LE MATEMATICHE
Vol. XLVI (1991) - Fasc. I, pp. 503-5024

ON THE EXISTENCE OF STEADY MOTIONS OF A VISCOUS FLOW WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS

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Introduction.

Let Ω be a bounded region of $\mathbb{R}^n$ $(n = 2, 3)$, delimited by $m+1$ closed, sufficiently smooth and separate surfaces (lines) Γ_i, $i = 1, \dots, m+1$. In this paper we shall consider the existence of solutions to the following Navier-Stokes boundary value problem:

$$(\mathcal{NS}) \qquad \left.\begin{aligned} v\Delta \mathbf{v} &= \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\mathbf{v} = \mathbf{a} \text{ at } \partial\Omega$$

As is well known, such a problem governs the distribution of velocity ($\mathbf{v}$) and pressure (p) fields in the steady motions of an incompressible, viscous fluid (of kinematical viscosity v) moving in the region Ω and subject to a prescribed velocity a ($\not\equiv 0$) at the boundary (1). In view

(1) For simplicity, we are assuming that there are no body forces acting on the fluld.

of the solenoidality of $\mathbf{v}$, the field a can *not* be prescribed *arbitrarily*; rather, it must satisfy the following *compatibility* condition:

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n} = \sum_{i=1}^{m+1}\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n} \equiv \sum_{i=1}^{m+1}\phi_i = 0, \tag{0.1}$$

that is, the *total* flux of a through $\partial\Omega$ must vanish. However, it is likewise well-established that $(\mathcal{NS})$ is known to have a (generalized) solution under the assumption that the fluxes ϕ_i of a along Γ_i satisfy the conditions (2)

$$\phi_i \equiv \int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n} = 0 \qquad i = 1, 2, \ldots, m+1, \tag{0.2}$$

cf. LERAY (1933), HOPF (1941), LADYZHENSKAYA (1959, 1965), FINN (1961), FUJITA (1961), VOROVICH & YOUDOVICH (1961), LIONS (1969). Observe that (0.2) is *stronger* than the compatibility condition (0.1) and that, in particular, it does not allow for the presence of separated sinks and sources of fluids into the region of flow.

The reason why one imposes (0.2) instead of (0.1) is related to the basic estimate which one needs in order to obtain a solution to the non-homogeneous stationary Navier-Stokes equations, *cf., e.g.*, FUJITA (1961), LIONS (1969); *cf.* also GALDI *(forthcoming)*. Let us briefly recall this point. One constructs a (sufficiently smooth) *solenoidal* extension $\mathbf{V}$, say, in Ω of the field a. Then, denoting by $\|\cdot\|_q$ the norm in the Lebesgue space $L^q(\Omega)$ and setting

$$\mathcal{D}(\Omega) = \{\mathbf{u} \in C_0^\infty(\Omega) : \nabla\cdot\mathbf{u} = 0\},$$

existence is easily established provided $\mathbf{V}$ satisfies the further crucial condition (3)

$$\left|\int_\Omega \mathbf{u}\cdot\nabla\mathbf{V}\cdot\mathbf{u}\right| \le \alpha\|\nabla\mathbf{u}\|_{1,2}^2 \tag{0.3}$$

(2) Throughout this paper, the infinitesimal volume or surface elements in the integrals will be generally omitted.

(3) As costumary, we set $\mathbf{A}\cdot\nabla\mathbf{B} = \sum_{i=1}^{3} A_i \partial\mathbf{B}/\partial x_i$

for some α strictly less than the coefficient of kinematical viscosity v and for all $\mathbf{u} \in \mathcal{D}(\Omega)$. If we do not want to impose restrictions from below on v, then Ω and a should verify the following *Extension Condition* (referred to by the abbreviation *EC*): *for any* $\alpha > 0$ *there exists a solenoidal extension* $\mathbf{V} = \mathbf{V}(\alpha)$ *of* a *satisfying* (0.3).

If $\mathbf{V}$ were *not* required to be solenoidal, every (sufficiently regular) Ω and a would satisfy *EC*. Actually for a given $\varepsilon > O$ we could choose

$$\mathbf{V} = \psi_\varepsilon \mathbf{W} \tag{0.4}$$

with $\mathbf{W}$ an extension of a and ψ_ε a "cut-off" function which is one in a neighbourhood of $\partial\Omega$ of width ε and zero outside a neighbourhood of width 2ε (*cf.* also Lemma 1. 2) . Thus, integrating by parts and using Hölder inequality together with a classical inequality of the Sobolev type it follows

$$\left|\int_\Omega \mathbf{u}\cdot\nabla(\psi_\varepsilon \mathbf{W})\cdot\mathbf{u}\right| \leq \left|\int_\Omega \psi_\varepsilon \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{W}\right| \leq ||\mathbf{u}||_4||\nabla\mathbf{u}||_2||\psi_\varepsilon\mathbf{W}||_4 \leq c||\nabla\mathbf{u}||_2^2||\psi_\varepsilon\mathbf{W}||_4$$

and since, by the properties of ψ_ε,

$$||\psi_\varepsilon\mathbf{W}||_4 \to 0 \text{ as } \varepsilon \to 0,$$

we recover (0.3) for any $\alpha > O$. Nevertheless, following the ideas of LERAY (1933, pp.40-41), successively completed and clarified by HOPF (1941) (*cf.* also HOPF (1957)), if Ω is of class C^2 (at least), instead of (0.4) one can take (4)

$$\mathbf{V} = \nabla \times (\psi_\varepsilon \mathbf{W}) \tag{0.5}$$

with a suitable choice of the field $\mathbf{W}$. Thus $\mathbf{V}$ is solenoidal and, by arguments slightly more complicated than those employed before, one can show that (0.3) is satisfied by any $\alpha > O$. Recalling that the incompressibility of the fluid requires (0.1), we may conclude that, if $\partial\Omega$ has only one connected component, the choice (0.5) ensures that any (sufficiently smooth) Ω and a satisfy *EC*. However, if $\partial\Omega$ has more than one connected component Γ_i, with the choice (0.5) we have, as

(4) If $n = 2$, one takes $\mathbf{V} = (\partial(\psi_\varepsilon w)/\partial x_2, -\partial(\psi_\varepsilon w)/\partial x_1) \equiv \nabla \times (\psi_\varepsilon w)$.

a consequence of the Stokes theorem, that Ω *and* a *satisfy EC if* (0.1) holds.

At this point we may think of constructing a field $\mathbf{V}$ in a form different from (0.5) so that Ω and a may satisfy *EC* under the *sole* condition (0.1). However, in a nice recent paper, TAKESHITA (1991, Theorems 1 and 2) has given examples of smooth domains Ω for which *EC holds (if and) only if* (0.2) *is satisfied, whatever the choice of a may be*. A typical domain considered by TAKESHITA is the spherical shell

$$S_{R_1,R_2} = \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}. \tag{0.6}$$

In the light of these considerations it appears that the LERAY-HOPF construction of a solution is possible, in general, only if the data obey the more stringent flux condition (0.2).

A few years ago, KAPITANSKIĬ & PILETSKAS (1984, §4) have proposed a different method which has the following two advantages: on the one side, it allows one to show existence to $(\mathcal{NS})$ also when the quantities $|\phi_i|$ are not zero, provided, however, they are less than *some* positive constant λ depending on v (5); on the other side, this method -unlike LERAY-HOPF's-requires little smoothness for the boundary $\partial\Omega$ and the simple lipschitz regularity is enough. However, the explicit value of λ can not be furnished.

The aim of the present paper is to prove that for any bounded lipschitzian domain Ω and any (sufficiently smooth) a ascribed on $\partial\Omega$ there exists a *computable* positive constant $C = C(n, \Omega)$ such that if

$$\sum_{i=1}^{m+1} |\phi_i| \leq cv, \tag{0.7}$$

problem $(\mathcal{NS})$ admits at least one solution. In particular, we can furnish an explicit bound from below for C when Ω is the spherical shell (0.6) and, for example, for $n = 2$ and $R_2 = 2R_1$ we find

$$C \geq 2v/(0.42 \cdot R_1 + 0.16).$$

(5) As a matter of fact, KAPITANSKIĬ & PILETSKAS give the proof for $\lambda = 0$; nevertheless, their argument goes through also if λ is positive and not too large.

The method we use is based upon a suitable coupling of the ideas of LERAY and HOPF with the results of BOGOVSKIĬ (1980) on the resolubility of the equation $\nabla \cdot \mathbf{v} = f$ in appropriate Sobolev spaces. It should be emphasized that our method works in any number of spatial dimensions $n \geq 2$.

The paper is organized as follows. In Section 1 we show some preparatory results concerning the existence of the solenoidal extension of a field a satisfying (0.2) when Ω is a lipschitz domain. Such existence was known in the literature only for domains of class C^2, *cf.* FOIAŞ, & TEMAM (1979). Successively, in Section 2, we establish our main theorem which ensures that condition (0.7) for a suitable, computable constant C depending only on n and Ω is sufficient for the existence of steady solutions to the Navier-Stokes problem with non-homogeneous boundary data. Finally, in Section 3, we give a lower bound for the constant C in the case when the relevant region of flow is the annulus delimited by circles of radii R and $2R$, respectively. Analogous results could be proved for a three-dimensional spherical shell.

1. On the solenoidal extension of certain vector fields defined on the boundary of a lipschitz domain.

Let us first recall some notations. Ω denotes a bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with a lipschitz boundary an $\partial\Omega$ [6]. By $L^q(\Omega)$, $1 \leq q \leq \infty$, we indicate the usual Lebesgue space with norm $\|\cdot\|_q$, while $W^{k,q}(\Omega)$, k a non-negative integer, stands for the Sobolev space of order (k, q), endowed with the natural norm which we denote by $\|\cdot\|_{k,q}$. In addition, $W_0^{k,q}(\Omega)$ is the completion in the norm $\|\cdot\|_{k,q}$ of the space $C_0^\infty(\Omega)$ of all indefinitely differentiable functions of compact support in Ω. Furthermore, the trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$, $1 < q < \infty$, is denoted by $W^{k-1/q,q}(\partial\Omega)$ and by $\|\cdot\|_{k-1/q,q}(\partial\Omega)$ we indicate the corresponding norm. Finally, $\mathcal{D}(\Omega)$ is the subset of $C_0^\infty(\Omega)$ constitued by solenoidal functions and $H^1(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{1,2}$. As is known, *cf.* BOGOVSKIĬ (1980,

(6) By this we mean that $\partial\Omega$, in the neighbourhood of any of its points, can be represented by a Lipschltz function and that Ω lies only on one side of $\partial\Omega$.

Theorem 2), it is

$$H^1(\Omega) = \{\mathbf{u} \in W_0^{1,q}(\Omega) : \nabla \cdot \mathbf{u} = O\}.$$

To show our main result we need some preparatory steps. The first step is to introduce a suitable "cut-off" function. This function involves the distance $\delta(x)$ of a point $x \in \Omega$ from the boundary $\partial\Omega$. We need to differentiate $\delta(x)$. However, if Ω has no or little smoothness, $\delta(x)$ is in general not more differentiable than the obvious Lipschitz-like condition $|\delta(x) - \delta(y)| \le K|x-y|$. However, we need more regularity on δ and, to this end, we introduce the so called *regularized distance* in the sense of STEIN (1970, p. 171). In this respect, we have the following lemma for whose proof we refer the reader to STEIN (1970, Chapter VI, Theorem 2).

LEMMA 1.1. *For $x \in \Omega$, set*

$$\delta(x) = dist(x, \partial\Omega).$$

Then, there is a function $\rho \in C^\infty(\Omega)$ such that for all $x \in \Omega$

$$\begin{aligned} &(i) \quad \delta(x) \le \rho(x) \le \kappa_1 \delta(x); \\ &(ii) \quad |\nabla \rho(x)| \le \kappa_2, \end{aligned}$$

where $\kappa_i = \kappa_i(n)$, $i = 1, 2$.

Remark 1.1. A simple estimate for the constant κ_1 is given by STEIN (1970, p. 169 and p. 171) and one has $\kappa_1 = (20/3)(12)^n$. Of course, if Ω is of class C^∞, we may take $\kappa_1 = \kappa_2 = 1$.

Owing to Lemma 1.1, we can prove the following result, *cf.* also HOPF (1941).

LEMMA 1.2. *Let δ be as in* Lemma 1.1. *For any $\varepsilon > 0$ set*

$$\gamma(\varepsilon) = \exp(-1/\varepsilon).$$

Then there exists a function $\psi_\varepsilon \in C^\infty(\bar{\Omega})$ *such that*

$$\begin{array}{lll} (i) & |\psi_\varepsilon(x)| \leq 1, & \textit{for all } x \in \Omega; \\ (ii) & \psi_\varepsilon(x) = 1, & \textit{if } \delta(x) < \gamma^2/2\kappa_1; \\ (iii) & \psi_\varepsilon(x) = 0, & \textit{if } \delta(x) \geq 2\gamma(\varepsilon); \\ (iv) & |\nabla\psi_\varepsilon(x)| \leq \kappa_2\varepsilon/\delta(x), & \textit{for all } x \in \Omega; \end{array}$$

where κ_1 *and* κ_2 *are the constants given in Lemma 1.1.* (7)

Proof. Consider the following function of IR into itself:

$$\varphi_\varepsilon(t) = \begin{cases} 1 & \text{if } t < \gamma^2(\varepsilon) \\ \varepsilon ln(\gamma(\varepsilon)/t) & \text{if } \gamma^2(\varepsilon) < t < \gamma(\varepsilon) \\ 0 & \text{if } t > \gamma(\varepsilon) \end{cases}$$

Clearly, choosing $\eta = \gamma^2/2$, the mollifier (in the sense of Friederichs) $\Phi_\varepsilon \equiv (\varphi_\varepsilon)_\eta$ of ϕ_ε verifies $\Phi_\varepsilon(t) = 1$ for $t < \gamma^2/2$, $\Phi_\varepsilon(t) = 0$ for $t > 2\gamma$ and

$$|\Phi'_\varepsilon(t)| \leq \varepsilon/t, \qquad \text{for all } t \in \mathrm{IR}. \tag{1.1}$$

In addition, $|\Phi(t)| \leq 1$. Setting

$$\psi_\varepsilon(x) \equiv \Phi_\varepsilon(\rho(x)),$$

where ρ is the regularized distance of Lemma 1.1, and recalling statements *(i), (ii)* of that lemma, we deduce

$$\psi_\varepsilon(x) = 1 \quad \text{if } \delta(x) < \gamma^2/2c_1$$

$$\psi_\varepsilon(x) = 0 \quad \text{if } \delta(x) > 2\gamma.$$

Moreover, from (1.1) and from Lemma 1.1 it follows for all $x \in \Omega$

$$|\nabla\psi_\varepsilon(x)| \leq \kappa_2\varepsilon/\rho(x) \leq \kappa_2\varepsilon/\delta(x).$$

(7) Notice that, by Remark 1.1, it is

$$\gamma(\varepsilon) < 2\kappa_1, \text{ for all } \varepsilon > 0. \tag{*}$$

Of course, whatever is the estimate for κ_1, we can always choose κ_1 such that (*) holds.

The result is therefore completely proved.

The following lemma provides the desired solenoidal extension of a boundary field a which satisfies the flux condition (0.1).

LEMMA 1.3. *Denote by* ω_i, $i = 1, \dots, m$, *the (bounded) connected components of* $\mathbb{R}^n \setminus \bar{\Omega}$ (8) *and set*

$$\omega \equiv \bigcup_{i=1}^{m} \omega_i.$$

Then, given $\mathbf{a} \in \mathbf{W}^{1/2,2}(\Omega)$ *verifying the condition*

$$\int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} = 0 \qquad i = 1, 2, \dots, m+1, \tag{1.2}$$

where $\mathbf{n}$ *is the outer normal to* $\partial\Omega$ *and*

$$\Gamma_i \equiv \partial\omega_i, \quad \textit{for } i = 1, 2, \dots, m, \qquad \Gamma_{m+1} \equiv \partial(\Omega \cup \bar{\omega}),$$

there exists $\mathbf{w} \in \mathbf{W}^{2,2}(\Omega)$ *if* $n = 3$ *[resp.* $\mathbf{w} \in \mathbf{W}^{2,2}(\Omega)$, *if* $n = 2$*] such that* $\mathbf{a} = \nabla \times \mathbf{w}$ *[resp.* $\mathbf{a} = \nabla \times \mathbf{w}$ (9)*] in the trace sense at* $\partial\Omega$. *Moreover, the following inequality holds*

$$\begin{gathered} \|\mathbf{w}\|_{2,2} \leq c \|\mathbf{a}\|_{1/2,2(\partial\Omega)} \\ [\textit{resp. } \|\mathbf{w}\|_{2,2} \leq c \|\mathbf{a}\|_{1/2,2(\partial\Omega)}] \end{gathered} \tag{1.3}$$

where $c = c(n, \Omega)$.

Proof. Since $\partial\Omega = \bigcup_{i=1}^{m} \Gamma_i$, from (1.2) it follows, in particular,

$$\int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} = 0. \tag{1.4}$$

By a well-known result of GAGLIARDO (1957), we can find a vector field $\mathbf{A} \in \mathbf{W}^{1,2}(\Omega)$ (not necessarily solenoidal) such that a is the trace of $\mathbf{A}$ at $\partial\Omega$ and, furthermore,

$$\|\mathbf{A}\|_{1,2} \leq \|\mathbf{a}\|_{1/2,2(\partial\Omega)}. \tag{1.5}$$

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(8) Clearly, the number m finite since $\partial\Omega$ is compact and, furthermore infdist $(\omega_i, \partial(\Omega \cup \bar{\omega})) > 0$.

(9) That is, $\mathbf{a} = (\partial w/\partial x_2, -\partial w/\partial x_1)$.

By Theorem 1 of BOGOVSKIĬ (1980), we may find a field $\mathbf{U}$ such that

$$
\begin{aligned}
&\nabla \cdot \mathbf{U} = -\nabla \cdot \mathbf{A} \\
(1.6)\qquad &\mathbf{U} \in \mathbf{W}_0^{1,2}(\Omega) \\
&\|\mathbf{U}\|_{1,2} \le c_2 \|\mathbf{A}\|_{1,2}.
\end{aligned}
$$

Thus, in view of (1.5), (1.6) we can extend a to a field $\mathbf{A} + \mathbf{U} \equiv \mathbf{v}_0 \in \mathbf{W}^{1,2}(\Omega)$ with $\nabla \cdot \mathbf{v}_0 = 0$ and verifying the inequality

$$
(1.7)\qquad \|\mathbf{v}_0\|_{1,2} \le c\|\mathbf{a}\|_{1/2,2(\partial\Omega)}.
$$

If $n = 2$, for a fixed $x_0 \in \Omega$ we define a function w through the line integral

$$
w(x) = \int_{x_0}^{x} (v_{01} dx_2 - v_{02} dx_1), \qquad x \in \Omega
$$

i.e., w is the stream function associated to $\mathbf{v}_0$. Since (1.2) holds, w is single-valued. Furthermore,

$$
\frac{\partial w}{\partial x_2} = v_{01}, \qquad \frac{\partial w}{\partial x_1} = v_{02}
$$

and so

$$
(1.8)\qquad |w|_{1,2} + |w|_{2,2} \le c_1 \|v_0\|_{1,2}
$$

Also, we can modify w by an additive constant in such a way that

$$
\int_{\Omega} w = 0,
$$

and so, fron Poincaré inequality, from (1.7) and (1.8) we deduce (1.3), proving the lemma if $n = 2$. To prove it for $n = 3$, we notice that, proceeding as before, we can extend a at $\partial\omega_i$, $i = 1, \dots, m$ into each ω_i to a solenoidal vector field $\mathbf{V}_i \in \mathbf{W}^{1,2}(\omega_i)$ satisfying the estimate

$$
(1.9)\qquad \|\mathbf{v}_i\|_{1,2,\omega_i} \le c_2 \|\mathbf{a}\|_{1/2,2(\partial\Omega)}, \qquad i = 1, \dots, m.
$$

Moreover, denoting by B an open ball with $B \supset \bar{\Omega}$, since

$$
\int_{\Gamma_{m+1}} \mathbf{a} \cdot \mathbf{n} = 0,
$$

we can extend a at $\partial(\Omega \cup \bar{\omega})$ to a solenoidal vector field $\mathbf{v}_{m+1} \in \mathbf{W}^{1,2}(\omega_{m+1})$, with $\omega_{m+1} \equiv B - (\Omega \cup \bar{\omega})$, such that

$$\|\mathbf{v}_{m+1}\|_{1,2,\omega_{m+1}} \leq c_2 \|\mathbf{a}\|_{1/2,2(\partial\Omega)}. \tag{1.10}$$

It is immediately checked that the vector field:

$$\mathbf{v} : x \in b \to \begin{cases} \mathbf{v}_0(x) & \text{if } x \in \Omega \\ \mathbf{v}_i(x) & \text{if } x \in \omega_i, \ i = 1, \ldots, m+1. \end{cases} \tag{1.11}$$

verifies the following properties

$$\begin{aligned} &(i) \quad \mathbf{v} \in \mathbf{W}^{1,2}(B) \\ &(ii) \quad \nabla \cdot \mathbf{v} = 0 \text{ in } B \\ &(iii) \quad \mathbf{v} = 0 \text{ at } \partial B, \end{aligned}$$

implying, $\mathbf{v} \in H^1(B)$. However, by means of an explicit representation formula, it can be easily proved that given $\mathbf{v} \in H^1(B)$ there exists $\mathbf{w} \in \mathbf{W}^{2,2}(B)$ such that

$$\begin{aligned} &\mathbf{v} = \nabla \times \mathbf{w} \\ &\|\mathbf{w}\|_{2,2} \leq c_1 \|\mathbf{v}\|_{1,2}. \end{aligned} \tag{1.12}$$

Actually, take first $\mathbf{v} \in \mathcal{D}(\Omega)$ and consider the function

$$\mathbf{Z}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{v}(y)}{|x-y|} dy.$$

Then $\mathbf{v} = \nabla \times \mathbf{w}$, where $\mathbf{w} = \nabla \times \mathbf{Z}$. By Calderón-Zygmund theorem on singular integrals and Young's inequality on convolutions it follows

$$\|\mathbf{w}\|_{2,2} \leq c \|\mathbf{v}\|_{1,2},$$

where $c = c(\Omega)$. Relations (1.12) are then a consequence of this ineqauality and of the density of $\mathcal{D}(\Omega)$ into $H^1(\Omega)$. (1.9)-(1.12) imply then (1.3) and so the restriction of $\mathbf{w}$ to Ω verifies all requirements stated in the lemma. The proof is therefore completed.

Remark 1.2. Using the same lines of proof, it is at once recognized that Lemma 1.3 is valid, more generally, with $\mathbf{w} \in \mathbf{W}^{2,q}(\Omega)$, $1 < q < \infty$, provided $\mathbf{a} \in \mathbf{W}^{1-1/q}(\partial\Omega)$. In particular, w obeys the following estimate

$$||\mathbf{w}||_{2,q} \leq c||\mathbf{a}||_{1-1/q,q(\partial\Omega)}$$

Remark 1.3. Lemma 1.3 admits of a suitable, immediate extension to arbitrary dimension $n \geq 4$, which will be appropriate to our purposes. Actually, if we set $\mathbf{W} = \nabla \mathbf{Z}$ (*i.e.*, $W_{ij} = \partial Z_j/\partial x_i$) with $\mathbf{Z}$ fundamental solution of Laplace' s equation in $\mathbb{R}^n$, then it is easily seen that $\mathbf{v} = \nabla \cdot \mathbf{W}$ and that $\mathbf{W}_{ij}$ satisfies an estimate of the type (1.3). More generally, if $\mathbf{a} \in \mathbf{W}^{1-1/q,q}(\partial\Omega)$, $1 < q < \infty$, then for all $i, j = 1, \ldots, n$ we have

$$||\mathbf{W}_{ji}||_{2,q} \leq c||\mathbf{a}||_{1-1/q,q(\partial\Omega)}.$$

2. Proof of the main result.

The aim of this section is to show that, if the fluxes ϕ_i of the boundary data a obey a restriction of the type (0.7), then there exists a solenoidal extension $\mathbf{V}$ of a such that condition (0.7) is verified for some $\alpha < v$. By what we said, this will be sufficient to ensure existence of a steady solution corresponding to a.

Let us introduce some further notation first. We denote by $c = c(n, \Omega)$ the constant entering the problem:

$$\begin{aligned} &\nabla \cdot \mathbf{b} = h \quad \text{in } \Omega \\ (2.1)\qquad &\mathbf{b} \in \mathbf{W}_0^{1,2}(\Omega) \\ &|\mathbf{b}|_{1,2} \leq c||h||_2. \end{aligned}$$

Moreover, if $\partial\Omega$ has more than one boundary, $\Gamma_1, \ldots, \Gamma_{m+1}$, -with Γ_i, $i = 1, \ldots, m$, the "interior" boundaries and Γ_{m+1} the "outer" one- we set

$$d \equiv \min \operatorname{dist}(\Gamma_i, \Gamma_j)$$

and

$$(2.2)\qquad \Omega_{i,d} = \{x \in \Omega : \operatorname{dist}(x, \Gamma_i) < d/2\}.$$

and indicating by ω_i, $i = 1, \dots, m$, the (bounded) connected components of $\mathbb{R}^n \setminus \bar{\Omega}$,

$$
\begin{aligned}
&\sigma_i(x) = -\nabla\mathcal{E}(x - x_i), \;\; x_i \in \omega_i, \;\; i = 1, \dots, m \\
(2.3) \qquad & \\
&\sigma_{m+1}(x) = -\rho_1(x)
\end{aligned}
$$

where $\mathcal{E}(\xi)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. Clearly, we have

$$
(2.4) \qquad \int_{\Gamma_i} \sigma_i \cdot \mathbf{n} = 1 \quad i = 1, \dots, m+1,
$$

where $\mathbf{n}$ denotes the outer normal to $\partial\Omega$.

The following fundamental extension lemma holds.

LEMMA 2.1. *For any* $\mathbf{a} \in \mathbf{W}^{1/2,2}(\partial\Omega)$ *satisfying*

$$
(2.5) \qquad \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} = 0,
$$

and for any $\eta > 0$ *there exists a solenoidal vector field*

$$
\mathbf{V} \in \mathbf{W}^{1,2}(\Omega) \textit{ with } \mathbf{V} = \mathbf{a} \textit{ at } \partial\Omega
$$

verifying

$$
(2.6) \quad \left| \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{V} \cdot \mathbf{u} \right| \le \left\{ \alpha + \sum_{i=1}^{m+1} \left(\kappa^2 \frac{4c\kappa_2}{d} \|\sigma_i\|_{2,\Omega_{i,d}} + \kappa \|\sigma_i\|_{4,\Omega_{i,d}} \right) |\phi_i| \right\} |\mathbf{u}|_{1,2}^2
$$

for all $\mathbf{u} \in \mathcal{D}(\Omega)$. *Here* κ, κ_2 *are constants depending on* n *and defined in (2.14) and* Lemma 1.1, *respectively,* (10) $c = c(n, \Omega)$ *is defined in (2.1) and*

$$
\phi_i = \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n}, \qquad i = 1, \dots, m+1.
$$

Furthermore, σ_i *and* $\Omega_{i,d}$ *are given in (2.2) and (2.3). Finally,*

$$
(2.7) \qquad \|\mathbf{V}\|_{1,2} \le c_1 \|\mathbf{a}\|_{1/2,2(\partial\Omega)},
$$

(10) The value of κ_2 depends on the regularlty of $\partial\Omega$. If Ω is, for example, of class C^∞ then we may take $\kappa_2 = 1$, cf. Remark 1.1.

where $c_1 = c_1(n, \Omega)$ ([11]).

Proof. We shall first consider the case $m > 0$. Let

$$\delta_i(x) \equiv \operatorname*{dist}_{x\in\Omega}(x, \Gamma_i), \qquad i = 1, \dots, m+1,$$

and denote by $\rho_i(x)$ the regularized distance of x from Γ_i, in the sense of STEIN (*cf.* Lemma 1.1). Set

$$\psi(t) = \begin{cases} 1 & t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

and define

$$\psi_i(x) \equiv \psi(4\rho_i(x)/d), \qquad i = 1, \dots, m+1. \tag{2.8}$$

Recalling the properties of $\rho_i(x)$, we have that $\psi_i(x)$ is piecewise differentiable and that, moreover,

$$\begin{aligned} &\psi_i(x) = 1 \quad \text{if } \sigma_i(x) < d/4\kappa_1 \\ &\psi_i(x) = 0 \quad \text{if } \sigma_i(x) \ge d/2 \\ &|\psi_i(x)| \le 1 \\ &supp(\nabla\psi_i) \subset \{x \in \Omega : d/4\kappa_i \le \sigma_i(x) \le d/2\} \end{aligned} \tag{2.8$'$}$$

where κ_1 and κ_2 are the constants introduced in Lemma 1.1. In view if (2.4) and (2.8)$'$ we recover that the field

$$\mathbf{v}_1(x) = \mathbf{a}(x) - \sum_{i=1}^{m+1} \phi_i\psi_i(x)\sigma_i(x), \quad x \in \partial\Omega \tag{2.9}$$

satisfies the $m+1$ conditions

$$\int_{\Gamma_i} \mathbf{v}_1 \cdot \mathbf{n} = 0, \qquad i = 1, \dots, m+1.$$

([11]) Strictly speaking, the validlty of inequality (2.7) is not necessary to show existence of solutions. However, it is fundamental to estimate the solutions (in suitable norm) in terms of the data.

By Lemma 1.3 we then have that, if $n = 3$, there exists $\mathbf{w} \in W^{2,2}(\Omega)$ [resp. $\mathbf{w} \in W^{2,2}(\Omega)$, if $n = 2$] such that $\mathbf{v}_1(x) = \nabla \times \mathbf{w}(x)$, $x \in \partial\Omega$ [*resp.* $\mathbf{v}_1(x) = \nabla \times w(x)$]. For given $\varepsilon > 0$ we set

$$\mathbf{V}_\varepsilon = \nabla \times (\psi_\varepsilon \mathbf{w}) \quad [resp.\ \mathbf{V}_\varepsilon = \nabla \times (\psi_\varepsilon w)]$$

where ψ_ε is the "cut-off" function defined in Lemma 1.1. From the properties of ψ_ε and $\mathbf{w}$ we easily realize that the field

$$\mathbf{U}(x) = \mathbf{V}_\varepsilon(x) + \sum_{i=1}^{m+1} \phi_i \psi_i(x) \rho_i(x), \quad x \in \Omega,$$

is an extension of a satisfying

$$\|\mathbf{U}\|_{1,2} \leq c_1 \|\mathbf{a}\|_{1/2,2(\partial\Omega)}. \tag{2.10}$$

However, $\mathbf{U}$ is not solenoidal and, therefore, in order to obtain the desired extension of a, we have to modify $\mathbf{U}$ appropriately. To this end, let us consider the field b defined by the following properties:

$$\begin{aligned} &\nabla \cdot \mathbf{b} = -\sum_{i=1}^{m+1} \sigma_i(x) \cdot \nabla(\phi_i \psi_i(x)) \equiv h(x) \\ &\mathbf{b} \in \mathbf{W}_0^{1,2}(\Omega), \\ &\|\nabla \mathbf{b}\|_2 \leq c\|h\|_2. \end{aligned} \tag{2.11}$$

Since, by (2.5) and (2.8)',

$$h \in L^q(\Omega), \quad \text{for all } q \in (1,\infty)$$
$$\int_\Omega h = 0,$$

Theorem 1 of BOGOVSKIĬ (1980) ensures the existence of at least one vector b satisfying (2.11). Furthermore, using $(2.8)'_{3,4}$ we obtain

$$\|\nabla \mathbf{b}\|_2 \leq \frac{4c\kappa_2}{d} \sum_{i=1}^{m+1} \|\sigma_i\|_{2,\Omega_{i,d}} |\phi_i|, \tag{2.11$'$}$$

The desired extension of a is then given by the field

$$\begin{aligned} \mathbf{V}(x) &\equiv \mathbf{V}_\varepsilon(x) + \sum_{i=1}^{m+1} \phi_i \psi_i(x) \sigma_i(x) + \mathbf{b}(x) \\ &\equiv \mathbf{V}_\varepsilon(x) + \mathbf{V}_\sigma(x) + \mathbf{b}(x). \end{aligned} \tag{2.12}$$

In fact, $\mathbf{V}$ is solenoidal, belongs to $\mathbf{W}^{1,2}(\Omega)$, its trace at $\partial\Omega$ is $\mathbf{a}$ and, moreover, in view of (2.10) and (2.11), $\mathbf{V}$ satisfies (2.7). Let us now estimate the trilinear form:

$$a(\mathbf{u},\mathbf{V},\mathbf{u}) \equiv \int_\Omega \mathbf{u}\cdot\nabla\mathbf{V}\cdot\mathbf{u}, \quad \mathbf{u}\in H^1.$$

In this respect, we recall the inequality:

$$\|\mathbf{u}\|_4 \le \kappa\|\nabla\mathbf{u}\|_2 \tag{2.13}$$

where

$$\kappa = \begin{cases} 2^{7/4}3^{-13/8}|\Omega|^{1/12} & \text{if } n=3 \\ |\Omega|^{1/4}/\sqrt{2} & \text{if } n=2. \end{cases} \tag{2.14}$$

cf, e.g., LADYZHENSKAYA (1969). By Hölder inequality, by (2.11)', (2.13) and (2.14) we obtain

$$|a(\mathbf{u},\mathbf{b},\mathbf{u})| \le \|\mathbf{u}\|_4^2\|\nabla\mathbf{b}\|_2 \le \sum_{i=1}^{m+1}\left(\kappa^2\frac{4c\kappa_2}{d}\|\sigma_i\|_{2,\Omega_{i,d}}|\phi_i|\right)\|\nabla\mathbf{u}\|_2^2. \tag{2.15}$$

Furthermore, by integration by parts we find

$$|a(\mathbf{u},\mathbf{V}_\sigma,\mathbf{u})| = |a(\mathbf{u},\mathbf{u},\mathbf{V}_\sigma)|$$

and so, again by Hölder inequality, by $(2.8)'_3$, (2.12) and by (2.13) it follows

$$|a(\mathbf{u},\mathbf{V}_\sigma,\mathbf{u})| \le \|\mathbf{u}\|_4\|\nabla\mathbf{u}\|_2\|\mathbf{V}_\sigma\|_4 \le \kappa\|\nabla\mathbf{u}\|_2^2\sum_{i=1}^{m+1}\|\sigma_i\|_{4,\Omega_{i,d}}|\phi_i|. \tag{2.16}$$

It remains to estimate the term

$$a(\mathbf{u},\mathbf{V}_\varepsilon,\mathbf{u}).$$

In this regard, we shall follow the argument of LIONS (1969). From the properties of the function ψ_i it follows

$$|\mathbf{V}_\varepsilon(x)| \le \frac{\varepsilon\kappa_2}{\delta(x)}|\mathbf{w}(x)| + |D\mathbf{w}(x)|, \quad \text{if } \delta(x) < 2\gamma(\varepsilon) \tag{2.17}$$

while

$$\mathbf{V}_\varepsilon(x) = 0, \quad \text{if } \delta(x) \geq 2\gamma(\varepsilon).$$

Moreover, from the Sobolev embedding theorem, it is

$$|\mathbf{w}(x)| \leq c_1 \|\mathbf{w}\|_{2,2}$$

$$\|\nabla \mathbf{w}\|_3 \leq c_1 \|\mathbf{w}\|_{2,2}$$

which, by Lemma 1.3, in turn implies

$$\|\nabla \mathbf{w}\|_3 + |\mathbf{w}(x)| \leq c_2 \|\mathbf{v}_1\|_{1/2,2(\partial\Omega)}. \tag{2.19}$$

Thus, (2.17) along with (2.9) and (2.13) gives for all $\mathbf{u} \in \mathcal{D}(\Omega)$

$$\begin{aligned} \| |\mathbf{u}| \, |\mathbf{V}_\varepsilon| \, \|_2 &\leq c_3 \left(\varepsilon \|\mathbf{v}_*\|_{1/2,2(\partial\Omega)} \|\mathbf{u}\delta^{-1}\|_2 + \int_{\delta(x)<2\gamma(\varepsilon)} u^2 |\nabla \mathbf{w}|^2 \right) \\ &\leq c_4 \left(\varepsilon \|\mathbf{v}_*\|_{1/2,2(\partial\Omega)} \|\mathbf{u}\delta^{-1}\|_2 + \|\nabla \mathbf{u}\|_2 \|\nabla \mathbf{w}\|_{3,\Omega} \right) \end{aligned} \tag{2.20}$$

where $\Omega_\varepsilon \equiv \{x \in \Omega : \delta(x) < 2\gamma(\varepsilon)\}$. In view of (2.19) we have

$$\zeta(\varepsilon) \equiv \|\nabla \mathbf{w}\|_{3,\Omega_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0;$$

furthermore, by a well-known Hardy-type inequality, we also have (*cf.* NEČAS (1967))

$$\|\mathbf{u}\delta^{-1}\|_2 \leq c \|\nabla \mathbf{u}\|_2,$$

so that (2.20) implies

$$\| |\mathbf{u}| \, |\mathbf{V}_3| \, \|_2 \leq c_5 (\varepsilon \|\mathbf{v}_1\|_{1/2,2(\partial\Omega)} + \zeta(\varepsilon)) \|\nabla \mathbf{u}\|_2 \equiv \zeta(\varepsilon) \|\nabla \mathbf{u}\|_2 \tag{2.21}$$

where

$$\chi(\varepsilon) \to \quad \text{as } \varepsilon \to 0. \tag{2.22}$$

From (2.2I) and from Schwarz inequality we then conclude.

$$|a(\mathbf{u}, \mathbf{V}_\varepsilon, \mathbf{u})| = |a(\mathbf{u}, \mathbf{u}\mathbf{V}_\varepsilon)| \leq \chi(\varepsilon) \|\nabla \mathbf{u}\|_2^2. \tag{2.23}$$

Collecting (2.15), (2.16) and (2.23) yields

$$\begin{aligned} |a(\mathbf{u}, \mathbf{V}\mathbf{u})| \leq \Bigg\{ \chi(\varepsilon) + \sum_{i=1}^{m+1} \Bigg(& \kappa^2 \frac{4c\kappa_2}{d} \|\sigma_i\|_{2,\Omega_{i,d}} \\ & + \max_{x \in \Omega_{i,d}} |\mathcal{E}(x - x_i)| + \kappa \frac{4\kappa_2}{d} \|\sigma_i\|_{4,\Omega_{i,d}} \Bigg) |\phi_i| \Bigg\} \|\nabla \mathbf{u}\|_2^2 \end{aligned} \tag{2.24}$$

which, in view of (2. 22), shows the result if $m > 0$. If $m = 0$ the proof is simpler, since, in such a case, one can take $\mathbf{V} = \mathbf{V}_\varepsilon$ and proceed as before to arrive formally at (2.24) with identically vanishing ϕ_i. The lemma is therefore completely proved.

Remark 2.1. It is simple to generalize Lemma 2.1 to space dimension $n \geq 4$, provided we make some changes in the proof. Actually, it suffices to use, instead of (2.13), the Sobolev inequality:

$$\|\mathbf{u}\|_{nq/(n-q)} \leq \gamma \|\nabla \mathbf{u}\|_q, \qquad 1 < q < n,$$

to take the field $\mathbf{b}$ as solution to the following problem

$$\nabla \cdot \mathbf{b} = h$$
$$\mathbf{b} \in \mathbf{W}_0^{1,n/2}(\Omega)$$
$$|\mathbf{b}|_{1,n/2} \leq c\|h\|_{n/2},$$

and, finally, to choose

$$\mathbf{V}_\varepsilon = \nabla \cdot (\psi_\varepsilon \mathbf{W})$$

as an extension of the field $\mathbf{v}_1$, with $\mathbf{W}$ defined in Remark 1.3. However, in order that $\mathbf{V}_\varepsilon$ satisfies the estimate needed in the lemma, we should require that $\mathbf{a}$ possesses slightly more regularity. Actually, in dimension higher than three (2.18) need not hold and we have, instead,

$$(*) \qquad \begin{array}{l} |\mathbf{W}(x)| \leq c_1 \|\mathbf{W}\|_{2,q} \\ \|\mathbf{W}\|_n \leq c_1 \|\mathbf{W}\|_{2,q} \end{array}, \quad q > n/2.$$

On the other hand, taking into account Remark 1.3, the right hand side of (*) is finite provided

$$(**) \qquad \mathbf{a} \in \mathbf{W}^{1-1/q,q}(\partial\Omega), \qquad q > n/2.$$

Therefore, if $n \geq 4$, under this additional condition on $\mathbf{a}$ the vector field (2.12) belongs to $\mathbf{W}^{1,q}(\Omega)$ (12), satisfies (2.23) and obeys the inequality

$$\|\mathbf{V}\|_{1,q} \leq c' \|\mathbf{v}_*\|_{1-1/q,q(\partial\Omega)}.$$

(12) Notice that, since $h \in L^r(\Omega)$, for all $r > 1$, we may take $\mathbf{b} \in \mathbf{W}^{1,q}(\Omega)$, see BOGOVSKIĬ (1980).

One can then show, that the trilinear form $a(\mathbf{u},\mathbf{V},\mathbf{u})$ satisfies the estimate

$$|a(\mathbf{u},\mathbf{V},\mathbf{u})| \leq \left\{ \chi(\varepsilon) + \sum_{i=1}^{m+1} \left(c_1\|\sigma_i\|_{n/2,\Omega_{i,d}} + c_2\|\sigma_i\|_{n,\Omega_{i,d}}\right)|\phi_i| \right\} \|\nabla\mathbf{u}\|_2^2$$

with c_1 and c_2 suitable constants.

In view of Lemma 2.1 and of the considerations developed in the introduction, we then derive the following existence result.

THEOREM 2.1. *Let Ω be a bounded, lipschitz domain of $\mathbb{R}^n$, $n = 2,3$, delimited by $m+1$ closed surfaces (lines) Γ_i, and let $\mathbf{a} \in \mathbf{W}^{1/2,2}(\partial\Omega)$ satisfy condition (0.2). Then, if*

$$\sum_{i=1}^{m+1} \left(\kappa^2\frac{4c\kappa_2}{d}\|\sigma_i\|_{2,\Omega_{i,d}} + \kappa\|\sigma_i\|_{4,\Omega_{i,d}}\right) \left|\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\right| < v,$$

there exists at least one (generalized) solution to the steady Navier-Stokes problem ($\mathcal{NS}$). In (2.25), the constants $\kappa = \kappa(n,\Omega)$ and $c = c(n,\Omega)$ are defined in (2.14), and (2.1) while σ_i and $\Omega_{i,d}$ are given in (2.2) and (2.3), respectively.

Remark 2.2. In view of the considerations developed in Remark 2.1, one can obtain an existence result in space dimension $n \geq 4$ provided a satisfies condition (**) of that remark and, in addition, the fluxes through Γ_i obey the following restriction

$$\sum_{i=1}^{m+1} \left(c_1\|\sigma_i\|_{n/2,\Omega_{i,d}} + c_2\|\sigma_i\|_{n,\Omega_{i,d}}\right) \left|\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\right| < v,$$

Remark 2.3. If $m = 0$, that is, if the connected components $\Gamma_1,\ldots,\Gamma_{M+1}$ of $\partial\Omega$ reduce to one, condition (2.25) is automatically satisfied, since, by the incompressibility condition, it is

$$\int_{\Gamma_1} \mathbf{a}\cdot\mathbf{n} \equiv \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n} = 0.$$

3. Application to flow in an annulus.

Condition (2.25) furnishes a *computable* bound on the fluxes ϕ_i in terms of v. It may be of a certain interest to evaluate this bound when Ω is a spherical shell. In fact, as we have noticed in the introductory section, such domains can not admit an extension field $\mathbf{V}(\alpha)$ of a obeying (0.3) for arbitrary $\alpha > 0$, *cf.* TAKESHITA (1992), and, as a consequence, the LERAY-HOPF construction of steady solutions would require identically vanishing ϕ_i. To fix the ideas, take Ω to be the annulus delimited by circles of radii R and $2R$, respectively. We then have

$$\Omega = \{x \in \mathrm{IR}^2 : R < |x| < 2R\}. \tag{3.1}$$

Thus, in the notation of Theorem 2.1, it results

$$d = 2R - R = R, \quad \kappa_2 = 1 \ (^{13}),$$

$$\sigma_1(x) = -\sigma_2(x) = -\nabla(\log|x|)/2\pi = -1/(2\pi|x|),$$

$$\Omega_{1,d} = \{x \in \mathrm{IR}^2 : R < |x| < 3R/2\},$$

$$\Omega_{2,d} = \{x \in \mathrm{IR}^2 : 3R/2 < |x| < 2R\}.$$

Moreover, we have to give explicit values to the constants κ and c defined in (2.11) and (2.14), respectively. Concerning κ, from (2.14) and (3.1) we find at once

$$\kappa = (3\pi)^{1/4}\sqrt{R/2} \simeq 1.238\sqrt{R}.$$

However, a sharper estimate can be obtained on κ. Actually, for all $\mathbf{u} \in \mathcal{D}(\Omega)$ it is, *cf.* LADYZHENSKAYA (1969),

$$\|u\|_4 \leq 2^{-1/4}\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}$$

and, therefore, setting

$$\mu = \mu(\Omega) = \max_{u \in W_0^{1,2}(\Omega)} (\|u\|_2^2/\|\nabla u\|_2^2).$$

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(13) Cf. Remark 1.1.

(Poincaré constant) it follows that we can choose

$$\kappa = 2^{-1/4}\mu^{1/4}.$$

Since Ω has a particularly simple shape, one can calculate (numerically) the value of μ from the formula

$$R/\sqrt{\mu} = \pi - 1/(16\pi) + 163/(3072\pi^2) - 93029/(491520\pi^5) + \dots$$

cf. McLACHLAN (1961, §1.62, eq.(4)) to recover $\mu \simeq 0.102 \cdot R^2$. Hence,

$$\kappa \simeq 0.47\sqrt{R}. \tag{3.2}$$

To evaluate the constant c, we observe that since the products a $\sigma_i \cdot \nabla\psi_i$, $i = 1, 2$ depend only on $r \equiv |x|$, the function h in (2.11) depends only on r too. Therefore, a solution $\mathbf{b}$ to (2.11) with Ω given in (3.1) can be chosen of the form:

$$\mathbf{b}(x) = \frac{\mathbf{x}}{r^2}\int_R^r \xi h(\xi)d\xi, \qquad x \in \Omega.$$

Since

$$\frac{\partial b_2}{\partial x_1} = \frac{\partial b_1}{\partial x_2}$$

by a direct computation we show

$$|\mathbf{b}|_{1,2} = \|\nabla \cdot \mathbf{b}\|_2 = \|h\|_2,$$

and we conclude $c = 1$. Collecting all these data and setting $\phi \equiv -\phi_1 = \phi_2$, condition (2.25) becomes

$$H|\phi| < v, \tag{3.3}$$

where

$$H \equiv \frac{\kappa}{2\pi}[4\kappa(A + B) + (C + D)],$$

$$A = \left(2\pi\int_R^{3/R2} \xi^{-1}d\xi\right)^{1/2}, \qquad B = \left(2\pi\int_{3R/2}^{2R} \xi^{-1}d\xi\right)^{1/2},$$

$$C = \left(2\pi\int_R^{3/R2} \xi^{-3}d\xi\right)^{1/4}, \qquad D = \left(2\pi\int_{3R/2}^{2R} \xi^{-3}d\xi\right)^{1/4},$$

and κ is given in (3.2). Evaluation of H furnishes

$$H \simeq 0.42 \cdot R + 0,16$$

and the flux condition (3.3) becomes

$$|\phi| < v/(0.42 \cdot R + 0,16).$$

Acknowledgment.

This work was supported by G.N.F.M. of Italian C.N.R. and by M. P. I. 40% and 60% contracts at the University of Ferrara.

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