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COMMON FIXED POINTS IN ORDERED BANACH SPACES

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In this paper we introduce the notion of g-weak isotone mappings in an ordered Banach space and we extend some common fixed point theorems of Dhage, O'Regan and Agarwal [1].

1. Introduction

In [1] Dhage, O'Regan and Agarwal introduced the class of weak isotone mappings and the class of countably condensing mappings in an ordered Banach space and they prove some common fixed point theorems for weak isotone mappings. In this paper we introduce the notion of g-weak isotone mappings which allows us to generalize some common fixed point theorems of [1].

We recall the definition of ordered Banach spaces.

Let B be a real Banach space with norm $\|\cdot\|$ and let P be a subset of B. By θ we denote the zero element of B and by $IntP$ the interior of P. The subset P is called a cone if:

1. P is closed, nonempty, and $P \neq \{\theta\}$;
2. $a,b \in \mathbb{R},\ a,b \geq 0,\ x,y \in P \Rightarrow ax+by \in P$;
3. $x,-x \in P \Rightarrow x = \theta$.

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Given a cone $P \subset B$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ if and only if $y - x \in IntP$. A cone P is called normal if there is a real number $c > 0$ such that for all $x, y \in B$, $\theta \leq x \leq y \implies \| x \| \leq c \| y \|$. The smallest positive number c satisfying the above condition is called the normal constant of P. A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in B$, then there is $x \in B$ with $\|x_n - x\| \to 0$, as $n \to +\infty$. Equivalently, a cone P is regular if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

2. Preliminaries

In the following we always assume that B is a Banach space, P is a cone in B and $\leq$ is a partial ordering with respect to P.

Definition 2.1. Let B be an ordered Banach space. A mapping $f : B \to B$ is said to be isotone increasing if for $x, y \in B$ and $x \leq y$ we have $f(x) \leq f(y)$.

Definition 2.2. Let B be an ordered Banach space and let $f, g : B \to B$ be two mappings. It is said that f and g are weakly isotone increasing if $f(x) \leq g(f(x))$ and $g(x) \leq f(g(x))$ for all $x \in B$. Similarly f and g are said to be weakly isotone decreasing if $f(x) \geq g(f(x))$ and $g(x) \geq f(g(x))$ for all $x \in B$. If f and g are either weakly isotone increasing or weakly isotone decreasing, then it is said that f and g are weakly isotone.

Definition 2.3. Let B be an ordered Banach space and let $f, g : B \to B$ be two mappings. The mapping f is said to be g-weakly isotone increasing if $f(x) \leq g(f(x)) \leq f(g(f(x)))$ for all $x \in B$. Similarly f is said to be g-weakly isotone decreasing if $f(x) \geq g(f(x)) \geq f(g(f(x)))$ for all $x \in B$. Also the mapping f is called g-weakly isotone if it is either g-weakly isotone increasing or g-weakly isotone decreasing.

Remark 2.1. Let B be an ordered Banach space. Two mappings $f, g : B \to B$ are weakly isotone if and only if f is g-weakly isotone and g is f-weakly isotone. But if f is g-weakly isotone this doesn't imply that f and g are weakly isotone, as follows by the next example.

Example 2.2. Let $B = \mathbb{R}^2$ and $P = \{x = (z, z) : z \in \mathbb{R}, z \geq 0\}$. Let $f, g : B \to B$ be defined by $f(x) = (1, 1)$ and let $g(x) = x$ for all $x \in B$, then f is g-weakly isotone. But f and g are not weakly isotone.

Definition 2.4. Let B be an ordered Banach space and let $x \in B$. A mapping $f : B \to B$ is said to be monotone-continuous in x if $f(x_n) \to f(x)$ for each increasing or decreasing sequence (x_n) that converges to x.

Condition D_Q: Let $Q \subset B$. Two mappings $f, g : Q \to Q$ are said to satisfy condition D_Q if for any countable set A of Q and for any fixed $a \in Q$ the condition

$$A \subset \{a\} \cup f(A) \cup g(A)$$

implies $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of A.

Weak-condition D_Q: Let $Q \subset B$. Two mappings $f, g : Q \to Q$ are said to satisfy weak-condition D_Q if for any monotone sequence (x_n) and for any fixed $a \in Q$ the condition

$$\{x_n\} \subset \{a\} \cup f(\{x_n\}) \cup g(\{x_n\})$$

implies (x_n) is convergent.

Remark 2.3. Let B be an ordered Banach space and let $Q \subset B$. If the mappings $f, g : Q \to Q$ satisfy condition D_Q, then they satisfy weak-condition D_Q.

For any subset A of B, $\operatorname{diam}(A)$ denotes the diameter of A, that is

$$\operatorname{diam}(A) := \sup\{d(x,y) : x, y \in A\},$$

and $\circ$ denotes the composition of mappings. The Kuratowski measure of noncompactness for a bounded subset A of B is defined by

$$\alpha(A) = \inf\{r > 0 : A \subset \bigcup_{i=1}^{n} A_i \text{ and } \operatorname{diam}(A_i) \leq r \text{ for } i \in \{1, 2, \ldots n\}\}.$$

Definition 2.5. Let B be an ordered Banach space and let $X \subset B$. Two mappings $f, g : X \to B$ are said to be a monotone-condensing pair if $f(X)$ and $(g \circ f)(X)$ are bounded and for every bounded monotone sequence (x_n) such that $\alpha(\{x_n\}) > 0$ and $\alpha(f(\{x_n\})) > 0$ we have $\alpha((g \circ f)(\{x_n\})) < \alpha(\{x_n\})$.

Definition 2.6. Let B be an ordered Banach space and let $X \subset B$. A mapping $f : X \to B$ is said to be countably condensing if $f(X)$ is bounded and if for every countably bounded set $A \subset X$ such that $\alpha(A) > 0$ we have $\alpha(f(A)) < \alpha(A)$.

Remark 2.4. Let B be an ordered Banach space and let $X \subset B$. If the mappings $f, g : X \to B$ are countably condensing, then they are a monotone-condensing pair.

3. Common fixed point theorems

Let B be an ordered Banach space and let X be a closed subset of B. Let $f,g : X \to X$ be two mappings such that f is g-weakly isotone. Given $x_0 \in X$ we define a sequence (x_n) in X as follows:

$$x_{2n-1} = f(x_{2n-2}), \quad x_{2n} = g(x_{2n-1})$$

for $n > 0$. We say that (x_n) is an f-g-sequence with initial point x_0. Later on, we denote by $F(f,g)$ the set of common fixed points of f and g.

Theorem 3.1. *Let X be a closed subset of an ordered Banach space B and let $f,g : X \to X$ be two monotone-continuous mappings. If f is a g-weakly isotone mapping and if f and g satisfy weak-condition D_X, then f and g have a common fixed point. Besides, for every $x_0 \in X$, we have $\lim_{n\to+\infty} x_n \in F(f,g)$ for every f-g-sequence with initial point x_0.*

Proof. Assume that f is g-weakly isotone increasing. Then every f-g-sequence (x_n) with initial point $x_0 \in X$ is increasing. In fact, $x_1 = f(x_0) \leq g(f(x_0)) = g(x_1) = x_2 \leq f(x_2) = x_3$ and so we have

$$x_1 \leq x_2 \leq x_3 \leq \cdots .$$

Now

$$\{x_n\} = \{x_1\} \cup \{x_1, x_3, ...\} \cup \{x_2, x_4, ...\} \subset \{x_1\} \cup f(\{x_n\}) \cup g(\{x_n\}).$$

For weak-condition D_X the sequence (x_n) converges to some $x \in X$. Since f,g are monotone-continuous we deduce

$$x = \lim_{n\to+\infty} x_{2n-1} = \lim_{n\to+\infty} f(x_{2n-2}) = f(x)$$

and

$$x = \lim_{n\to+\infty} x_{2n} = \lim_{n\to+\infty} g(x_{2n-1}) = g(x).$$

It follows that $x = f(x) = g(x)$ and thus x is a common fixed point for f and g. The case when f is g-weakly isotone decreasing is similar. □

Example 3.2. Let $B = \mathbb{R}^2$ and $P = \{x = (z,z) : z \in \mathbb{R}, z \geq 0\}$. Let

$$X = \{x \in B : \| x \| \leq 2\} \cup \{(u,0) : u \in \mathbb{R}, u > 2\}$$

and $f,g : X \to X$ be defined by $f(x) = (1,1)$ if $\| x \| \leq 2$ and $f(x) = \theta$ if $\| x \| > 2$, and $g(x) = x$ for all $x \in X$. The mappings f and g satisfy weak-condition D_X, but do not verify the condition D_X. Besides the mappings f and g have the common fixed point $x = (1,1)$.

Corollary 3.3. *Let X be a closed subset of an ordered Banach space B and let $f,g : X \to X$ be two monotone-continuous mappings. If f is a g-weakly isotone mapping, f and g satisfy weak-condition D_X and*

$$\lim_{n\to+\infty} \operatorname{diam}((g\circ f)^n(X)) = 0,$$

then f and g have a unique common fixed point.

Theorem 3.4. *Let X be a closed subset of an ordered Banach space B and let $f,g : X \to X$ be monotone-continuous mappings and a monotone-condensing pair. If f is a g-weakly isotone mapping, then f and g have a common fixed point. Besides, for every $x_0 \in X$, we have $\lim_{n\to+\infty} x_n \in F(f,g)$ for every f-g-sequence with initial point x_0.*

Proof. Assume that f is g-weakly isotone increasing. Then every f-g-sequence (x_n) with initial point $x_0 \in X$ is increasing. From

$$\{x_2,x_4,...\} \subset \{x_2\} \cup (g\circ f)(\{x_2,x_4,...\}),$$

we obtain that the monotone sequence (x_{2n}) is bounded and thus $\alpha((g\circ f)(\{x_{2n}\})) < \alpha(\{x_{2n}\})$, if $\alpha(\{x_{2n}\}) > 0$ and $\alpha(f(\{x_{2n}\})) > 0$. It follows that $\alpha(\{x_{2n}\}) = 0$ and consequently there exists $x \in X$ such that $x_{2n} \to x$. Since (x_n) is increasing we have $x_n \to x$. Now, the monotone-continuity of f and g assure that x is a common fixed point for f and g. □

With the same argument of Theorem 3.1 we can prove the following theorem.

Theorem 3.5. *Let X be a closed subset of an ordered Banach space B and let $f,g : X \to X$ be monotone-continuous mappings. Assume that the partial ordering $\leq$ is induced by a regular order cone and that f is a g-weakly isotone mapping. Assume moreover that f and g satisfy the following condition:*

(i) for any monotone sequence (x_n) and for any fixed $a \in Q$ the condition

$$\{x_n\} \subset \{a\} \cup f(\{x_n\}) \cup g(\{x_n\})$$

implies (x_n) is order bounded.

Then f and g have a common fixed point. Besides, for every $x_0 \in X$, we have $\lim_{n\to+\infty} x_n \in F(f,g)$ for every f-g-sequence with initial point x_0.

4. Common fixed point for multivalued mappings

Let B be an ordered Banach space and let 2^B be the family of all nonempty subsets of B. Let $X, Y \in 2^B$. Then $X \leq Y$ means $x \leq y$ for all $x \in X$ and $y \in Y$.

Definition 4.1. A mapping $F : B \to 2^B$ is said to be isotone increasing if for $x, y \in B$ with $x \leq y$ we have $F(x) \leq F(y)$.

Definition 4.2. Let B be an ordered Banach space and let $F, G : B \to 2^B$ be mappings. The mapping F is said to be G-weakly isotone increasing if $F(x) \leq G(y) \leq F(z)$ for all $x \in B$, $y \in F(x)$ and $z \in G(y)$. F is said to be G-weakly isotone decreasing if $F(x) \geq G(y) \geq F(z)$ for all $x \in B$, $y \in F(x)$ and $z \in G(y)$. The mapping F is said to be G-weakly isotone if it is either G-weakly isotone increasing or G-weakly isotone decreasing.

Definition 4.3. A mapping $F : B \to 2^B$ is said to be monotone-closed if for each increasing or decreasing sequence $(x_n) \subset B$ with $x_n \to x_0$, and for each sequence (y_n) with $y_n \in F(x_n)$ and $y_n \to y_0$, we have $y_0 \in F(x_0)$.

Condition D_Q: Let $Q \subset B$. Two mappings $F, G : Q \to 2^Q$ are said to satisfy condition D_Q if for any countable set A of Q and for any fixed $a \in Q$ the condition

$$A \subset \{a\} \cup F(A) \cup G(A)$$

implies $\overline{A}$ is compact, where $F(A) = \cup_{x \in A} F(x)$.

Weak-condition D_Q: Let $Q \subset B$. Two mappings $F, G : Q \to 2^Q$ are said to satisfy weak-condition D_Q if for any monotone sequence (x_n) and for any fixed $a \in Q$ the condition

$$\{x_n\} \subset \{a\} \cup F(\{x_n\}) \cup G(\{x_n\})$$

implies that (x_n) is convergent.

Let B be an ordered Banach space and let X be a closed subset of B. Given $F, G : X \to 2^X$ such that F is a G-weakly isotone mapping, and given $x_0 \in X$ we define a sequence (x_n) in X as follows:

$$x_{2n-1} \in F(x_{2n-2}), \quad x_{2n} \in G(x_{2n-1})$$

for $n > 0$. We say that (x_n) is an F-G-sequence with initial point x_0.

With the same argument of Theorem 3.1 we can prove the following theorem.

Theorem 4.1. *Let X be a closed subset of an ordered Banach space B and let $F, G : X \to 2^X$ be two monotone-closed mappings. If F is a G-weakly isotone mapping and if F and G satisfy weak-condition D_X, then F and G have a common fixed point.*

Definition 4.4. Let X be a closed subset of an ordered Banach space B. It is said that $F, G : X \to 2^X$ are a pair of monotone-condensing mappings if $F(X)$ and $(G \circ F)(X)$ are bounded and for every bounded monotone sequence (x_n) such that $\alpha(\{x_n\}) > 0$ and $\alpha(F(\{x_n\})) > 0$ we have $\alpha((G \circ F)(\{x_n\})) < \alpha(\{x_n\})$.

Theorem 4.2. *Let X be a closed subset of an ordered Banach space B and let $F, G : X \to 2^X$ be monotone-closed and a pair of monotone-condensing mappings . If F is a G-weakly isotone mapping, then F and G have a common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Then every F-G-sequence (x_n) with initial point x_0 is monotone. From

$$\{x_2, x_4, \ldots\} \subset \{x_2\} \cup (G \circ F)(\{x_2, x_4, \ldots\}),$$

we obtain that the monotone sequence (x_{2n}) is bounded and thus

$$\alpha((G \circ F)(\{x_{2n}\})) < \alpha(\{x_{2n}\}) \quad \text{if} \quad \alpha(\{x_{2n}\}) > 0 \quad \text{and} \quad \alpha(F(\{x_{2n}\})) > 0.$$

It follows that $\alpha(\{x_{2n}\}) = 0$ and consequently there exists $x \in X$ such that $x_{2n} \to x$. Since (x_n) is monotone we have $x_n \to x$. Now, since F, G are monotone-closed and since $x_{2n+1} \in F(x_{2n})$ and $x_{2n} \in G(x_{2n-1})$, we deduce that $x \in F(x)$ and $x \in G(x)$. Thus x is a common fixed point for F and G. □

5. Applications

Let $\mathbb{R}$ be the real line, E be a Banach space with norm $\|.\|$ and let $C(E)$ denote the class of all nonempty closed subsets of E. Given a closed and bounded interval $J = [0,1] \subset \mathbb{R}$, consider the integral inclusions

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s) S(s, x(s))\, ds \tag{1}$$

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s) T(s, x(s))\, ds \tag{2}$$

for $t \in J$, where $\sigma : J \to J$, $q : J \to E$, $k : J \times J \to \mathbb{R}$ are continuous and $S, T : J \times E \to C(E)$. By a common solution for the integral inclusions (1) and (2), we mean a continuous function $x : J \to E$ such that

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s) v_1(s)\, ds$$

and

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s) v_2(s)\, ds$$

for some $v_1, v_2 \in B(J,E)$ satisfying $v_1(t) \in S(t,x(t))$ and $v_2(t) \in T(t,x(t))$, for all $t \in J$, where $B(J,E)$ is the space of all E-valued Bochner integrable functions on J.

In [2] Turkoglu and Altun proved an existence theorem of common solutions for the integral inclusions (1) and (2) via, a common fixed point theorem of Dhage, O'Regan and Agarwal [1]. A similar result we can obtain as consequence of Theorem 4.2.

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