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INSTABILITY PROBLEMS IN EHD, FHD AND MHD

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Classes of linear instability and nonlinear stability problems are discussed in the fields of electrohydrodynamics, ferrohydrodynamics and magnetohydrodynamics.

1. Introduction.

The letters EHD, FHD and MHD stand for, respectively, electrohydrodynamics, ferrohydrodynamics and magnetohydrodynamics. Basically, these subjects are concerned with the interaction of electric, magnetic and temperature fields in a fluid. Due to the many practical applications these topics have been attracting much recent attention. More precisely, electrohydrodynamics is the branch of fluid mechanics which is concerned with electric force effects, ferrohydrodynamics deals with the mechanics of fluid motion induced by strong forces of magnetic polarization and magnetohydrodynamics studies the interaction between magnetic fields and fluid conductors of electricity.

In this paper we review some findings concerning problems of hydrodynamic instability in these areas. We pay particular attention to topics within the scope of the conference, namely waves and stability. Uses of an integral relation technique nowadays referred to as the energy method for establishing criteria for nonlinear stability are outlined as are applications of nonlinear acceleration wave methods. Detailed discussions of energy theory in these and related fields may be found in the book by Straughan [42].

2. Electrohydrodynamic instability problems

Early theoretical studies of convection-like instabilities in insulating fluid layers subject to temperature gradients and electrical potential differences across the layer are those of Roberts [31] and Turnbull [43].

Roberts [31] studies two models, one which allows the dielectric constant of the fluid to vary with temperature, and a second which is effectively also investigated independently by Turnbull [43] and which has the electrical conductivity temperature dependent. The temperature dependent conductivity model is studied further by Martin & Richardson [19].

These papers had some success in agreeing with experimental work, but were not totally successful. This work is reviewed at length in [42].

Concurrently another approach hed been followed by P.Atten and his co-workers, cf. Lacroix *et al.* [15]. These writers concentrated on mobility models of charge transport. Indeed, current thinking would appear to follow a mixture of the ideas of Atten, Roberts and Turnbull, see e.g. Castellanos *et al.* [5,6], González & Castellanos [12], Gonzáles *et al.* [13], Hoburg [14], McCluskey & Atten [18], Mohamed & El Shehawey [20], Sneyd [39], Worraker & Richardson [45]. For an account of the application of energy methods to study nonlinear stability in these areas see e.g. [42].

Particularly interesting articles in EHD are those of González *et al.* [13], and Rosensweig *et al.* [35]. The work of [13] is concerned with the problem of confinement of a liquid and may have application to the floating zone technique for growing high quality crystals of electronic materials such as Silicon, Gallium Arsenide and Indium Phosphide. This paper studies the shape bifurcation and stability of a liquid bridge of dielectric fluid under the influence of an applied electric field. The labyrinthine instability observed by Rosensweig *et* *al.* [35] is a fascinating one and occurs also in magnetic fluids (FHD).

3. Ferrohydrodynamic instability problems

Ferrohydrodynamics (FHD) may be looked at from two points of view. In the first the fluids of concern possess a giant magnetic response and this gives rise to several striking phenomena with important applications, see e.g. [33-35]. The first approach treats the magnetic field as quasi-static whereas the second deals with the movement of magnetic field lines although both employ a nonlinear constitutive law linking the magnetic field to the magnetic induction, see e.g. [21,32,33].

The relevant equations of FHD according to Rosensweig [33], for the thermo-ferro convection problem are now discussed.

In the first approach to the thermo-ferro convection problem the free charge and electric displacement are neglected and the field equations are employed in a quasi-static limit in that Maxwell's equations are taken as

$$\nabla \times \mathbf{H} = 0, \quad \nabla . \mathbf{B} = 0, \tag{3.1}$$

where **H**,**B** denote magnetic field and magnetic induction. The fields **B** and **H** are, in general, related by a constitutive equation of form

$$\mathbf{B} = \mu(\mathbf{H}; T, \rho)\mathbf{H}, \tag{3.2}$$

where T and ρ are temperature and density.

The magnetization **M** is defined by the relation

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{3.3}$$

where $\mu_0(= 4\pi \times 10^{-7}$ Henry $\mathrm{m}^{-1})$ is the permeability of free space.

The fluid is taken to be an incompressible Newtonian one and then with v_i denoting the velocity field, the momentum and continuity equations are:

$$\frac{\partial v_i}{\partial t} + v_m v_{i,m} = -\frac{1}{\rho} p_{,i} + \nu \Delta v_i + f_i, \tag{3.4}$$

$$v_{i,i} = 0, \tag{3.5}$$

where p is the pressure, ρ is the (constant) density, and $\mathbf{f}$ is the total force: (in this case the ρ dependence in (3.2) may be supressed). The magnetic field effect is incorporated through the force which is comprised of a buoyancy force and a contribution from the magnetic field. The magnetic body force $\mathbf{f}_m$ may have several forms and these are described in Rosensweig [33], pp. 110–119, Landau *et al.* [17], p. 127.

For the thermal convection problem, the body force may be chosen so that the momentum equation may be written

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla\hat{p} + \nu\Delta\mathbf{v} - g\big(1 - \alpha[T - T_R]\big)\mathbf{k} + \mu_0 M\nabla H, \tag{3.6}$$

where the magnetic body force has been included and where the modified pressure $\hat{p}$ has form:

$$\hat{p} = p - \mu_0\int_0^H \left(\frac{\partial Mv}{\partial v}\right)_{H,T} dH. \tag{3.7}$$

A further equation is needed for the temperature and this is provided by the energy balance law which is taken by Finlayson [10], see also Curtis [8] and Shliomis [38], as:

$$\begin{aligned}\left[\rho C_{VH} - \mu_0\,\mathbf{H}.\left(\frac{\partial\mathbf{M}}{\partial T}\right)_{V,\mathbf{H}}\right]\frac{dT}{dt}& \\ + \mu_0 T\left(\frac{\partial\mathbf{M}}{\partial T}\right)_{V,\mathbf{H}}\cdot\frac{d\mathbf{H}}{dt} &= k\Delta T,\end{aligned} \tag{3.8}$$

where C_{VH} and k are heat capacity at constant volume and magnetic field, and (constant) thermal conductivity. Thus, the complete system of equations is (3.1)-(3.3) and (3.6)-(3.8). The terms involving the derivatives of $\mathbf{M}$ and $\mathbf{H}$ in (3.8) have a pronounced effect on the convection process and are departures from the classical theory of heat conduction due to the magnetic field effects: certainly they would considerably complicate any attempt to procure a nonlinear energy stability analysis of the above system.

Finlayson [10], Curtis [8] and Shliomis [38] concentrate on linear instability although Lalas & Carmi [16] do speak of nonlinear energy stability. They take the magnetization to have form

$$M = M_0\big[1 - \gamma(T - T_R)\big], \tag{3.9}$$

and argue that the magnetic terms may be discarded in the energy balance equation and so this may be reduced to

$$\rho C_{VH}\frac{dT}{dt} = k\Delta T, \tag{3.10}$$

a form which is certainly much more tractable from an energy stability analysis point of view.

Another approach is adopted in Straughan [42], chapter 12, where a rigorous energy analysis is developed for a material in which the permeability μ depends only on T. For this case the magnetic force assumes a form due to Korteweg and Helmholtz, namely

$$\mathbf{f}_m = \nabla\left[\frac{H^2}{2}\rho\left(\frac{\partial\mu}{\partial\rho}\right)_T\right] - \frac{H^2}{2}\nabla\mu. \tag{3.11}$$

It is worth pointing out that Rosensweig [33] remarks that there has been some concern in the literature over this representation, but he pointedly writes, ... *the Korteweg-Helmholtz expression stands out for its ability to explain experimental results in a straightforward way.*

Thus, the model studied in Straughan [42] is based on the equations,

$$\nabla\times\mathbf{H} = 0, \qquad \nabla.\mathbf{B} = 0, \tag{3.12}$$

$$\mathbf{B} = \mu(T)\mathbf{H}, \tag{3.13}$$

the equation for the temperature field being the usual one, namely,

$$\frac{\partial T}{\partial t} + v_iT_{,i} = \kappa\Delta T. \tag{3.14}$$

The momentum equation is

$$\frac{\partial v_i}{\partial t} + v_jv_{i,j} = -\omega_{,i} + \nu\Delta v_i - g\big[1 - \alpha(T - T_R)\big]k_i - \frac{1}{2}H^2\mu_{,i}, \tag{3.15}$$

where now

$$\omega = p - \frac{1}{2}H^2\rho\left(\frac{\partial\mu}{\partial\rho}\right)_T, \tag{3.16}$$

and the continuity equation is

$$v_{i,i} = 0. \tag{3.17}$$

By introducing a magnetic potential (with perturbation ϕ) and using a Boussinesq approximation, the *nonlinear* perturbation equations to the steady conduction solution are reduced in [42] to:

$$\begin{aligned} u_{i,t} + u_j u_{i,j} = & -\omega_{,i} + \Delta u_i + R\theta k_i \\ & + \delta L k_i \Big(\phi_{,z} - \delta \frac{Pr}{R}\theta\Big) - \frac{1}{2}\delta L k_i |\nabla\phi|^2 \\ & + L\delta \frac{Pr}{R}\phi_{,z}\theta_{,i} - \frac{1}{2} L\delta \frac{Pr}{R}\theta_{,i}|\nabla\phi|^2, \end{aligned} \tag{3.18}$$

$$Pr(\theta_{,t} + u_i\theta_{,i}) = -Rw + \Delta\theta, \tag{3.19}$$

$$\Delta\phi = \delta\frac{Pr}{R}\theta_{,z}, \tag{3.20}$$

$$u_{i,i} = 0, \tag{3.21}$$

where δ, L, R^2 are a depth parameter, a parameter measuring the strength of the magnetic field, and the Rayleigh number.

We do not include details of the arguments necessary to produce a nonlinear energy analysis since these are contained in [42]. However, we observe that it is necessary to employ elliptic estimates to control the nonlinearities and the energy functional required has form

$$\mathcal{E} = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}Pr\|\theta\|^2 + \frac{1}{6}Pr\|\theta^3\|^2, \tag{3.22}$$

where $\|.\|, < . >$, denote the $L^2(V)$ norm and integral over V, V being a period-cell of the perturbation.

We might point out that an analysis of thermo- ferro- hydrodynamic instability in a porous medium is given in the recent paper of Vaidyanathan *et al.* [44]. Unfortunately, no practical application of their work is discussed there.

The second model which may be thought of as dealing with a ferrohydrodynamic material is discussed in Roberts [32] and

Muzikar & Pethick [21] in connection with the stability of an equilibrium configuration in a complex magnetic material, such as a superconductor, and, in particular, to configurations of a neutron star. The deductions of these papers are based on linear analyses. The writer in [41] used a fully nonlinear approach to rederive Roberts' criterion, by using the theory of nonlinear acceleration waves.

In this model the material is compressible and so the continuity equation is

$$\rho_{,t} + (\rho v_i)_{,i} = 0. \tag{3.23}$$

The magnetic induction satisfies the equations

$$B_{i,t} + v_a B_{i,a} = v_{i,s} B_s - v_{m,m} B_i, \qquad B_{i,i} = 0, \tag{3.24}$$

and the momentum equation is

$$\rho(v_{i,t} + v_j v_{i,j}) = \sigma_{ki,k}, \tag{3.25}$$

where we have absorbed the body force in the pressure and the stress tensor is

$$\sigma_{ik} = -(p + H_r B_r)\delta_{ik} + H_i B_k. \tag{3.26}$$

The pressure has form

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}, \tag{3.27}$$

ψ being the free energy, and the relation $B_i = \mu H_i$ of freespace is replaced by

$$H_i = \rho \frac{\partial \psi}{\partial B_i}. \tag{3.28}$$

In [41] it is shown that wavespeeds of a nonlinear acceleration wave satisfy a sixth order equation which may be factorized to yield Alfven waves with speeds given by

$$U^2 = B_n^2 \frac{1}{B} \frac{\partial \psi}{\partial B}, \tag{3.29}$$

where $B = |\mathbf{B}|$ and B_n is the normal component of $\mathbf{B}$ to the wave surface, and fast and slow waves whose wavespeeds satisfy the equation

$$U^4 - U^2\left(p_\rho + B\frac{\partial\psi}{\partial B} + B_T^2\{2\frac{\rho}{B}\frac{\partial^2\psi}{\partial\rho\partial B} + \frac{1}{B}\frac{\partial\psi}{\partial B} + \frac{\partial^2\psi}{\partial B^2}\}\right) + p_\rho B_n^2\left(\frac{1}{B}\frac{\partial\psi}{\partial B} + \frac{\zeta}{\rho}B_T^2\right) - \xi^2 B_n^2 B_T^2 = 0, \tag{3.30}$$

where B_T is the tangential component of $\mathbf{B}$ to the wave surface and ζ, ξ are functions of B, ρ.

By requiring that the wavespeeds be real and positive it is then shown in [41] that a necessary condition for stability is

$$B^2\frac{\partial^2\psi}{\partial B^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right) \geq \left[B\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial B})\right]^2. \tag{3.31}$$

The above analysis is extended by Abdullah & Lindsay [1] who calculate the evolutionary behaviour of the wave amplitudes. The Bénard problem according to a viscous analogue of Roberts' [32] theory is studied by Abdullah & Lindsay [2,3].

4. Magnetohydrodynamic instability problems

The magnetohydrodynamic convection problem is very important due to its connection with the behaviour of planetary and stellar interiors. This is critically reviewed by Fearn *et al.* [9]. Chronologically this topic was developed before those of sections 2 and 3 and should, in some sense, appear first. However, we here deal with them alphabetically.

In fact, the dynamo problem has a long history and the use of integral relations in this field goes back at least as far as Chandrasekhar [7]. Backus [4] includes a survey of previous work on this subject, but his paper is very important in that he uses a variety of "energies" (cf. [42]) to establish exponential decay of various magnetic quantities. His paper is one of the first which develops a variational energy stability theory in the context of the dynamo problem.

Proctor [22] gives bounds on the temperature difference across a spherical layer and on the heat supply for a convecting sphere which he interprets as necessary conditions for dynamo action, [22] p. 139. Straughan [40] derives a lower bound for Γ_1 defined by

$$\Gamma_1 = \inf \frac{\|\mathrm{curl}\,\mathbf{B}\|^2}{\|B^2\|^2},$$

for V a bounded domain in $\mathbf{R}^3$, and applies this bound to the theory of Proctor [22]. Again, the basic tools are those of integral relations.

In fundamental work, Rionero [23-29] systematically developed the energy method in magnetohydrodynamics. A very important contribution of his is the fact that he was the first to establish the existence of a maximising solution in the energy variational problem. Rionero [24] shows how a thermal field may be incorporated into a study of nonlinear energy stability in magnetohydrodynamics, in [26] he shows how a variety of boundary conditions may be handled in the thermo-magnetohydrodynamic nonlinear stability problem, and in [28] he even shows how the Hall effect may be dealt with in the magnetohydrodynamic Bénard problem for a heat conducting viscous fluid; the last topic is highly non trivial from a nonlinear energy stability point of view. Further nonlinear analyses in magnetohydrodynamics along the lines laid out by Rionero are those of Galdi [11] and Rionero & Mulone [30].

The energy balance law in MHD is usually taken to be the convective heat equation, but there are instances in which other effects from the energy balance are important. One of these is the Thomson effect which is discussed at length by Shercliff [37]. Shercliff [37] is concerned with liquid metal coolants in nuclear reactors or molten metals in industrial metallurgy and also considers a generalization of Ohm's law to include current generation by a temperature gradient. I am unaware of nonlinear analyses emphasizing the instability properties of the Shercliff model.

As a final remark we draw attention to the interesting work of Salan & Guyon [36] on instabilities in nematic liquid crystals under the influence of magnetic fields even while heated from above.

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