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GROWTH AND DECAY OF SHOCK WAVES IN TRANSVERSELY ISOTROPIC RODS

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In this paper we study the propagation of shock waves in linear hyperelastic rods, transversely isotropic in the reference configuration.

1. Introduction.

In section 2 we briefly recall the basic equations governing the propagation of weak shock waves in hyperelastic rods and the method exposed in [1] for the integration of the vector decay equation for elastic rods with multiple wave speeds.

In section 3 we apply this method to the study of shock waves in hyperelastic transversely isotropic rods. For untwisted straight rods, twisted straight rods and untwisted planar-curved rods we obtain explicit expressions for wave speeds and wave mode vectors. In particular we emphasize their dependence on the geometrical properties of the rod.

An exhaustive treatment of the results exposed in this paper and further results for twisted spatial rods can be found in [4].

2. Formulation of the basic equations.

We briefly recall the equations governing the propagation of shock waves, within the framework of the isothermal linear theory of hyperelastic rods, according to the notion of a rod as a Cosserat continuum.

We denote by $\mathcal{E}$ and $\mathcal{V}$ the three - dimensional Euclidean point space and its associated translation space of vectors. We set: $\mathcal{U} = \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.

The Euclidean inner product $<\,,\,>$ on $\mathcal{V}$ induces a Euclidean inner product $\{\,,\,\}$ on $\mathcal{U}$, defined by $\{u, v\} = \sum_{i=1}^{3} < \mathbf{u}_i, \mathbf{v}_i >$, $\forall u, v \,\varepsilon\, \mathcal{U}$, $u = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, $v = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

We denote by $\mathbf{r} = \mathbf{r}(s,t)$ and $\mathbf{d}_1 = \mathbf{d}_1(s,t), \mathbf{d}_2 = \mathbf{d}_2(s,t)$ the position vector of a curve representing the rod axis and a pair of directors, respectively; s is a material coordinate and t is time.

If we set $p = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2)$, the generalized position vector $p = p(s,t) \,\varepsilon\, \mathcal{U}$ describes the state of the rod.

We introduce the decomposition $p = \hat{p} + u$, where $\hat{p}$ is an arbitrary equilibrium state and u a small deformation superimposed upon it.

The propagation condition and decay-induction equation for linear shock waves are, respectively

$$(2.1) \qquad (\mathcal{K}^{-1}\mathcal{A} - V^2\mathcal{I})[u'] = 0$$

$$(2.2) \qquad (\mathcal{K}^{-1}\mathcal{A} - V^2\mathcal{I})[u''] + (\mathcal{K}^{-1}\mathcal{A}' - \mathcal{K}^{-1}\mathcal{B} + \tilde{V}\mathcal{I})[u'] + 2V[\tilde{u'}] = 0.$$

In the above formulas V is the wave speed, [ ] the jump of a quantity across the point of discontinuity, $\mathcal{K}$ the inertia tensor, $\mathcal{I}$ the unit tensor; moreover we have set: $\cdot' \equiv \dfrac{\partial\cdot}{\partial s}, \tilde{\cdot} \equiv V\cdot'$. $\mathcal{A}$ and $\mathcal{B}$ are defined in terms of the strain energy density $W = W(p, p'; s)$ as follows: $\mathcal{A} = W_{,p'p'}\,|_{p=\hat{p}}$, $\mathcal{B} = W_{,pp'}\,|_{p=\hat{p}} - (W_{,pp'})^T\,|_{p=\hat{p}}$.

The squares of the wave speeds are solutions of the characteristic equation

$$(2.3) \qquad \det\,(\mathcal{K}^{-1}\mathcal{A} - V^2\mathcal{I}) = 0.$$

We introduce a second inner product $\{\ ,\ \}_{\mathcal{K}}$ in $\mathcal{U}$: $\forall u,\ v \in \mathcal{U}$, $\{u,v\}_{\mathcal{K}} = \{u, \mathcal{K}v\}$; for each root V_α^2 of (2.3) we denote by ${}^\alpha\mathcal{U}_s \subset \mathcal{U}$ the eigenspace associate to V_α^2 and by ${}^\alpha\mathcal{P}_s : \mathcal{U} \to {}^\alpha\mathcal{U}_s$ the orthogonal projection with respect to $\{\ ,\ \}_{\mathcal{K}}$. By projecting equation (2.2) on ${}^\alpha\mathcal{U}_s$ for each V_α^2 and setting $[u'] = a$, we obtain

$$(2.4) \qquad 2V_\alpha^{2\,\alpha}\mathcal{P}_s\, a_\alpha' + {}^\alpha\mathcal{P}_s(\mathcal{K}^{-1}\,\mathcal{A}' - \mathcal{K}^{-1}\,\mathcal{B} + V_\alpha V_\alpha'\mathcal{I})a_\alpha = 0$$

(α non summed).

In the case of multiple eigenvalues of (2.3), the decay equation (2.4) determines both the direction (indetermined by the propagation condition) and the amplitude of the wave mode vector $a_\alpha \in {}^\alpha\mathcal{U}_s$, when we assigne a_{α_0} at $s = 0$.

In [1] the solution of (2.4) is written in the form

$$(2.5) \qquad a = {}_\alpha\oplus a_\alpha = {}_\alpha\oplus{}^\alpha\mathcal{L}a_{\alpha 0} = {}_\alpha\oplus{}^\alpha\mathcal{G}(\frac{V_{\alpha_0}}{V_\alpha})^{\frac{3}{2}}\,{}^\alpha\mathcal{I}a_{\alpha 0};$$

${}^\alpha\mathcal{I}$ is the identity on ${}^\alpha\mathcal{U}_0$, $\left(\dfrac{V_{\alpha 0}}{V_\alpha}\right)^{\frac{3}{2}}{}^\alpha\mathcal{I}$ is a dilatation on ${}^\alpha\mathcal{U}_0$, ${}^\alpha\mathcal{G}$ is an orthonormal transformation with respect to $\{\ ,\ \}_{\mathcal{K}}$. The steps which lead to the construction of ${}^\alpha\mathcal{G}$ are the following:

(i) we denote by ω_I $(I = 1, 2, \dots, 9)$ an orthonormal basis of wave mode vectors with respect to $\{\ ,\ \}_{\mathcal{K}}$. We set

$$(2.6) \qquad \omega_I' = \Gamma_I^{\cdot J}\omega_J \quad , \omega_I = Q_{\cdot I}^{J0}\omega_{J0};$$

the coefficients $\Gamma_I^{\cdot J}$ describe the rod geometry; $Q_{\cdot I}^{J0}$ is an orthonormal transformation with respect to $\{\ ,\ \}_{\mathcal{K}}$. From $(2.6)_{1,2}$, we obtain a matrix differential equation for Q

$$(2.7) \qquad (Q_{\cdot I}^{I0})' = Q_{\cdot K}^{I0}\Gamma_I^{\cdot K} \quad \text{(initial conditions } Q_{\cdot I}^{I0} = \delta_{\cdot I}^{I0} \text{ at } s = 0).$$

(ii) If we set ${}^\alpha\mathcal{G} = {}^\alpha\mathcal{G}_{\cdot j}^{i}\omega_i \otimes \omega^j$ $(i, j = 1, 2\ \dots,\ \dim({}^\alpha\mathcal{U}))$, equation (2.4), with the use of (2.5), gives the following matrix differential equation for the components ${}^\alpha\mathcal{G}_{\cdot j}^{i}$

$$(2.8) \qquad ({}^\alpha\mathcal{G}_{\cdot j}^{i})' + \frac{1}{2}\left\{\Gamma_k^{\cdot i} - \Gamma_{\cdot k}^{i} - \frac{1}{V_\alpha^2}(\mathcal{K}^{-1})_{\cdot K}^{i}\mathcal{B}_{\cdot k}^{K}\right\}{}^\alpha\mathcal{G}_{\cdot j}^{k} = 0,$$

with initial conditions ${}^{\alpha}\mathcal{G}^{i}_{\cdot j} = \delta^{i}_{\cdot j}$ at $s = 0$.

We set $\mathcal{G} = {}_{\alpha}\oplus{}^{\alpha}\mathcal{G}$; for distinct speeds $\mathcal{G}^{I}_{\cdot J} = \delta^{I}_{\cdot J} \Rightarrow \mathcal{G}^{I0}_{\cdot J} = Q^{I0}_{\cdot J}$; for multiple speeds, since $Q =_{\alpha} \oplus^{\alpha} Q$, from $(2.6)_2$ we have $\omega_i = {}^{\alpha}Q^{j0}_{\cdot i}\omega_{j0} \Rightarrow {}^{\alpha}\mathcal{G}^{i0}_{\cdot j} = {}^{\alpha}Q^{i0}_{\cdot i}\, {}^{\alpha}\mathcal{G}^{i}_{\cdot j}$.

3. Shock waves in transversely isotropic rods.

We study shock waves in a linear homogeneous hyperelastic rod, transversely isotropic in the reference configuration. The explicit form for the appropriate strain energy density W can be found in [2]. For the inertia tensor $\mathcal{K}$ we adopt the form

$$\mathcal{K}^{I}_{\cdot J} = \rho \ \mathrm{diag}(1,1,1,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha), \tag{3.1}$$

where $\rho =$ const. is the linear mass density and $\alpha =$ const. denotes the two equal inertia moments of the section.

In order to emphasize the effects of the rod geometry on wave speeds and wave modes, following [3] we introduce the skew-symmetric connection coefficients

$$\begin{gathered} \wedge_{12} =< \hat{\mathbf{d}}_1', \hat{\mathbf{d}}_2 >= \tau + f', \\ \wedge_{13} =< \hat{\mathbf{d}}_1', \hat{\mathbf{r}}' >= -k \sin f, \wedge_{23} =< \hat{\mathbf{d}}_2', \hat{\mathbf{r}}' >= k \cos f; \end{gathered} \tag{3.2}$$

k and τ are the principle curvature and torsion, respectively, while f' is the relative twist.

The characteristic equation leads to single or double speeds; then we use the method exposed in section 2. We define a basis of wave mode vectors, orthonormal with respect to $\{\ ,\ \}_{\mathcal{K}}$

$$\begin{gathered} \omega_I = \{0, \cdots, 0, (\alpha_I \rho)^{-\frac{1}{2}}, 0, \cdots, 0\} \\ (\alpha_I = 1 \text{ for } I = 1,2,3; \alpha_I = \alpha \text{ for } I = 4,5,\cdots,9); \end{gathered} \tag{3.3}$$

from $(2.6)_1, (3.2)$ and (3.3) we have: $\Gamma^{\cdot J}_{I} = -\Gamma^{\cdot I}_{J}$ and

$$\Gamma^{\cdot 2}_{1} = \Gamma^{\cdot 5}_{4} = \Gamma^{\cdot 8}_{7} = \wedge_{12}, \Gamma^{\cdot 3}_{1} = \Gamma^{\cdot 6}_{4} = \Gamma^{\cdot 9}_{7} = \wedge_{13}, \Gamma^{\cdot 3}_{2} = \Gamma^{\cdot 6}_{5} = \Gamma^{\cdot 9}_{8} = \wedge_{23}.$$

For different choices of $\Gamma^{\cdot J}_{I}$, we give the following results:

the wave speeds and their multiplicity, the reference map Q satisfying equation (2.7) and the transformation $\mathcal{G}$ in (2.8). In the following the k's are material constants appearing in the strain energy function W. For brevity, the explicit expression for some wave speeds is omitted; for more detailed results we refer to [4].

a) Untwisted straight rods: $\wedge_{12} = \wedge_{13} = \wedge_{23} = 0$

$$(3.4) \quad \begin{cases} \text{shear waves}: V_1^2 = V_2^2 = \dfrac{k_5}{\rho} \\ \text{extension wave}: V_3^2 = \dfrac{4k_3}{\rho} \\ \text{bending waves}: V_6^2 = V_9^2 = \dfrac{k_{15}}{\rho\alpha} \\ \text{cross-sectional extensional and shear waves}: \\ V_4^2 = \dfrac{k_{10}}{\rho\alpha} + \dfrac{k_{17}}{2\rho\alpha}, V_5^2 = V_8^2 = \dfrac{k_{10}}{\rho\alpha} - \dfrac{k_{17}}{2\rho\alpha}, V_7^2 = \dfrac{2k_{12}}{\rho\alpha} - \dfrac{k_{10}}{\rho\alpha} + \dfrac{k_{17}}{2\rho\alpha} \end{cases}$$

$$(3.5) \qquad Q^{I0}_{.J} = \delta^{I0}_{.J} \quad ; \quad \mathcal{G}^{I}_{.J} = \delta^{I}_{.J}.$$

b) Twisted straight rods: $\wedge_{12} = f', \wedge_{13} = \wedge_{23} = 0$

$$(3.6) \quad \begin{cases} \text{shear waves: } V_1^2 = V_2^2 \text{ (twist dependent)} \\ \text{extension wave: } V_3^2 = \dfrac{4k_3}{\rho} \\ \text{bending waves: } V_6^2 = V_9^2 \text{ (twist dependent)} \\ \text{cross-sectional extensional and shear waves: as in } (3.4)_4 \end{cases}$$

$$(3.7) \qquad Q^{I0}_{.J} = \operatorname{diag}(\bar{Q}, 1, \bar{Q}, 1, \bar{Q}, 1),$$

$$\bar{Q} = \begin{pmatrix} \cos \displaystyle\int_0^s f'(x)dx & -\sin \displaystyle\int_0^s f'(x)dx \\ \sin \displaystyle\int_0^s f'(x)dx & \cos \displaystyle\int_0^s f'(x)dx \end{pmatrix}$$

$$(3.8) \qquad \mathcal{G}^{I}_{.J} = \operatorname{diag}(\bar{F}, 1, 1, \bar{G}, 1),$$

$$\bar{F} = \begin{pmatrix} \cos A(s) & \sin A(s) \\ -\sin A(s) & \cos A(s) \end{pmatrix},$$

$$\bar{G} = \begin{pmatrix} \cos B(s) & 0 & 0 & -\sin B(s) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin B(s) & 0 & 0 & \cos B(s) \end{pmatrix},$$

where

$$A(s) = \int_0^s f' \left[1 + \frac{k_5 + k_{15}(f')^2}{\rho V_1^2} \right] dx$$

and

$$B(s) = \frac{2k_{10} - 4k_{12} - k_{17}}{4\rho\alpha V_5^2} \int_0^s f'(x)dx.$$

c) **Untwisted planar-curved rods:** $\Lambda_{13} = -k, \ \Lambda_{12} = \Lambda_{23} = 0$

(3.9)
$$\begin{cases} \text{shear waves: } V_1^2 = V_2^2 = \dfrac{k_5}{\rho} \\ \text{extension wave: } V_3^2 \text{ (curvature dependent)} \\ \text{bending wave: } V_6^2 \text{ (curvature dependent)} \\ \text{bending wave: } V_9^2 = \dfrac{k_{15}}{\rho\alpha} \\ \text{cross-sectional extensional and shear waves: as in } (3.4)_4 \end{cases}$$

(3.10)
$$Q^{I0}_{\cdot J} = \text{diag}\,(\bar{R}, \bar{R}, \bar{R}) \quad ,$$

$$\bar{R} = \begin{pmatrix} \cos \int_0^s k(x)dx & 0 & \sin \int_0^s k(x)dx \\ 0 & 1 & 0 \\ -\sin \int_0^s k(x)dx & 0 & \cos \int_0^s k(x)dx \end{pmatrix}$$

(3.11)
$$\mathcal{G}^I_{\cdot J} = \delta^I_{\cdot J}.$$

Remarks.

Case a): since the rod is uniform, the wave strengths remain constant. In agreement with our hypothesis of transverse isotropy, we have no coupling between shear and bending effects.

Case b): the shearing wave mode vectors are twist dependent and coupled, while the bending wave mode vectors are uncoupled, according to the constitutive equations assumed. Moreover, we observe that the twist affects the mode vectors of cross-sectional extensional and shear waves, while the corresponding speeds are the same as in case a). Finally, for a rod with uniform twist ($f'' = 0$) all speeds are constant and there is not growth or decay.

Case c): the wave speeds and wave mode vectors of cross-sectional extensional and shear waves are the same as in case a), but in virtue of (3.10) the components $\mathcal{G}^{I0}_{:J}$ of $\mathcal{G}$ are different. Finally, if $k' = 0$ all speeds are constant and the wave strengths are also constant.

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