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AN INEQUALITY FOR CONVEX FUNCTIONALS AND ITS APPLICATION TO A MAXWELLIAN GAS

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We study the trend towards equilibrium of the solution of the spatially homogeneous Boltzmann equation for a gas of Maxwellian molecules. The cases of axially symmetric and plane initial densities are invéstigated. In these situations, the strong L_1 convergence to equilibrium follows by a suitable use of some convex and isotropic functionals, with monotonic behaviour in time along the solution. The initial density is required to have finite energy and entropy. It is shown that the functionals satisfy a common convolution inequality.

1. Introduction.

One of the most interesting problems in the kinetic theory of rarefied gases is represented by the study of the asymptotic behaviour of the solution of the Boltzmann equation.

In the spatially homogeneous case, as far as hard intermolecular potentials are concerned, the problem has been satisfactory solved by Arkeryd [1] and Gustafsson [5]. In paper [1] Arkeryd proved exponential convergence to equilibrium together with stability for arbitrary initial densities with finite entropy and sufficiently many moments (higher than two). His proof was based on the spectral properties of the linearized collision operator.

The paper by Gustafsson studied the L_p-behaviour in time ($p \geq 1$) of the solution to the Boltzmann equation, proving global boundedness in time for the L_p moments that initially exist, and strong convergence in L_p to equilibrium. The proof was based on the translation continuity of the solution.

With regards to the Boltzmann equation for Maxwellian molecules, the first result about convergence to equilibrium, namely the exponential convergence of higher moments to the corresponding moments of the equilibrium density, was obtained by Truesdell [13]. Some year later, McKean [7] proved the strong convergence to equilibrium and studied the speed of approach for the Kac's caricature of the Maxwellian gas. More recently Tanaka [9] investigated the trend to equilibrium and the stability properties of the solution to the Boltzmann equation for a Maxwell gas without any cut-off, in a metric equivalent to the weak* convergence of measures. Finally, in [11] we gave a new proof of the weak trend towards equilibrium for a gas with finite initial energy, and a proof of the strong trend if in addition the initial entropy is finite, and the model is two-dimensional.

To obtain the result of [11], we used some arguments already contained in the paper of McKean [7], as well as new ideas we introduced in [10] to give a new proof of the central limit theorem of probability theory. The main result was based on the monotonicity property of the functional

$$J(f) = \int_{\mathbf{R}^d} \frac{(\nabla f)^2}{f} d\mathbf{v} \tag{1}$$

which, evaluated in correspondance to the solution of the Boltzmann equation for Maxwellian pseudomolecules with planar velocities has been discovered to decrease with time [12].

From the previous history of the problem, it appears clearly that the Boltzmann H-functional is not the unique functional which is monotonically decreasing in time (at least for a Maxwell gas). In the present talk, we will outline the importance of finding decreasing functionals to solve the problem of the trend towards equilibrium.

2. Generalization of the Boltzmann H-theorem.

When the velocity space has dimension d, a gas of Maxwellian molecules is constituted by molecules which repel each other with a force inversely proportional to the $(2d-1)$-th power of their distance. The evolution equation for the distribution function $f(\mathbf{v},t)$, $t \in \mathbf{R}^+$ reads [2], [14]

$$(2) \qquad \frac{\partial f}{\partial t} = \int_{\mathbf{R}^d \times S^{d-1}} d\mathbf{w} d\mathbf{n} g\left(\frac{\mathbf{qn}}{q}\right) \{f(\mathbf{v}_1)f(\mathbf{w}_1) - f(\mathbf{v})f(\mathbf{w})\}$$

In the above expression, $\mathbf{w} \in \mathbf{R}^d$, $\mathbf{n}$ is a unit vector, so that $d\mathbf{n}$ is an element of area of the surface of the unit sphere S^{d-1} in $\mathbf{R}^d$. Moreover $\mathbf{q} = \mathbf{v} - \mathbf{w}$ is the relative velocity, whereas $(\mathbf{v}_1, \mathbf{w}_1)$ represent the post-collisional velocities, defined by

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{v} + \mathbf{w} + q\mathbf{n})$$

$$\mathbf{w}_1 = \frac{1}{2}(\mathbf{v} + \mathbf{w} - q\mathbf{n})$$

The function $g(\nu)$, when $d = 3$, has a singularity of the form $(1-\nu)^{-\frac{5}{4}}$ as ν tends to 1 [14]. To eliminate this difficulty, one requires generally a stronger condition on g, namely the condition that $g(\nu)$ should be summable in $(-1,1)$. One time this "cut-off" is introduced, we say that we are dealing with Maxwellian pseudomolecules.

It is known that equation (2) can by simplified by passing to Fourier transforms [2]. Denoting by ϕ the characteristic function

$$(3) \qquad \phi(\mathbf{k},t) = \int_{\mathbf{R}^d} d\mathbf{v} f(\mathbf{v},t) e^{-i\mathbf{kv}}$$

we obtain, instead of equation (2) the simpler equation

$$(4) \qquad \frac{\partial \phi}{\partial t} = \int_{S^{d-1}} d\mathbf{n} g\left(\frac{\mathbf{kn}}{k}\right) \left\{\phi\left(\frac{\mathbf{k} + k\mathbf{n}}{2}\right) \phi\left(\frac{\mathbf{k} - k\mathbf{n}}{2}\right) - \phi(0)\phi(\mathbf{k})\right\}$$

which is considerably easier to handle, thanks to the reduction in the multeplicity of the integration in the collisional operator.

Since we are interested in finding the functionals $F(f)$ decreasing monotonically in time along the solution $f(\mathbf{v},t)$ to the Boltzmann equation (2), we will derive sufficient conditions for the corresponding functionals $\widehat{F}(\phi)$ defined by the formula

$$\widehat{F}(\phi) = F(f)$$

where ϕ and f are the solutions to (2) and (4) respectively, thus related by (3).

We assume that the functionals to be considered are the convex ones, and that they are invariant under the translations and the rotations in the velocity space $\mathbf{R}^d$.

For the sake of simplicity, we also assume the normalized conditions

$$\begin{aligned} &\int_{\mathbf{R}^d} d\mathbf{v} f_0(\mathbf{v}) = 1; \quad \int_{\mathbf{R}^d} d\mathbf{v}\mathbf{v} f_0(\mathbf{v}) = 0 \\ &\int_{\mathbf{R}^d} d\mathbf{v} v^2 f_0(\mathbf{v}) = d; \quad \int_{\mathbf{R}^d} d\mathbf{n} g(\mathbf{nm}) = 1 \end{aligned} \tag{5}$$

Then, the Boltzmann equation (2) can be written in the simpler form

$$\frac{\partial f}{\partial t} = f_+ - f \tag{6}$$

where

$$f_+(\mathbf{v}, t) = \int_{\mathbf{R}^d \times S^{d-1}} d\mathbf{w} d\mathbf{n} g\left(\frac{\mathbf{qn}}{q}\right) \ f(\mathbf{v}_1) f(\mathbf{w}_1) \tag{7}$$

and

$$\int_{\mathbf{R}^d} d\mathbf{v} f_+(\mathbf{v}, t) = 1$$

The discrete in time form of equation (6)

$$\frac{\hat{f} - f}{\tau} = f_+ - f$$

can be considered as a finite difference scheme for the Boltzmann equation with time step $0 < \tau < 1$, where the functions $\hat{f}$ and f approximate the functions $f(\mathbf{v}, t + \tau)$ and $f(\mathbf{v}, t)$ respectively.

The solution to the above equation on each time step is

$$\hat{f} = \tau f_+ + (1 - \tau) f$$

For any convex functional $F(f)$,

$$F(\widehat{f}) \leq \tau F(f_+) + (1-\tau)F(f)$$

is valid. Hence the inequality

$$F(f_+) \leq F(f) \tag{8}$$

is the sufficient condition for the monotonic behaviour in time

$$F(\widehat{f}) \leq F(f)$$

Owing to the Wild's form of the solution to (2), we can conclude that condition (8) is a sufficient condition also for the monotonic behaviour in time for the continuous Boltzmann equation.

To formulate more or less simple (for the verification) sufficient conditions for the functionals in such a way that inequality (8) be valid for an arbitrary indicatrix $g(\nu)$ in equation (2), we remark that (8) is equivalent to

$$\widehat{F}(\phi_+) \leq \widehat{F}(\phi) \tag{9}$$

where

$$\phi_+(\mathbf{k}, t) = \int_{S^{d-1}} d\mathbf{n} g\left(\frac{\mathbf{kn}}{k}\right) \phi\left(\frac{\mathbf{k}+k\mathbf{n}}{2}\right) \phi\left(\frac{\mathbf{k}-k\mathbf{n}}{2}\right) \tag{10}$$

Let us start with the simple case in which $\phi = \phi(|\mathbf{k}|)$, and g takes the form of a delta function

$$g(\nu) = \delta(\nu - \cos\theta) 2 \| S^{d-1} \|^{-1}$$

which satisfies the normalization condition (5). Then

$$\phi_+(k) = \phi\left(k \sin\frac{\theta}{2}\right) \phi\left(k \cos\frac{\theta}{2}\right)$$

Thus, the inequality

$$\widehat{F}\left[\phi\left(k \sin\frac{\theta}{2}\right) \phi\left(k \cos\frac{\theta}{2}\right)\right] \leq \widehat{F}\left[\phi(k)\right]$$

for any $0 \leq \theta \leq \pi$ is a necessary condition for (9) in the case of isotropic solutions to (4).

For a convex functional this inequality is also a sufficient condition for (9), since in the isotropic case

$$\phi_+(k) = \int_0^\pi d\theta g^*(\cos\theta)\sin\theta\phi\left(k\sin\frac{\theta}{2}\right)\phi\left(k\cos\frac{\theta}{2}\right) \tag{11}$$

where

$$g^*(\nu) = \|S^{d-1}\|\frac{g(\nu)}{2}; \quad \int_{-1}^1 d\nu g^*(\nu) = 1$$

Since (11) represent an averaging with respect to the unit measure g^*, the convexity gives the result.

Let $f \circ g$ denote the convolution operation between f and g.

In [3] we used this argument to obtain the following results

THEOREM 1. *Let $f_0(\mathbf{v})$ be an axially symmetric function, i.e. $f_0 = f_0(v, \mathbf{sm})$, where $\mathbf{s} = \frac{\mathbf{v}}{v}$ and $\mathbf{m}$ is a fixed unit vector. Let also the inequality for the convex and isotropic functional $F(f)$*

$$F\left\{\left[\frac{1}{\cos^d\alpha}f\left(\frac{v}{\cos\alpha}, \mathbf{sm}_+\right)\right] \circ \left[\frac{1}{\sin^d\alpha}f\left(\frac{v}{\sin\alpha}, \mathbf{sm}_-\right)\right]\right\} \le F[f(v, \mathbf{sm})] \tag{12}$$

be valid for any $0 < \alpha < \frac{\pi}{2}$ and any pair of orthogonal unit vectors $(\mathbf{m}_+, \mathbf{m}_-)$ Then the inequality (8) is also valid, and $F(f)$ is nonincreasing in time for both the discrete and continuous Boltzmann equation.

THEOREM 2. *In the plane case, the inequality*

$$F\left\{\left[\frac{1}{\cos^2\alpha}f\left(\frac{v}{\cos\alpha}, \theta\right)\right] \circ \left[\frac{1}{\sin^2\alpha}f\left(\frac{v}{\sin\alpha}, \theta + \frac{\pi}{2}\right)\right]\right\} \le F[f(v, \theta)] \tag{13}$$

for any $0 < \alpha < \frac{\pi}{2}$, is the sufficient condition for the convex isotropic functional $F(f)$ be nonincreasing in time for both the discrete and continuous Boltzmann equation.

Since we are dealing with convex and isotropic functionals, the inequalities (12) and (13) are verified if the following general inequality holds

$$F\left\{\left[a^{-d}f(a^{-1}\mathbf{v})\right] \circ \left[b^{-d}g(b^{-1}\mathbf{v})\right]\right\} \le a^2F(f) + b^2F(g) \tag{14}$$

for all $a, b > 0$ such that $a^2 + b^2 = 1$, $\mathbf{v} \in \mathbf{R}^d$, and f and g are normalized distributions (satisfying the first condition in (5)). In many cases, the validity of the above inequality for a functional F contains as further result the fact that the equality sign into (14) is possible if and only if f and g are Maxwellian distributions.

We give now some important examples of functionals satisfying (14), with the aim of presenting in the next section the consequences of their monotonicity in the study of the trend towards equilibrium. The first example refers to a functional introduced by Tanaka [8],[9]. Let us denote by G the class of all functions in $\mathbf{R}^d$ which satisfy conditions (5). Given $f \in G$, let us denote with $\mathbf{F}(f)$ the family of all probability distributions F on $\mathbf{R}^{2d}$ satisfying, for any Borel set A

$$F(A \times \mathbf{R}^d) = \int_A d\mathbf{v} f(\mathbf{v}); \quad F(\mathbf{R}^d \times A) = \int_A d\mathbf{v} \omega(\mathbf{v})$$

where

$$\omega_\theta(\mathbf{v}) = (2\pi\theta)^{-\frac{d}{2}} exp\left\{\frac{-v^2}{2\theta}\right\}$$

and $\omega = \omega_1$.

We define, for $k \geq 1$

$$E_k(f) = E_k(f, \omega) = \left\{\inf_{F \in \mathbf{F}} \int_{\mathbf{R}^d \times \mathbf{R}^d} |\mathbf{v} - \mathbf{w}|^k F(d\mathbf{v}, d\mathbf{w})\right\}$$

Then, it can be easily seen that $E_2(f)$ satisfies inequality (14). Moreover Tanaka [9] was able to prove that, given the initial densities $f_{0,1}$ and $f_{0,2}$, the solutions f_1 and f_2 to equation (2) satisfy

$$E_2[f_1(t), f_2(t)] \leq E_2[f_1(s), f_2(s)] \ ; \quad s \leq t$$

The second example we give refers to the J functional introduced by Linnik [6], defined by equation (1). In [3] we proved in a straightforward way that J satisfies (14). It is interesting to note that J and the Boltzmann H-functional are related by

$$\frac{\partial H(f_\theta)}{\partial \theta} = -\frac{1}{2} J(f_\theta) \tag{15}$$

where $f_\theta = f \circ \omega_\theta$.

The relation (15) is the starting point to prove that the Boltzmann H-functional satisfies (14) [3].

Thus, at least the three quoted functionals are monotonically decreasing along the solution to equation (2). It would be interesting to find (if any) other examples of convex and isotropic functionals with the same property. In [7] McKean introduced , starting from (15) a sequence of functionals defined as the subsequent derivatives of $H(f_\theta)$ with respect to the θ variable. Unfortunately, these functionals, which seemed to be good candidates, are not all convex.

3. Trend to equilibrium.

Here we outline the main arguments which are needed to prove that the solution to equation (2) manifests a strong trend towards equilibrium.

In what follows, we suppose that the inital density satisfies the normalization conditions (5).

To start with, we recall the following existence and uniqueness result for the Cauchy problem [5]

THEOREM 3. *Let* $(1+v^2)f_0 \in L_1^+(\mathbf{R}^d)$. *Then there exists a unique global solution to the initial value problem for equation (2). This solution conserves the first two moments. If further* $f_0 \log f_0 \in L_1(\mathbf{R}^d)$, *then the* H*-functional of the solution is nonincreasing as a function of time.*

An interesting property of a gas of Maxwellian molecules has been found in [11]. It extends a previous result by Bobylev [2] related to the equation for the Fourier transform.

THEOREM 4. *Let* $f(t)$ *be the solution of the Cauchy problem for equation (2), corresponding to the initial density* f_0 *satisfying the hypotheses of Theorem 3. Then* $f_\theta = f \circ \omega_\theta$ *is the solution to the Cauchy problem for equation (2) corresponding to the initial value* $f_{0,\theta} = f_0 \circ \omega_\theta$

A first consequence of Theorem 4 is that, starting from an initial density $f_0 \circ \omega_\theta$, the sequence $\{f_\theta(t)\}_{t\geq 0}$ of the solutions is relatively compact in $L_1(\mathbf{R}^d)$, and therefore there is a subsequence which converges strongly in L_1 to a function f_∞.

A classical argument of Carleman [4], based on the H-theorem, permits to identify the limit. If the initial density possesses finite moments up to the order $2+\alpha$, $\alpha > 0$ the limit is the Maxwellian function $\omega_{1+\theta}(\mathbf{v})$. In this case the whole sequence of the H-functions converges to the minimum (which is attained at $\omega_{1+\theta}(\mathbf{v})$), and this implies [11] that the sequence $\{f_\theta(t)\}_{t\geq 0}$ converges in L_1 to $\omega_{1+\theta}$.

If only the first two moments are initially finite, we cannot identify the energy of the limit, concluding that the limit of the subsequence $\{f(t_{n_k})\}$ is the Maxwellian function $\omega_\nu(\mathbf{v})$, $\nu \leq 1+\theta$. By contradiction, we can easily proof that we arrive to the same Maxwellian also in this second case. Suppose in fact that $\nu < 1+\theta$, so that, for some $\epsilon > 0$ the Tanaka's functional $E_1(\omega_\nu, \omega_{1+\theta}) > \epsilon$

We have

$$E_1(\omega_\nu, \omega_{1+\theta}) \leq E_1(\omega_\nu, f_\theta(t)) + E_2(f_\theta(t), f_\theta^\epsilon(t))^{\frac{1}{2}} + E_1(f_\theta^\epsilon(t), \omega_{1+\theta}) \tag{16}$$

Considering that the L_1-convergence of a sequence $\{f_n\}$ with finite second moments to f implies $E_1(f_n, f) \to 0$, we choose f_0^ϵ in such a way that it possesses finite moments up to the order 4, and in addition $E_2(f_{0,\theta}, f_{0,\theta}^\epsilon) \leq \frac{1}{3}\epsilon$. Moreover, we choose $t = t_{nk}$ so large that

$$E_1(\omega_\nu, f_\theta(t)) + E_1(f_\theta^\epsilon(t), \omega_{1+\theta}) < \frac{1}{3}\epsilon$$

and the contradiction arises.

The previous reasoning permits to conclude with the following [11]

THEOREM 5. *Let $f(\mathbf{v},t)$ ($f_\theta(\mathbf{v},t)$, $\theta > 0$) be the solution to the initial value problem for equation (2), with initial data $f_0(\mathbf{v})$ ($f_{0,\theta}(\mathbf{v})$) that satisfies (5). Then $f(\mathbf{v},t)$ converges weakly * in L_1 ($f_\theta(\mathbf{v},t)$ converges strongly in L_1) towards the Maxwellian function $\omega(\mathbf{v})$ as time goes to infinity.*

We remark that no entropy is needed for the weak * convergence. When the initial entropy is finite, strong convergence for general initial values follows if we are able to estimate, for $\theta > 0$, the difference

$$|H(f_\theta(t)) - H(f(t))|$$

This can be easily done in the case of plane velocities and axially symmetric solutions, due to Theorems 1 and 2. In this case, in fact, (15) implies

$$|H(f_\theta(t)) - H(f(t))| = \frac{1}{2}\int_0^\theta J(f_{0,\nu}(t))d\nu$$

and we have a uniform in time estimation. Thus in these cases we conclude

THEOREM 6. *Let $f(\mathbf{v},t)$ be the solution to the Cauchy problem for equation (2), with initial data that are axially symmetric in $\mathbf{R}^d$, or with plane initial data ($\mathbf{v} \in \mathbf{R}^2$). Then, if the initial energy and entropy are finite, the solution converges strongly in L_1 towards the Maxwellian function $\omega(\mathbf{v})$ as time goes to infinity.*

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