LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 493-502

NON-AXISYMMETRIC NECKING

YI-CHAO CHEN (Houston)

Experimental observations have revealed certain non-axisymmetric deformations at the initial stage of necking. This phenomenon is studied in this paper by using an energy stability criterion. It is shown that before the onset of axisymmetric necking, a non-axisymmetric and piecewice homogeneous deformation may have a lower energy than the axisymmetric deformations.

1. Introduction.

Necking is a well-known instability phenomenon that has stimulated many theoretical studies. Antman [1] uses an elastic rod theory to study tension of bars, and finds solutions that appear to represent necking deformations. The stability of Antman's solution is discussed by Owen [5]. Ericksen [4] studies weak solution (piecewise C^1) of the equilibrium equations and shows the existence of a piecewise homogeneous deformation that represents an idealized neck and is an absolute minimum of the strain energy in piecewise C^1 functions.

In the above works, only axisymmetric deformations are considered. However, experimental observations have revealed certain non-axisymmetric deformations at the initial stage of the development of necks. For example, Buisson and Ravi-Chandar [3] observe, in a uniaxial tension experiment with polycarbonate bars, the formation of a shear band preceding the necking deformation. The shear band forms a certain angle with the axis of the bar, which increases as the tension progresses. The shear band eventually evolves into an axisymmetric neck.

Antman and Carbone [2] have used non-linear elasticicty to analyze such non-axisymmetric deformations. With a convergent perturbation process, they show the existence of bifurcation points on the trivial solution branch, that correspond to shear instability. As pointed out by them, a formal justification of shear bifurcation branches is rather difficult.

In this paper, instead of trying to find non-axisymmetric euquilibrium solutions, we study a class of non-axisymmetric deformations that appear to resemble the shear band and nevertheless have very simple structures. In particular, the total strain energy can be minimized in this class of deformations by elementary methods. It is shown that, under certain conditions, one of such deformations has a lower strain energy than the axisymmetric equilibrium solutions discussed by Antman [1] and Owen [5].

2. Preliminary.

In this work, we consider a thin elastic bar of unit width, which is modelled as a two-dimensional body, represented by $\Omega \equiv (0, L) \times (0, 1)$ in a rectangular cartesian coordinate system, with L denoting the length of the undeformed bar.

The body is deformed under the action of a hard loading device that specifies the longitudinal displacement of the two ends of the bar, but imposes no kinematic constraints in the transverse direction. We shall consider the family of invertible, continuous and piecewise C^1 deformations:

$$\mathcal{A} \equiv \{x \in C^0 \cap \text{ piecewise } C^1(\bar{\Omega}; R^2) : \det \nabla x > 0 \quad \text{a.e. } x_1(0, X_2) = 0, x_1(L, X_2) = L\lambda\} \tag{1}$$

where $\lambda > 1$ is the overall stretch controlled by the loading device.

The body is comprised of a homogeneous isotropic elastic material that is associated with a smooth strain-energy function $W = W(v_1, v_2)$, where v_1 and v_2, the principal stretches, are eigenvalues of the stretch tensor $(\nabla x^T \nabla x)^{1/2}$. We shall make the following physicall reasonable assumptions on the strain-energy function:

$$W_{11} \geq k_1 > 0, \quad \frac{v_1 W_1 - v_2 W_2}{v_1^2 - v_1^2} \geq k_2 > 0, \tag{2}$$

where a subscript i of W denotes the derivative with respect to v_i.

The total strain energy stored in the deformed body is given by

$$E[x] = \int_\Omega W(\nabla x).$$

An energy stability criterion will be employed in this work. By this criterion, a deformation is stable if it minimizes the total strain energy E in a subset of $\mathcal{A}$.

3. Homogeneous deformations.

A deformation $\bar{x}$ is homogeneous if it is of C^1 and has a constant gradient in Ω. By equilibrium and isotropy,

$$\nabla \bar{x} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \tag{3}$$

The value of the strain-energy function associated with this deformation is given by

$$\bar{W}(\lambda) \equiv W(\lambda, \lambda^{-\frac{1}{2}}). \tag{4}$$

Such a homogeneous deformation is an absolute minimum of E in the class of all homogeneous deformations.

It has been known[1] that necking phenomenon is related to certain characteristics of $\bar{W}(\lambda)$ as depicted in Figure 1: $\bar{W}'$ is monotone increasing in λ in the ranges of small and large deformations, and

(1) See, for example, Ericksen [4] and Owen [5].

monotone decreasing in λ in the intermediate deformation range. There exist two values λ_1 and λ_2, that satisfy

$$\bar{W}'(\lambda_1) = \bar{W}'(\lambda_2) = \frac{\bar{W}(\lambda_2) - \bar{W}(\lambda_1)}{\lambda_2 - \lambda_1} > 0. \tag{5}$$

It can be readily shown that

$$(\lambda - \lambda_1)\bar{W}'(\lambda_1) \leq \bar{W}(\lambda) - \bar{W}(\lambda_1). \tag{6}$$

For a given stretch λ satisfying $\lambda \leq \lambda_1$ or $\lambda \geq \lambda_2$, the homogeneous deformation given by (3) is an absolute minimum of E. Such a deformation may not be a minimum when $\lambda \in (\lambda_1, \lambda_2)$. In the latter case, if one considers a broader class of deformations than $\mathcal{A}$, a special piecewise homogeneous deformation proves to be of great importance in connection with the necking phenomenon.

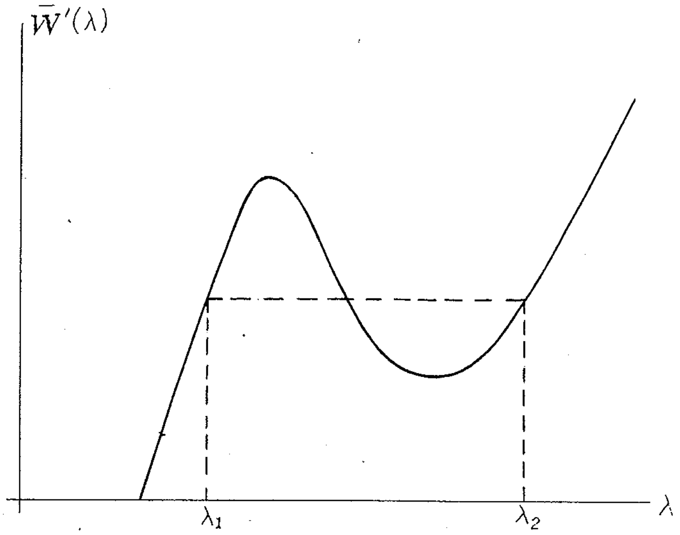

Fig. 1 - Constitutive behavior of strain-energy function.

Let $\breve{\mathcal{A}}$ be the family of deformations x that satisfy all conditions specified in (1) except that x_2, the second component of x, needs only to be piecewise continuous in $\bar{\Omega}$. That is, a deformation in $\breve{\mathcal{A}}$ is continuous in the longitudinal direction but may suffer jump discontinuity in the transverse direction.

For a given $\lambda \in (\lambda_1, \lambda_2)$, we consider a deformation

$$\breve{x}(X) = \begin{cases} \mathbf{F}_1\mathbf{X} & \text{if } 0 \leq X_1 \leq L(\lambda_2 - \lambda)/(\lambda_2 - \lambda_1), \\ \mathbf{F}_2\mathbf{X} + \mathbf{a} & \text{if } L(\lambda_2 - \lambda)/(\lambda - \lambda_1) < X_1 \leq L. \end{cases} \tag{7}$$

where

$$\mathbf{F}_1 \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-\frac{1}{2}} \end{pmatrix}, \ \mathbf{F}_2 \equiv \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-\frac{1}{2}} \end{pmatrix} \ \mathbf{a} \equiv (-L(\lambda_2 - \lambda), 0).$$

Such a deformation, belonging to $\check{A}$, represents an idealized hald-neck deformation as shown in Figure 2.

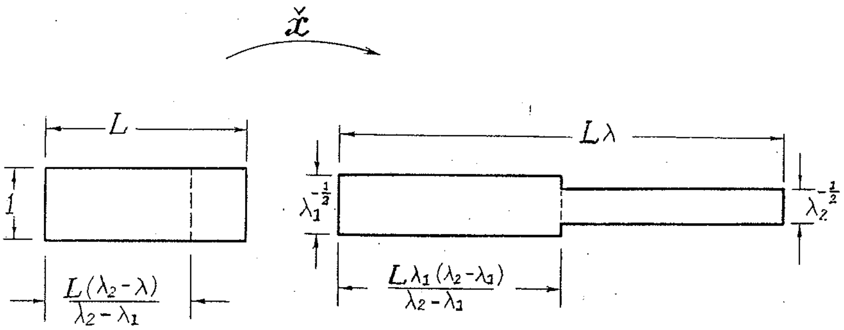

Fig. 2 - An idealized half-neck deformation.

THEOREM 1. *The total strain energy E attains an absolute minimum at $\check{x}$ in $\check{A}$.*

Proof. For $\lambda \in (\lambda_1, \lambda_2)$, let an $x \in \check{A}$ be given. The first argument x_1 of x continuous and piecewise C^1 in $\bar{\Omega}$ with

$$x_1(0, X_2) = 0, \ x_1(L, X_2) = L\lambda. \tag{8}$$

Let v_1 and v_2 be the principal stretches associated with x, ordered as $v_1 \geq v_2$. Then,

$$x_{1,1} \leq v_1 \text{ on } \bar{\Omega}, \tag{9}$$

where a comma followed by i denotes the derivative with respect to X_i. By using (7), (8), (5), (9) and (6), we find that

$$E[\check{x}] = L\bar{W}(\lambda_1) - (L\lambda_1 - \int_\Omega x_{1,1})\bar{W}'(\lambda_1) \leq \int_\Omega \bar{W}(v_1) \leq E[x]. \qquad \square$$

Although discontinuous deformation is physically unrealistic, the piecewise homogeneous deformation described above gives much important insight into the necking phenomenon. As suggested by the works of Antman [1] and Owen [5], a continuous necking deformation, after rescaled to a fixed length, will approach one such piecewise homogeneous deformation as L tends to infinity.

4. A rod theory.

By using a rod theory, Antman [1] has found non-homogeneous solutions that appear to represent necking deformations for rods of circular cross sections. Owen [5] shows that for a fixed λ, one of the Antman's solutions, corresponding to a half neck, is stable in the sense that it minimizes the total strain energy among all deformations considered in the rod theory, provided that the initial length L is sufficiently large. This last condition is crucial. In this section, we derive a necessary condition for a non-homogeneous deformation to have a lower strain energy than the homogeneous deformation. In particular, it provides an upper bound for L in order that the half-neck solution can be a minimum.

Following Antman [1] and Owen [5], we consider a special class of deformations

$$\tilde{\mathcal{A}} \equiv \{x \in \mathcal{A} : x_1(X_1, X_2) = x(X_1), x_2(X_1, X_2) = X_2 y(X_1), (x, y) \in C^0 \cap$$

$$\text{piecewise } C^1([0, L]; R^2)\}.$$

For this class of deformations, we define a one-dimensional strain energy function

$$\tilde{W}(x', y, y') \equiv \int_0^1 W(v_1, v_2) dX_2, \tag{10}$$

where

$$v_{1,2} = \frac{1}{2}\left[\sqrt{(x' + y)^2 + X_2^2 y'^2} \pm \sqrt{(x' - y)^2 + X_2^2 y'^2}\right]. \tag{11}$$

By (2) and the mean-value theorem, we have

$$\tilde{W}(x', y, y') - \tilde{W}(x', y, 0) \geq \frac{k_2}{6} y'^2. \tag{12}$$

Minimizing E in $\tilde{A}$ then leads to the following one-dimensional variational problem:

$$\text{Minimize} \int_0^L \tilde{W}(x', y, y')dX_1 \text{in } \mathcal{C} \equiv \{(x,y) \in C^0 \cap$$
$$\cap \text{ piecewise } C^1([0,L];R^2) : x(0) = 0, x(L) = L\lambda\}.$$

This minimization problem is studied in [1] and [5]. Of particular interest is the existence of the half-neck solution that is related to the piecewise homogeneous solution depicted in Figure 2.Indeed, it consists of two approximately homogeneous sections joined by a smooth transition. Such a solution is a minimum of E in $\tilde{A}$ if L is sufficiently large. However, for L not sufficiently large, the total strain energy associated with the half-neck solution may be higher than that with the homogeneous solution because a certain penalty must be paid to have the continuous transition between the two approximately homogeneous sections.

PROPOSITION 1. *For a fixed* λ, *if*

$$\frac{k_1 k_2(\lambda_L - \lambda_0)(\lambda_L^{-\frac{1}{2}} - \lambda_0^{-\frac{1}{2}})^2}{3k_1 L^2(\lambda_L - \lambda)(\lambda - \lambda_0) + 2k_2(\lambda_L - \lambda_0)^2} \geq \frac{\bar{W}(\lambda) - \bar{W}(\lambda_0)}{\lambda - \lambda_0} - \frac{\bar{W}(\lambda_L) - \bar{W}(\lambda)}{\lambda_L - \lambda} \tag{13}$$

for all λ_0 *and* λ_L, *then the homogeneous deformation* $\bar{x}$ *given by (3) is an absolute minimum of* E *in* $\tilde{A} \cap C^2(\bar{\Omega}; R^2)$.

Proof. Let an $\tilde{x} \in \tilde{A} \cap C^2(\bar{\Omega}; R^2)$ be given with $\tilde{x}(\mathbf{X}) = (x(X_1), X_2 y(X_1))$. By (10), (11) and (12), we find that

$$E[\tilde{x}] - E[\bar{x}] = \int_0^L [\tilde{W}(x', y, y') - \tilde{W}(x', y, 0) + W(x', y) - \bar{W}(x') + \bar{W}(x') - \bar{W}(\lambda)]dX_1$$
$$\geq \int_0^L \left[\frac{k_2}{6}y'^2 + \frac{k_1}{2}(y - x'^{-\frac{1}{2}})^2 + \bar{W}(x') - \bar{W}(\lambda)\right] dX_1.$$

It follows from a straightforward calculation that the last expression is non-negative under the hypothesis. □

5. Non-axisymmetric deformations.

The analyses in the previous sections lead to the following observation concerning the bar subject to increasing overall stretch λ: When λ is smaller than λ_1, the homogeneous deformation has the lowest energy. As λ exceeds λ_1, a part of the bar tends to shift to a state that provides a larger stretch so that the remaining bar can stay at the state of small stretch. The increase in strain energy due to the shifting is less than that required by deforming the entire bar homogeneously. The energy reduction depends on the value of $\lambda - \lambda_1$. When this value is small, the energy reduction is also small. On the other hand, the shifting is companied by a rapid change in radial deformation occuring in the transition section, which incurs a penalty by increasing the total strain energy. Such an increase is virtually independent of the value of $\lambda - \lambda_1$, and might well offset the reduction in the strain energy due to the shifting when $\lambda - \lambda_1$ is small.

It then follows, naturally that a deformation consisting of two homogeneous streches λ_1 and λ_2 but without transition may have a lower total strain energy. One such deformation is constructed as follows.

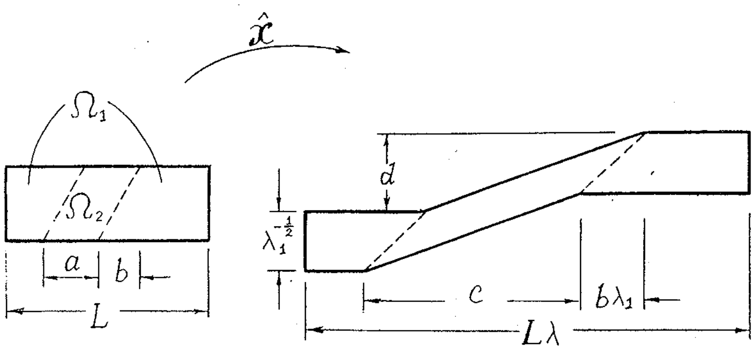

Fig. 3 - A non axisymmetric deformation.

Let $\lambda \in (\lambda_1, \lambda_2)$ be given. We consider a continuous, piecewise homogeneous deformation $\hat{x}$ illustrated in Figure 3. The reference configuration Ω is divided into two parts Ω_1 and Ω_2, each undergoing a homogeneous deformation. The principal stretches of $\hat{x}|_{\Omega_1}$ are λ_1 and $\lambda_1^{-\frac{1}{2}}$, with the principal axes coinciding with the coordinate axes, while those of $\hat{x}|_{\Omega_2}$ are λ_2 and $\lambda_2^{-\frac{1}{2}}$, with the principal axes making a certain angle with the coordinate axes. In Figure 3, parameters

a, b, c and d are to be chosen so that $\hat{x}$ has the lowest total strain energy in this special class of deformations. It then follows from a straightforward calculation that

$$
\begin{aligned}
a &= \frac{L\lambda_2^{\frac{1}{2}}(\lambda - \lambda_1)(1 + \lambda_1^{\frac{3}{2}})}{\lambda_1^{\frac{1}{2}}(\lambda_2 - \lambda_1)(1 + \lambda_1\lambda_2^{\frac{1}{2}})}, \\
b^2 &= \frac{1 + \lambda_1^{\frac{1}{2}}\lambda_2}{\lambda_1(\lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}})(1 + \lambda_1\lambda_2^{\frac{1}{2}})}, \\
c &= L(\lambda - \lambda_1) + a\lambda_1, \; d = Lb(\lambda - \lambda_1).
\end{aligned} \tag{14}
$$

The non-axisymmetric deformation $\hat{x}$ appears quite close to the equilibrium deformation observed in Buisson and Ravi-Chandar's [3] experiment. It is then reasonable to expect the strain energy of $\hat{x}$ to serve as an adequate upper bound for the strain energy of that equilibrium deformation. The following proposition gives a condition under which this upper bound is lower than the strain energy of the homogeneous deformation, and consequently lower than that of the half-neck deformation, provided that inequality (13) holds.

PROPOSITION 2. *For a given strech $\lambda \in (\lambda_1, \lambda_2)$, the total strain energy associated with the non-axisymmetric deformation $\hat{x}$ is not higher than that with the homogeneous deformation $\bar{x}$ if and only if*

$$
\frac{\bar{W}(\lambda) - \bar{W}(\lambda_1)}{\lambda - \lambda_1} \geq \frac{\sqrt{\lambda_2} + \sqrt{\lambda_1^3\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_1^3\lambda_2}} \bar{W}'(\lambda_1). \tag{15}
$$

Proof. By (14), (5) and the construction of $\hat{x}$, we have

$$
E[\bar{x}] - E[\hat{x}] = L[\bar{W}(\lambda) - \bar{W}(\lambda_1)] - \frac{L(\lambda - \lambda_1)(\sqrt{\lambda_2} + \sqrt{\lambda_1^3\lambda_2})}{\sqrt{\lambda_1} + \sqrt{\lambda_1^3\lambda_2}} \bar{W}'(\lambda_1). \quad \square
$$

Whether or not inequality (15) will hold for some λ is completely determined by the behavior of $\bar{W}$. It is an easy matter to construct strain-energy fucntions for which (15) does hold for some λ. Also, inequalities (15) and (13) appear to be independent of each other as the behavior of $\bar{W}$ is not restricted by the values of k_1 and k_2. One can construct a strain-energy function that satisfies both (15) and (13). The corresponding non-axisymmetric deformation then would have a total strain energy not higher than those associated with the homogeneous deformation and the half-neck deformation.

REFERENCES

[1] Antman S.S., *Nonuniqueness of equilibrium states for bars in tension,* J. Math. Anal. Appl. **44** (1973), 333-349.

[2] Antman S.S., Carbone E.R., *Shear and necking instabilities in nonlinear elasticity,* J. Elasticity **7** (1977), 125-151.

[3] Buisson G., Ravi-Chandar K., *On the constitutive behaviour of polycarbonate under large deformation.* Polymer **31** (1990), 2071-2076.

[4] Ericksen J.L., *Equilibrium of bars,* J. Elasticity **5** (1975), 191-201.

[5] Owen N.C., *Existence and stability of necking deformation for nonlinearly elastic rods.* Arch. rational Mech. Anal. **98** (1987), 357-383.

Department of Mechanical Engineering
University of Houston
Houston, Texas