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ON HILBERT FUNCTION UNDER LIAISON

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A relation between Hilbert functions of linked ideals is established and discussed by various examples. Then it is applied to compute the degree of the intersection of two projective subschemes in a special situation.

We consider the following problem. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subset R := K[x_0, \ldots, x_r]$ be homogeneous ideals of pure dimension d such that $R/\mathfrak{c}$ is a graded Gorenstein K-Algebra and $\mathfrak{c} : \mathfrak{a} = \mathfrak{b}, \mathfrak{c} : \mathfrak{b} = \mathfrak{a}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by $\mathfrak{c}$. Our question is: What can be said on the Hilbert function of $\mathfrak{b}$ in terms of the Hilbert function of $\mathfrak{a}$ and $\mathfrak{c}$. This problem has already been studied in [1] in the case that $R/\mathfrak{a}$ is Cohen-Macaulay. The theorem below is our answer. It also extends results of [5] and [6]. Our approach is quite different from that of [1]. We use duality theory as a main tool.

Moreover, we will discuss our theorem by several examples and give an application to intersection theory.

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1. Notations and preliminary results.

Let K be an infinite field. By a graded K-algebra we always mean a Noetherian graded ring $A = \bigoplus_{t \in \mathbb{N}} A_t$ with $A_0 = K$ and which is generated by A_1. The irrelevant ideal $\bigoplus_{t>0} A_t$ is denoted by $\mathfrak{m}_A$ or simply by $\mathfrak{m}$.

Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a graded A-module. The K-vector spaces M_t are also denoted by $[M]_t$. If M is Noetherian or Artinian $\operatorname{rank}_K [M]_t < \infty$. In that case we denote by $h_M(t) := \operatorname{rank}_K [M]_t$ the Hilbert function of M. It is well known that $h_M(t)$ equals the Hilbert polynomial $p_M(t)$ for $t \gg 0$ if M is Noetherian and for $t \ll 0$ if M is Artinian.

Let M be a Noetherian graded A-module of Krull dimension $d > 0$. Then $p_M(t)$ has the form

$$p_M(t) = h_0(M)\binom{t}{d-1} + h_1(M)\binom{t}{d-2} + \ldots + h_{d-1}(M)$$

where the integers $h_0(M) > 0,\ h_1(M), \ldots, h_{d-1}(M)$ are the so-called Hilbert coefficients of M. We define the index of regularity of M to be

$$r(M) := \min\{t \in \mathbb{Z} : h_M(i) = p_M(i) \text{ for all } i \geq t\}.$$

For $j \in \mathbb{Z}$ we denote by $M(j)$ the graded module given by $[M(j)]_i = [M]_{i+j}$. All homomorphisms between graded moduls are considered to be graded of degree zero. Thus for graded modules M, N we have $f \in \underline{Hom}_A(M,N)$ if f is a homomorphism such that $f([M]_t) \subset [N]_t$ for all $t \in \mathbb{Z}$. We define the graded A-module $\underline{Hom}_A(M,N)$ by $[\underline{Hom}_A(M,N)]_t = \underline{Hom}_A(M,N(t))$ for all $t \in \mathbb{Z}$. As usual $\underline{Ext}^i_A(M,N)$ $(i \in \mathbb{N})$ denotes the i-th right derived functor of $\underline{Hom}_A(M,N)$ and $H^i_{\mathfrak{m}}(M) = \varinjlim_n \underline{Ext}^i_A(A/\mathfrak{m}^n, M)$ the i-th local cohomology module of M. If M is Noetherian $\underline{H}^i_{\mathfrak{m}}(M)$ is an Artinian graded A-module for all $i \in \mathbb{N}$. The local cohomology modules can be used to describe the difference between Hilbert function and Hilbert polynomial [9].

LEMMA 1. *If M is a Noetherian graded A-module we have for* *all* $t \in \mathbb{Z}$

$$h_M(t) - p_M(t) = \sum_{i \geq 0} (-1)^i \mathrm{rank}_K [\underline{H}^i_{\mathfrak{m}}(M)]_t.$$

The dual of a graded A-module M is defined by $M^\vee := \bigoplus_{t \in \mathbb{Z}} Hom_K([M]_{-t}, K)$. If $\mathrm{rank}_K[M]_t < \infty$ for all $t \in \mathbb{Z}$ there is a canonical isomorphism $M \cong M^{\vee\vee}$. The graded K-Algebra $A^\vee$ is the injective hull of $K^\vee \cong K = A/\mathfrak{m}$ in the category of graded A-modules.

The graded K-algebra is said to be Gorenstein if it has a finite injective resolution. Now we can state the duality theorem (cf. [13], Theorem 0.4.14 and [11], Theorem (3.4)).

DUALITY THEOREM *Let S be a Gorenstein ring of dimension d and let A be a factor of S. Then we have*

(i) (graded case) If S is a graded K-algebra and M a graded A-module there are for all $i \in \mathbb{Z}$ natural isomorphisms

$$\underline{H}^i_{\mathfrak{m}}(M)^\vee \cong \underline{Ext}^{d-i}_S(M, S)(r(S) - 1),$$

(ii) (local case) If S is a regular local ring with maximal ideal $\mathfrak{m}$ and M an A-module there are for all $i \in \mathbb{Z}$ natural isomorphisms

$$D(H^i_{\mathfrak{m}}(M)) \cong Ext^{d-i}_S(M, S)$$

where $D(\_) := Hom_s(\_, E)$ and E denotes the injective hull of $k = S/\mathfrak{m}$ (as an S-module).

Let now $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subset R = K[x_0, \ldots, x_r]$ be homogeneous ideals such that $\mathfrak{c} \subset \mathfrak{a} \cap \mathfrak{b}$ and $R/\mathfrak{c}$ is Gorenstein. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by $\mathfrak{c}$ (cf. [10], [7]) if

(i) $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are of pure dimension $\dim R/\mathfrak{c}$

(ii) $\mathfrak{c} : \mathfrak{a} = \mathfrak{b}$ and $\mathfrak{c} : \mathfrak{b} = \mathfrak{a}$.

If in addition $\mathfrak{a}$ and $\mathfrak{b}$ have no primary components in common we have $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$. In that case it is said that $\mathfrak{a}$ and $\mathfrak{b}$ are linked geometrically by $\mathfrak{c}$ in the sense of [10] and [7].

Note that in condition (ii) each relation is equivalent to the other because $\mathfrak{a}$ and $\mathfrak{c} : \mathfrak{a}$ are linked by $\mathfrak{c}$ if $\mathfrak{c} \subsetneq \mathfrak{a}$, $R/\mathfrak{a}$ is of pure dimension $\dim R/\mathfrak{c}$ and $R/\mathfrak{c}$ is Gorenstein [10], Proposition 2.2. If $\mathfrak{a}$ and $\mathfrak{b}$ are linked by $\mathfrak{c}$ then $R/\mathfrak{a}$ is Cohen-Macaulay, Buchsbaum or locally Cohen-Macaulay, respectively iff $R/\mathfrak{b}$ has the corresponding property [10] (cf. also [7], [13]).

A homogeneous ideal $\mathfrak{c} \subset R$ or $R/\mathfrak{c}$ is said to be a complete intersection of type $(d_1, \ldots, d_s)$ if $\mathfrak{c}$ is generated by an R-sequence $\{F_1, \ldots, F_s\}$ of homogeneous forms of degree $d_1, \ldots, d_s$. In that case we have $r(R/\mathfrak{c}) = d_1 + \ldots + d_s - r$.

For unexplained facts and notations we refer to the monography [13].

2. Main result.

The purpose of this note is to prove and to discuss the following theorem on Hilbert function under liaison.

THEOREM *Let* $S := R/\mathfrak{c}$ *be a graded Gorenstein* K*-algebra of Krull dimension* $d > 0$ *and let* $\mathfrak{a}, \mathfrak{b} \subset R$ *be homogeneous ideals linked by* $\mathfrak{c}$. *Then*

(i) $h_0(R/\mathfrak{b}) = h_0(R/\mathfrak{c}) - h_0(R/\mathfrak{a})$ *and if* $d > 1$

$$h_1(R/\mathfrak{b}) = (r(S) - d + 1)h_0(R/\mathfrak{a}) + h_1(R/\mathfrak{a}) + h_1(R/\mathfrak{c}).$$

(ii) If $R/\mathfrak{a}$ *is locally Cohen-Macaulay for all* $t \in \mathbb{Z}$ *we obtain*

$$p_{R/\mathfrak{b}}(t) = p_S(t) + (-1)^d p_{R/\mathfrak{a}}(r(S) - 1 - t).$$

(iii) If $R/\mathfrak{a}$ *is Cohen-Macaulay for all* $t \in \mathbb{Z}$ *we get*

$$h_{R/\mathfrak{b}}(t) = h_S(t) + (-1)^{\alpha+1}[h_{R/\mathfrak{a}}(r(S) - 1 - t) - p_{R/\mathfrak{a}}(r(S) - 1 - t)].$$

(iv) If $R/\mathfrak{a}$ *is Gorenstein for all* $t \in \mathbb{Z}$ *we have*

$$h_{R/\mathfrak{b}}(t) = h_S(t) - h_{R/\mathfrak{a}}(r(R/\mathfrak{a}) - r(S) + t).$$

Remarks (1) Since the situation in (i), (ii), (iii) is symmetric we may interchange $\mathbf{a}$ and $\mathbf{b}$ in these formulas. Using (i) we thus obtain

$$h_1(R/\mathbf{c}) = -\frac{1}{2}(r(S) - d + 1)h_0(R/\mathbf{c}).$$

(2) Let C_1, $C_2 \subseteq \mathbb{P}^3$ be smooth curves of genus p_1 and p_2 respectively linked by a complete intersection of type (d_1, d_2). Then Theorem (i) provides

$$p_1 - p_2 = \left(\frac{d_1 + d_2}{2} - 2\right)(deg(C_1) - deg(C_2)),$$

a relation which is well known (cf., for example, [7]) but usually the fact that C_1, C_2 have codimension 2 is essentially used in the proofs.

(3) For a function $f : \mathbb{Z} \to \mathbb{Z}$ we define $\Delta^0 f = f$ and $\Delta^{i+1} f(t) = \Delta^i f(t) - \Delta^i f(t-1)$ $(i \geq 0, t \in \mathbb{Z})$. Then we obtain from Theorem (iii) if $R/\mathbf{a}$ is Cohen-Macaulay

$$\Delta^d h_{R/\mathbf{b}}(t) = \Delta^d h_S(t) - \Delta^d h_{R/\mathbf{a}}(r(s) + d - 1 - t)$$

This is the content of Theorem 3 of [1].

(4) For a further discussion of the Theorem we refer to the next section.

In order to state the starting point for the proof of our Theorem we define the canonical module $K_{\mathbf{a}}$ of $R/\mathbf{a}$ to be $K_{\mathbf{a}} = \underline{Ext}_R^{r+1-d}(R/\mathbf{a}, R)(-r-1)$

LEMMA 2. *We have the following exact sequence*

$$O \to K_{\mathbf{a}}(-r(S) + 1) \to S \to R/\mathbf{b} \to 0.$$

Proof. Since $S = R/\mathbf{c}$ is Gorenstein the duality theorem shows $K_{\mathbf{a}}(1 - r(S)) \cong (\underline{H}^d_{\mathbf{m}}(R/\mathbf{a}))^\vee(1 - r(S)) \cong (\underline{H}^d_{\mathbf{m}\cdot S}(R/\mathbf{a}))^\vee(1 - r(S)) \cong \underline{Hom}_S(R/\mathbf{a}, S) \cong \underline{Hom}_R(R/\mathbf{a}, R/\mathbf{c}) \cong \mathbf{c} : \mathbf{a}/\mathbf{c} = \mathbf{b}/\mathbf{c}$. Q.E.D.

Let X, V, $W \subset \mathbb{P}^r$ be the subschemes defined by $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{b}$. Applying the "sheafification" functor we get from Lemma 2 the exact sequence

$$0 \to \omega_V(1 - (X)) \to \mathcal{O}_X \to \mathcal{O}_W \to 0$$

where ω_V is the dualizing sheaf of V and $r(X) := r(S)$.

Using the duality theorem we obtain from Lemma 2.

COROLLARY 3. *If* $\mathbf{a}, \mathbf{b} \subset R$ *are linked by* $\mathbf{c}$ *we get for all* $t \in \mathbb{Z}$:

$$h_{R/\mathbf{b}}(t) = h_S(t) - \operatorname{rank}_K [\underline{H}^d_{\mathfrak{m}}(R/\mathbf{a})]_{r(S)-1-t}.$$

Since we have with the above notations $h^0(\mathbb{P}^r, \mathcal{J}_X(t)) = \binom{r+t}{r} - [H^{\alpha+1}_{\mathfrak{m}} h_{R/\mathbf{c}}(t)$, the corresponding relation for W and $[\underline{H}^d_{\mathfrak{m}}(R/\mathbf{a})]_t \cong [\underline{H}^{\alpha+1}_{\mathfrak{m}} (\mathbf{a})]_t \cong H^d(\mathbb{P}^r, \mathcal{J}_V(t))$ we can rewrite Corollary 3 if $d < r$:

$$h^0(\mathbb{P}^r, \mathcal{J}_W(t) = h^d(\mathbb{P}^r, \mathcal{J}_V(r(X) - 1 - t)) + h^0(\mathbb{P}^r, \mathcal{J}_X(t))$$

This is the main result of [5] and proved there if V, W are locally Cohen-Macaulay.

Corollary 3 shows that it suffices to compute $\operatorname{rank}_K [\underline{H}^d_{\mathfrak{m}}(R/\mathbf{a})]_t$ in order to prove the Theorem. We need the following result. In the proof we use ideas of [12].

LEMMA 4. *Let* $A = R/\mathbf{a}$ *be a graded* K*-algebra of pure dimension* $d > 0$ *then*

$$\dim \underline{H}^i_{\mathfrak{m}}(A)^\vee = \begin{cases} < i & \text{if } 1 \le i < d \\ d & \text{if } i = d. \end{cases}$$

Moreover, we have for all $t \ll 0$

$$\operatorname{rank}_K [\underline{H}^d_{\mathfrak{m}}(A)]_t = (-1)^{d+1} p_A(t) + o(t^{d-2}).$$

where we set $o(t^{-1}) := 0$.

Proof. Since a has grade $r + 1 - d$ we find an R-regular sequence $F_0, \ldots, F_{r-d} \in \mathbf{a}$ and put $S := R/(F_0, \ldots, F_{r-d})R$. From the duality theorem we obtain $\underline{H}^i_{\mathfrak{m}}(A)^\vee \cong \underline{H}^i_{\mathfrak{m}_S}(S/\mathbf{a}S)^\vee \cong \underline{Ext}^{d-i}_S(S/\mathbf{a}S, S)(r(S) - 1)$. Therefore the assertion is proved for $1 \le i < d$ if we can show $\dim \underline{Ext}^{d-i}_S(S/\mathbf{a}S, S) < i$ $(1 \le i < d)$.

Assume $\dim \underline{Ext}^{d-i}_S(S/\mathbf{a}S, S) \ge i$ for some i $(1 \le i < d)$. Let $\mathbf{p} \in \operatorname{Supp} \underline{Ext}^{d-i}_S(S/\mathbf{a}S, S)$ with $\dim S/\mathbf{p} \ge i$. Then the duality theorem yields

$$0 \ne \underline{Ext}^{d-i}_S(S/\mathbf{a}S, S)_{\mathbf{p}} \cong Ext^{d-i}_{S_{\mathbf{p}}}(S_{\mathbf{p}}/\mathbf{a}S_{\mathbf{p}}, S_{\mathbf{p}}) = D(H^{\dim S_{\mathbf{p}} - d + i}_{\mathbf{p}S_{\mathbf{p}}}(S_{\mathbf{p}}/\mathbf{a}S_{\mathbf{p}})).$$

Since $\dim S_{\mathfrak{p}} - d + i = \dim S - \dim S/\mathfrak{p} - d + i = i - \dim S/\mathfrak{p} \leq 0$ we get $\dim S/\mathfrak{p} = i$ and $0 \neq H^0_{\mathfrak{p}S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{a}S_{\mathfrak{p}})$, i.e. $\mathfrak{p} \in Ass S/\mathfrak{a}S$. But this contradicts the assumption that $R/\mathfrak{a}R \cong S/\mathfrak{a}S$ is of pure dimension $d > i$.

It remains to show the assertion for $i = d$. We have proved for $i < d$ that $\mathrm{rank}_K[\underline{H}^i_{\mathfrak{m}}(A)]_t$ is either zero for all $t \ll 0$ or a polynomial of degree $\leq i - 2$. Thus Lemma 1 completes the proof. Q.E.D.

Now we are in a position to prove the theorem.

Proof of the theorem: (i) Combining Corollary 3 and the second assertion of Lemma 4 a routine computation shows (i).

(ii) If $R/\mathfrak{a}$ is locally Cohen-Macaulay $\underline{H}^i_{\mathfrak{m}}(R/a)$ is of finite length for all $i < d$. Thus Lemma 1 gives for $t \ll 0$

$$-p_A(t) = (-1)\mathrm{rank}_K[\underline{H}^d_{\mathfrak{m}}(R/a)]_t.$$

Now (ii) follows from corollary 3.

(iii) Since $H^i_{\mathfrak{m}}(R/\mathfrak{a}) = 0$ for all $i < d$ iff $R/\mathfrak{a}$ is Cohen-Macaulay (iii) is again a consequence of corollary 3.

(iv) We have the isomorphism $(R/\mathfrak{a})^{\vee} \cong H^d_{\mathfrak{m}}(R/a)(r(R/a) - 1))$ if $R/\mathfrak{a}$ is Gorenstein (cf. for example [13], page 58). Q.E.D.

3. Examples and miscellaneous results.

We want to show that we can not weaken the assumptions on $R/\mathfrak{a}$ in Theorem (ii), (iii), (iv) even if we consider geometrical linkage.

EXAMPLE 5. We consider the following homogeneous ideals $\mathfrak{c}, \mathfrak{p}, \mathfrak{q} \subset R = K[x_0, \ldots, x_4]$:

$$\mathfrak{c} = (x_1x_4 - x_2x_3, x_0x_3x_4 - x_0x_4^2 + x_3^3)$$

$$\mathfrak{p} = (x_1x_4 - x_2x_3, x_0x_1x_2 - x_0x_2^2 + x_1^2x_3, x_0x_2x_3 - x_0x_2x_4 + x_1x_3^2, x_0x_3x_4 - x_0x_4^2 + x_3^3)$$

$$\mathfrak{q} = (x_1x_4 - x_2x_3, x_3^2, x_3x_4, x_4^2)$$

Note that $\mathfrak{q}$ is a (x_3, x_4)-primary ideal of length 2 ([13], Theorem III.3.2) and $\mathfrak{p}$ defines a surface in $\mathbb{P}^4$ given parametrically by

$\{u^3, u^2v, uvw, uw(w-u), w^2(w-u)\}$ ([8], page 234) which was first studied by R. Hartshorne. Furthermore $\mathfrak{c}$ has primary decomposition $\mathfrak{c} = \mathfrak{p} \cap \mathfrak{q}$. Thus $\mathfrak{p}$ and $\mathfrak{q}$ are linked geometrically by $\mathfrak{c}$. Moreover, $R/\mathfrak{p}$ is locally Buchsbaum but not locally Cohen-Macaulay [13], Example V.5.2. For the Hilbert polynomials we obtain:

$$p_{R/\mathfrak{q}}(t) = 2\binom{t}{2} + 4t + 1$$

$$p_{R/\mathfrak{p}}(t) = 4\binom{t}{2} + 5t \quad \text{([8], page 234) and}$$

$$p_{R/\mathfrak{c}}(t) = 6\binom{t}{2} + 3t + 2 \quad \text{and} \quad r(R/\mathfrak{c}) = 1.$$

Since $p_{R/\mathfrak{q}}(t) \neq p_{R/\mathfrak{c}}(t) - p_{R/\mathfrak{p}}(-t)$ this example shows that the assumption in Theorem (ii) is not superfluous.

EXAMPLE 6. Now we want to show that we cannot replace the assumption in Theorem (iii) by the weaker assumption that $R/\mathfrak{a}$ is Buchsbaum. We consider the following primary decomposition

$$\mathfrak{c} = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3) = \mathfrak{p} \cap (x_0, x_1) \cap (x_2, x_3) \subset R = K[x_0, \ldots, x_3]$$

where $\mathfrak{p} = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3, x_0^2x_2 - x_1^3, x_1x_3^2 - x_2^3)$ is the defining prime ideal of the twisted quartic curve in $\mathbb{P}^3$ given parametrically by $\{s^4, s^3t, st^3, t^4\}$. We put $\mathfrak{a} = (x_0, x_1) \cap (x_2, x_3)$. $R/\mathfrak{a}$ and thus also $R/\mathfrak{p}$ are known to be Buchsbaum [13], Lemma I.2.14 and page 15. The above primary decomposition shows that $\mathfrak{p}$ and $\mathfrak{a}$ are linked geometrically by $\mathfrak{c}$. Moreover we have $p_{R/\mathfrak{a}}(t) = 2t + 2$ and $r(R/\mathfrak{c}) = 2$. Thus we obtain

$$h_{R/\mathfrak{p}}(1) = 4 \neq 4 - [1 - 2] = h_{R/\mathfrak{c}}(1) - [h_{R/\mathfrak{a}}(0) - p_{R/\mathfrak{a}}(0)],$$

i.e., the assertion of Theorem (iii) is not true for $t = 1$.

On the other hand we wish to point out that there are Buchsbaum K-algebras $R/\mathfrak{a}$ such that the assertion of Theorem (iii) is true. A careful study of the method of Evans and Griffith [2] (cf. also [13]) to construct rings with prescribed local cohomology shows that we can find graded Buchsbaum K-algebras $R/\mathfrak{a}$ of dimension

$d \geq 3$ with

$$[\underline{H}^1_{\mathfrak{m}}(R/\mathfrak{a})]_t \cong [\underline{H}^2_{\mathfrak{m}}(R/\mathfrak{a})]_t \cong \begin{cases} K & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{H}^i_{\mathfrak{m}}(R/\mathfrak{a}) = 0 \quad \text{for} \quad 2 < i < d.$$

and $\mathfrak{a}$ is of pure codimension 2. Thus we get by Lemma 1 for all $t \in \mathbb{Z}$ that $h_{R/\mathfrak{a}}(t) - p_{R/\mathfrak{a}}(t) = (-1)^d \text{rank}[\underline{H}^d_{\mathfrak{m}}(R/\mathfrak{a})]_t$ and $\mathfrak{a}$ contains a complete intersection $\mathfrak{c}$ of codimension 2. Put $\mathfrak{b} = \mathfrak{c} : \mathfrak{a}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are linked by $\mathfrak{c}$ ([10], Proposition 2.2) and we obtain in view of Corollary 3

$$h_{R/\mathfrak{b}}(t) = h_{R/\mathfrak{c}}(t) + (-1)^{d+1}[h_{R/\mathfrak{a}}(r(R/\mathfrak{c}) - 1 - t) - p_{R/\mathfrak{a}}(r(R/c) - 1 - t)].$$

Thus the assumption in Theorem (iii) on $R/\mathfrak{a}$ is not necessary.

The next example shows that the assumption in Theorem (iv) can not be weakened but is also not necessary.

EXAMPLE 7. Let $R/\mathfrak{a}$ be a graded Cohen-Macaulay K-algebra. Compairing Theorem (iii) and (iv) we see that the assertion of (iv) is true iff for all $t \in \mathbb{Z}$

$$(*) \qquad h_{R/\mathfrak{a}}(t) = (-1)^d[h_{R/\mathfrak{a}}(r(R/\mathfrak{a}) - 1 - t) - p_{R/\mathfrak{a}}(r(R/\mathfrak{a}) - 1 - t)].$$

This is a condition on (the symmetry of) the Hilbert function of $R/\mathfrak{a}$. It is satisfied if $R/\mathfrak{a}$ is Gorenstein.

Let now $X \subseteq \mathbb{P}^r (r \geq 2)$ be a finite set of (simple) points spanning $\mathbb{P}^r$ and let $\mathfrak{a} \subset R$ be the defining ideal of X. Moreover we take $\mathfrak{b}, \mathfrak{c} \subset R$ from $\mathfrak{a}$ as above. Then $R/\mathfrak{a}$ is a Cohen-Macaulay K-algebra of dimension 1. If, for example, X consists of $r+3$ points then (*) cannot be satisfied and therefore $R/\mathfrak{a}$ is not Gorenstein and the assertion of Theorem (iv) is not true. However this assertion and (*) are true if X has $r+2$ points. But in this case $R/\mathfrak{a}$ is Gorenstein iff any $r+1$ points of X span $\mathbb{P}^r$ [1], Theorem 5 or [4], Theorem C.

We will conclude with an application of our Theorem to intersection theory. Let V, W, be pure-dimensional projective subschemes of $\mathbb{P}^r$. Since Bezout has stated his famous theorem a lot of work has been spent to understand the degree of $V \cap W$, especially to understand the situations where $\deg V \cdot \deg W \neq \deg(V \cap W)$ (cf. for example [3]). In the special situation of geometrical linkage we can often compute $\deg(V \cap W)$. Note that a subscheme X is said to be Gorenstein if its coordinate ring $R/I(X)$ is Gorenstein and for any subscheme $\deg X = h_0(R/I(X))$. Moreover we set $h_1(X) = h_1(R/I(X))$ and $r(X) = r(R/I(X))$.

As preparation we first note a consequence of Theorem (i) and Remark (1).

COROLLARY 8. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \subset R$ *be as in the Theorem and additionally* $h_0(R/\mathbf{a}) \neq h_0(R/\mathbf{b})$. *Then*

$$h_1(R/\mathbf{c}) = \frac{h_1(R/\mathbf{a}) - h_1(R/\mathbf{b})}{h_0(R/\mathbf{a}) - h_0(R/\mathbf{b})} \cdot (h_0(R/\mathbf{a}) + h_0(R/\mathbf{b})).$$

PROPOSITION 9. *Let* V, W *be projective subschemes of* $\mathbb{P}^r$ *of pure dimension* $n \geq 1$ *with no primary components in common and different degree such that* $V \cup W$ *is Gorenstein of dimension* n. *Then*

$$\alpha = 2\frac{h_0(V) \cdot h_1(W) - h_1(V) \cdot h_0(W)}{h_0(V) - h_0(W)} \geq 0$$

and $\alpha > 0$ *iff* $V \cap W$ *has dimension* $n-1$. *In that case* $\deg(V \cap W) = \alpha$.

Proof. Let $R = K[x_0, \ldots, x_r]$. By the assumptions $R/I(V \cup W)$ is Gorenstein and $I(V)$, $I(W)$ are linked geometrically by $I(V \cap W)$. The exact sequence

$$0 \to R/I(V \cup W) \to R/I(V) \oplus R/I(W) \to R/I(V \cap W) \to 0$$

gives us with $p_V(t) := p_{R/I(V)}(t)$ for all t and analogous notations for the other subschemes

$$p_{V \cap W}(t) = p_V(t) + p_W(t) - p_{V \cup W}(t).$$

Applying Theorem (i) we see that $p_{V \cap W}(t)$ is a polynomial of the form

$$p_{V \cap W}(t) = \alpha_0 \binom{t}{n-1} + \alpha_1 \binom{t}{n-2} + \ldots + \alpha_{n-1}.$$

with integers $\alpha_0 \geq 0$, $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_0 = h_1(V) + h_1(W) - h_1(V \cup W)$. Thus Corollary 8 yields $\alpha_0 = \alpha \geq 0$. This proves the proposition because the degree of $p_{V \cap W}(t)$ equals the dimension of $V \cap W$. Q.E.D.

In Corollary 8 (and Proposition 9) we excluded the case $h_0(R/\mathfrak{a}) = h_0(R/\mathfrak{b})$ since then $h_1(R/\mathfrak{c})$ is not uniquely determined by $h_0(R/\mathfrak{a})$, $h_0(R/\mathfrak{b})$, $h_1(R/\mathfrak{a})$ and $h_1(R/\mathfrak{b})$ as we demonstrate now.

EXAMPLE 10. We consider the following ideals in $R = K[x_0, \dots x_3]$:

$\mathfrak{a} = (x_0x_1, x_2x_3^2)$,

$\mathfrak{c} = (x_0x_1, x_2x_3^5)$,

$\mathfrak{c}' = (x_0x_1^3, x_2x_3^2)$,

$\mathfrak{b}' = \mathfrak{c}' : \mathfrak{a}$, $\mathfrak{b} = \mathfrak{c} : \mathfrak{a}$.

As complete intersections $R/\mathfrak{c}$ and $R/\mathfrak{c}'$ are Gorenstein. Thus $\mathfrak{a}$ and $\mathfrak{b}$ are linked by $\mathfrak{c}$ and $\mathfrak{a}$ and $\mathfrak{b}'$ are linked by $\mathfrak{c}'$ according to [10], Proposition 2.2. Moreover we have $h_0(R/\mathfrak{c}) = h_0(R/\mathfrak{c}') = 2h_0(R/\mathfrak{a}) = 12$ and therefore $h_0(R/\mathfrak{b}) = h_0(R/\mathfrak{b}') = 6$. Since $h_0(R/\mathfrak{a}) = h_0(R/\mathfrak{b}') = h_0(R/b')$ we get from Theorem (i) $h_1(R/a) = h_1(R/\mathfrak{b}) = h_1(R/\mathfrak{b}')$ but $-42 = h_1(R/\mathfrak{c}) \neq h_1(R/\mathfrak{c}') = -36$.

Note that in Example 10 we did not consider geometrical linkage. Thus it remains open whether $h_0(R/\mathfrak{c})$ can be computed in terms of $h_0(R/\mathfrak{a})$ and $h_1(R/\mathfrak{a})$ if $h_0(R/\mathfrak{a}) = h_0(R/\mathfrak{b})$. In that case we have $h_1(R/\mathfrak{a}) = h_1(R/\mathfrak{b})$ by Theorem (i) and Remark (1).

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