_{Q_T} + \int_0^T |\tilde{w}^{(j)}|_{Q_t}^{(2+\alpha,1+\frac{\alpha}{2})}dt] \leq$$

$$\leq \varepsilon|\tilde{w}^{(j)}|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})} + 2C_{16}\int_0^T |\tilde{w}^{(j)}|_{Q_t}^{(2+\alpha,1+\frac{\alpha}{2})}dt + C_{16}\delta_{j1}|\tilde{v}_0|_{\Omega_0},$$

hence, (2.4) implies

$$
\begin{aligned}
&N_T[\tilde{w}^{(j+1)}, S^{(j+1)}] \leq C_{14}(1+\Theta_1(T))^{3+\alpha}\{(\delta+2\varepsilon)N_T[\tilde{w}^{(j)}, S^{(j)}]+\\
(2.5)\qquad &+(C_{15}+2C_{16}+1)\int_0^T N_t[\tilde{w}^{(j)}, S^{(j)}]dt + \delta_{j1}[(|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)})^2+\\
&+(C_{15}+C_{16})|\tilde{v}_0|_{\Omega_0}]\}.
\end{aligned}
$$

After the summation with respect to $j = 1, \dots, m$ we obtain the following estimate for $\sum_{m+1}(T) = \sum_{j=1}^{m+1} N_T[\tilde{w}^{(j)}, S^{(j)}]$:

$$
\begin{aligned}
&\sum\nolimits_{m+1}(T) \leq N_T[\tilde{w}^{(1)}, S^{(1)}] + C_{14}(1+\Theta_1(T))^{3+\alpha}\{(\delta+2\varepsilon)\times\\
&\times N_T[\omega^{(j)}, S^{(j)}] + (C_{15}+2C_{16}+1)\int_0^T \sum\nolimits_{m+1}(t)dt+\\
&+(|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)})^2 + (C_{15}+C_{16})|\tilde{v}_0|_{\Omega_0}
\end{aligned}
$$

Suppose that δ is so small and ε is chosen so small that

$$
(2.6)\qquad C_{14}(1+\Theta_1(T))^{3+\alpha}(\delta+2\varepsilon) \leq \frac{1}{2}
$$

Then

$$
\sum\nolimits_{m+1}(T) \leq 2N_T[\tilde{w}^{(1)}, S^{(1)}] + C_{17}\left\{\int_0^T \sum\nolimits_{m+1}(t)dt + (|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)})^2 + |\tilde{v}_0|_{\Omega_0}\right\}
$$

and the Gronwall's lemma yields

$$
(2.7)\qquad \sum\nolimits_{m+1}(T) \leq e^{C_{17}T}\{2N_T[\tilde{u}^{(1)}, q^{(1)}] + C_{17}[(|\tilde{v}_0|_{\Omega_0}^{(2+\alpha)})^2 + |\tilde{v}_0|_{\Omega_0}]\}
$$

The conditions (1.8) for $\tilde{u}^{(m+1)}$ will be fulfilled, if

$$
(2.8)\qquad (T+T^{\frac{1}{2}})e^{C_{17}T}\{2N_T[\tilde{u}^{(1)}, q^{(1)}] + C_{17}|\tilde{v}_0|_{\Omega_0} + C_{17}(|v_0|_{\Omega_0}^{(2+\alpha)})^2\} \leq \delta
$$

The left-hand sides of (2.6), (2.8) do not depend on m, hence there exists $T' > 0$ such that (1.8), (2.7) hold for $T \leq T'$. From (2.3), (2.7) it follows that the sequence $\{\psi^{(m)}, \tilde{u}^{(m)}, q^{(m)}\}$ is convergent to the solution of (1.5).

To prove the uniqueness, we repeat the above calculations and show that the difference $\omega = \psi - \psi'$, $\tilde{u} - \tilde{u}' = \tilde{w}$, $q - q' = s$ of two solutions satisfies the inequalities analogous to (2.3), (2.5), namely

$$N_T[\tilde{w}, s] \leq C_{14}(1 + 9\Theta_1(T))^{3+\alpha}\{(\delta + 2\varepsilon)N_T[\tilde{w}, s]+$$

$$+ (C_{15} + 2C_{16} + 1)\int_0^T N_t[\tilde{w}, s]dt\}$$

$$|\omega|_{Q_T}^{(2+\omega,1+\frac{\alpha}{2})} \leq C_{13}\Theta_1(T)\{M_T[\tilde{w}] + T^{\frac{1-\alpha}{2}}|\nabla w|_{Q_T} + \delta|w|_{Q_T}^{(2+\alpha,1+\frac{\alpha}{2})}$$

If (2.6), (2.8) hold, it follows that $\tilde{w} = 0$, $s = 0$ and as a consequence $\omega = 0$, $\psi - \psi' = 0$. Theorem 1 is proved.

The proof of theorem 2 is close to the proof of theorem 1.2 in V.A. Solonnikov [7].

We observe at the conclusion that theorems 1 and 2 hold also in the case when the visconsity μ and heat conductivity κ are positive functions of the temperature θ. Necessary estimates for the solution of a linearized problem of the type (1.5) with a variable viscosity are obtained by I. Sh. Mogilevskii [3].

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