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CONTRACTIBILITY OF CURVES

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Results concerning contractibility of curves (equivalently: of dendroids) are collected and discussed in the paper. Interrelations between various conditions which are either sufficient or necessary for a curve to be contractible are studied. A full discussion is provided of numerous possibilities, and consequently several new implications are proved and examples are constructed giving a wide spectrum of both results and open questions concerning the subject.

0. Introduction.

During the last two decades contractibility of curves, i.e., of one-dimensional continua was studied by a number of authors (see References). Various conditions were considered which either imply or are implied by contractibility of a curve. They were formulated using various techniques, were expressed in different ways, so

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sometimes it is not easy to compare them. In the present paper some interrelations between these conditions are investigated.

All spaces considered in the paper are metric continua, and all mappings are assumed to be continuous. A property of a continuum X is said to be *hereditary* provided each subcontinuum of X has this property. A continuum is said to be *hereditarily unicoherent* if the intersection of each two its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid.* If a dendroid is locally connected, it is called a *dendride.* The unique arc joining points a and b in a dendroid is denoted by a b. A point of a dendroid X is called an end point of X if it is an *end point* of each arc contained in X and containing it. By a *ramification point* of a dendroid X we understand a point which is the centre of a simple triod contained in X , i.e., a point $p \in X$ such that there are three arcs pa, pb and pc in X, the intersection of each two of them being just the singleton $\{p\}$. A dendroid having exactly one ramification point v is called a *fan*, and v is then called its *top*. If a fan has countably many end points, it is said to be *countable*.

We use the symbols Li, Ls and Lim to denote the lower limit, the upper limit and the topological limit of a sequence of sets as they are defined in the Kuratowski monograph [38], §29, I, III and VI, pp. 335, 337 and 339; and the symbol lim denotes the limit of a sequence of points of a space. If two points p and q of either the plane or Euclidean 3-space are given, then $\overline{pq}$ denotes the straight line segment with ends p and q. As usual, $\mathbb{N}$ denotes the set of all positive integers.

Recall that a continuum X is said to be *uniformly arcwise connected* provided it is arcwise connected and for every positive number ε there is a number $k \in \mathbb{N}$ such that every arc contained in X can be divided into at most k subarcs whose diameters do not exceed ε (see [5], p. 193; compare also [37]).

A continuum X is said to be *contractible* provided there exists a mapping $H : X \times [0,1] \to X$ (called a *homotopy*) such that for each point x in X we have $H(x,0) = x$ and $H(x,1) = p$ for some point $p \in X$. It is known (see [8], Proposition 1, p. 73) that

(0.1) *if a continuum is one-dimensional and contractible, then it is a dendroid,* and moreover, ([3], Proposition 4, p. 73; cf. [15], Theorem 3, p. 94)

(0.2) *each contractible dendroid is uniformly arcwise connected.*

Furthermore, as an immediate consequence of the definitions we see that

(0.3) *a locally connected curve is contractible if and only if it is a dendrite.*

1. Homotopically fixed sets.

Some general conditions which imply noncontractibility of dendroids are considered e.g. in [16]. We recall them here for the reader's convenience.

A homotopy $H : X \times [0,1] \to X$ for which the condition $H(x,0) = x$ holds for each point x of X is called a *deformation*. The following proposition is known (see [16], Proposition 1, p. 230).

PROPOSITION 1.1. (Charatonik, Grabowski). *If a space X contains some two subsets A and B such that*

(1.2) $$\emptyset \neq A \subset B \neq X,$$

and

(1.3) for every deformation $H : X \times [0,1] \to X$ we have $H(A \times [0,1]) \subset B$,

then X is not contractible.

It is of some interest to know if a converse is true to the above proposition in the following sense.

Question 1.4. Does every noncontractible dendroid X contain some two subsets A and B satisfying (1.2) and (1.3)?

Recall that a nonempty subset A of a space X is said to be *homotopically fixed* provided condition (1.3) holds with $B = A$ (see [16], p. 230). As a particular case of 1.4 above we have

Question 1.5. Does every noncontractible dendroid X contain a homotopically fixed proper subset?

Another cause of noncontractibility of dendroids, which resembles the one given in 1.1 is the following.

PROPOSITION 1.6. *If a nondegenerate space X contains a nonempty subset A such that*

(1.7) *for every deformation $H : X \times [0,1] \to X$ and for every $t \in [0,1]$ we have $A \subset H(X \times \{t\})$,*

then X is not contractible.

Proof. Since $A \subset H(X \times \{1\})$, if A is nondegenerate, then the set $H(X \times \{1\})$ cannot be a singleton. If A is degenerate, say $A = \{a\}$, take $b \in X \setminus \{a\}$ and suppose that X is contractible, i.e., there is a deformation $H_0 : X \times [0,1] \to X$ with $H_0(X \times \{1\}) = \{a\}$. Then X is arcwise connected ([39], §54, VI, Theorem 1, p. 374), and if $h : [0,1] \to ab$ is a homeomorphism with $h(0) = a$ and $h(1) = b$, we define $H : X \times [0,1] \to X$ by $H(x,t) = H_0(x,2t)$ for $t \in [0,1/2]$, and $H(x,t) = h(2t-1)$ for $t \in [1/2,1]$. Then $H(X \times \{1\}) = \{b\}$, so (1.7) is not satisfied.

Question 1.8. Does every noncontractible dendroid X contain a nonempty subset A satisfying (1.7)?

2. Three conditions.

Now we recall some special concepts which are closely related to contractibility of dendroids (especially of fans).

A dendroid X is said to be of *type* N (between points p and q) provided there exist in X two sequences of arcs $p_n p'_n$ and $q_n q'_n$, and points $p''_n \in q_n q'_n \setminus \{q'_n, q'_n\}$ and $q''_n \in p_n p'_n \setminus \{p, p'_n\}$, such that the following conditions are satisfied:

(2.1) $$pq = \operatorname{Lim} p_n p'_n = \operatorname{Lim} q_n q'_n;$$

(2.2) $$p = \lim p_n = \lim p'_n = \lim p''_n;$$

(2.3) $$q = \lim q_n = \lim q'_n = \lim q''_n.$$

The above concept is due to L. G. Oversteegen ([47], p. 837) and is related to the following condition of B. G. Graham (see [33], p. 78). A dendroid X is said to *contain a zigzag* provided there exist in X: an arc pq, a sequence of arcs $p_n q_n$ and two sequences of points p'_n and q'_n situated in these arcs in such a manner that $p_n < q'_n < p'_n < q_n$ (where $<$ denotes the natural order on $p_n q_n$ from p_n to q_n), for which the following conditions hold:

(2.4) $$pq = \mathrm{Lim}\, p_n q_n;$$

(2.5) $$p = \lim p_n = \lim p'_n;$$

(2.6) $$q = \lim q_n = \lim q'_n.$$

It is evident that if a dendroid contains a zigzag, then it is of type N ([49], p. 393) but not conversely, even for fans, as it can be seen from an example below.

EXAMPLE 2.7. There is a countable plane fan of type N which contains no zigzag.

Proof. Let v be the pole (i.e. the origin) of a polar coordinate system in the Euclidean plane. For each $n \in N$ put in the polar coordinates (ρ, ϕ):

$$a = (1,0),\ a_n = (1, 2^{1-n}),\ p_n = (1/3, (3/4)\cdot 2^{1-n}),$$

$$q_n = (2/3, (3/4)\cdot 2^{1-n}),\ r_n = (2/3, 2^{1-n}).$$

Let

$$X = \overline{va} \cup \bigcup\{\overline{va_{2n-1}} \cup \overline{a_{2n-1}p_{2n-1}} \cup \overline{p_{2n-1}q_{2n-1}} : n \in \mathbb{N}\}$$
$$\cup \bigcup\{\overline{vr_{2n}} \cup \overline{r_{2n}p_{2n}} : n \in \mathbb{N}\}$$

Then X is a fan with the top v. (see Fig. 1). Putting $p = \lim p_n = (1/3, 0)$ and $q = \lim q_n = (2/3, 0)$, we see that X is of type N between p and q, while it contains no zigzag.

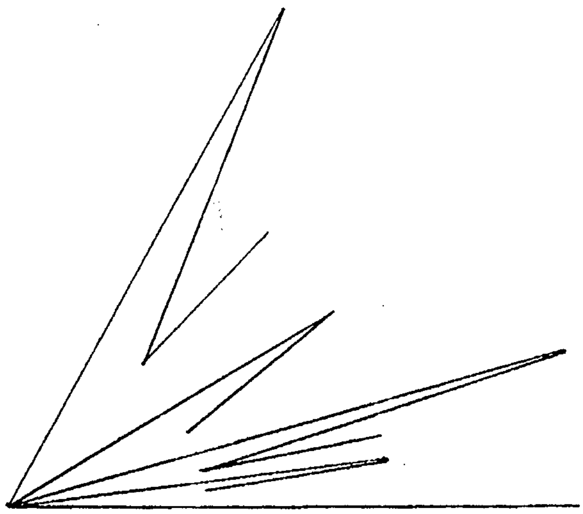

Fig. 1

A point p of a dendroid X is called a *Q-point* of X provided there exists a sequence of points p_n of X converging to p such that Ls $pp_n \neq \{p\}$ and, if for each $n \in \mathbb{N}$ the arc $p_n q_n$ is irreducible between p_n and the continuum Ls pp_n, then the sequence of points q_n converges also to p. This concept is due to R. B. Bennett [3] and it was intensively exploited in investigations of contractibility of fans, e.g. in [33] and [49].

The third concept we recall here is pairwise smoothness; it was formulated by B. G. Graham in [33], p. 78, and was shown to be an important tool in studies of contractibility of fans in [33] and [49]. Let two sequences of points r_n^1 and r_n^2 of a dendroid X be given, both converging to a common limit point r. We say that the former sequence *dominates* the latter one provided that whenever there is a point s in X and a sequence of points s_n^1 of X converging to s with the property that the arcs $r_n^1 s_n^1$ converge to the arc rs, then it follows that there also exists a sequence of points s_n^2 of X converging to s such that the arcs $r_n^2 s_n^2$ converge to rs.

Remark 2.8. If, for each point s of a dendroid X and for each sequence s_n^1 tending to s the limit of the sequence of arcs $r_n^1 s_n^1$ is *not* an arc, then the sequence r_n^1 dominates all the sequences r_n^2 whatsoever.

A dendroid X is said to be *pairwise smooth* provided that whenever a pair of sequences converge to a common limit point, then one of the pair dominates the other.

The following internal characterization of contractibility of fans is due to L. G. Oversteegen (see [49], Theorem 3.4. p. 393; compare also Theorem 3.4 below).

THEOREM 2.9. (Oversteegen). *For every fan X the following conditions are equivalent:*

(2.10) X is contractible;

(2.11) X is not of type N, contains no Q-point and is pairwise smooth;

(2.12) X contains no zigzag, contains no Q-point and is pairwise smooth.

The above characterization describes three possible reasons for the noncontractibility of a fan:

(2.13) being of type N (in particular containing a zigzag),

(2.14) containing a Q-point, and

(2.15) being not pairwise smooth.

Let us note that no one of the above three conditions (2.13), (2.14) and (2.15) implies any of the other two. Namely a fan of type N without any Q-point and being pairwise smooth is shown in [33], Fig. 5 (also Fig. 6), p. 92. A fan which is not pairwise smooth but contains no Q-point and is not of type N is pictured in [33], Fig. 3 (also Fig. 4), p. 91. And finally the third needed example is constructed below.

EXAMPLE 2.16. There is a countable plane fan which contains a Q-point, is not of type N and is pairwise smooth.

Proof. In the Cartesian rectangular coordinates in the plane consider the segment $A = \{(x, 0) : 0 \leq x \leq 1\}$ and at each of its points of the form $(m/2^n, 0)$, where $n \in \mathbb{N}$ and $m \in \{1, 3, 5, \ldots, 2^{n-1}\}$ erect a segment $\{(m/2^n, y) : 0 \leq y \leq 1/2^n\}$. Denote by D the union of A and of all erected segments. Then D is a dendrite for which A is the closure of the set of all its ramification points (see Fig. 6 of [39], §49, VI, Remark, p. 247). Let p and q denote the end points of A. Take, in the upper half plane $\{(x,y) : y \geq 0\}$, a sequence of arcs pp_n such that: (1) $q = \lim p_n$. (2) for each distinct m, $n \in \mathbb{N}$ we have $pp_n \cap pp_m = \{p\} = pp_n \cap D$, and (3) the arcs pp_n approximate D without folding back so that $D = \operatorname{Lim} pp_n$. Next consider the union $D \cup \bigcup\{pp_n : n \in \mathbb{N}\}$ (see Fig. 2), and shrink the arc $A \subset D$ to the point p. The resulting space X (pictured here in Fig. 3) is a fan with its top p being a Q-point. It follows from the definition that X is not of type N. The argument showing that X is pairwise smooth is based on 2.8.

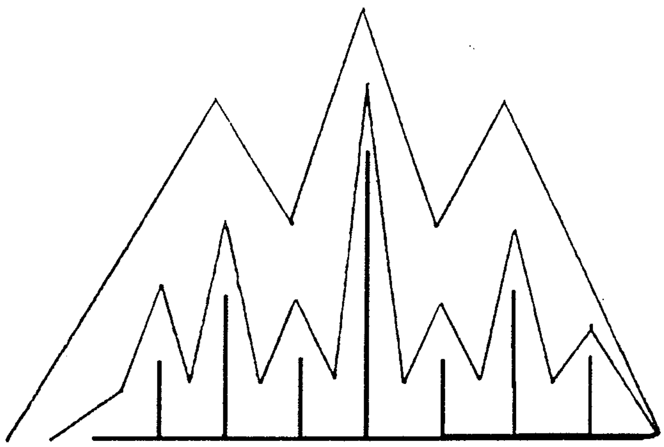

Fig. 2

Another sufficient condition of noncontractibility of dendroids which is related in some way to (2.13) has been defined in [29], p. 121. It runs as follows.

Let a surjective mapping $g : X \to Y$ from a continuum X onto Y be given. We say that the triad (X, g, Y) *has property* (*) provided that

(2.17) the continuum X is of type N between some points p and q,

(2.18) the continuum Y is hereditarily unicoherent,

(2.19) $g(p) \neq g(q)$,

(2.20) $g(p_n q_n'') \cap g(q_n'' p_n') = \{g(q_n'')\}$,

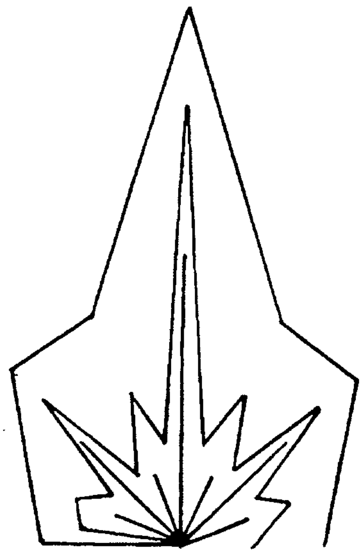

Fig. 3

(2.21) $g(q_n p_n'') \cap g(p_n'' q_n') = \{g(p_n'')\}$,

where the notation in (2.20) and (2.21) agrees with the definition of a continuum of type N (see conditions (2.1)-(2.3)). Then the following result holds (see [29], Theorem, p. 121).

THEOREM 2.22. (Czuba). *If a continuum Y is a continuous image of a continuum X under a mapping g such that the triad (X, g, Y) has property (*), then Y noncontractible.*

Remark 2.23. Let us recall that there exist fans having a surprising property that is much stronger than (2.13), namely those which are of type N between each pair of their points (see [48], (3), p. 386). An uncountable collection of such fans has been used in [48] to answer the question P 788 of [15], p. 97, viz. to show that there is no countable family of fans with the property that a (semismooth) fan is noncontractible if and only if it contains a member of this family ([48], Theorem 2.3, p. 388).

Now let us come back to the other two conditions each of which suffices for noncontractibility of a fan, i.e. to (2.14) and (2.15) and try to examine if these conditions also suffice when applied to some wider classes of dendroids.

Concerning (2.14) it is known that each fan containing a Q-point is not contractible ([49], Theorem 3.2, p. 393; [33], Theorem 2.3, p. 81), but it is conjectured that the implication holds for all dendroids.

Question 2.24. Let a dendroid X contain a Q-point. Is it true that X is not contractible?

Condition (2.15) seems to be the weakest one among the three, because (2.15) implies noncontractibility for fans ([33], Theorem 2.4, p. 82; [49], Theorem 3.3., p. 393), while not for arbitrary dendroids. This last statement can be seen by the following example.

EXAMPLE 2.25. There exists a contractible plane dendroid which is not pairwise smooth and has two ramification points only.

Proof. Again in the Cartesian rectangular coordinates in the plane put $a = (-1, 0)$, $b = (0, 1)$, $c = (0, 0)$, $d = (1, 0)$ and, for each $n \in \mathbb{N}$, let $a_n = (-1/n, 1/n)$, $b_n = (0, 1 + 1/n)$, $c_n = (1/n, 1/n)$, $d_n = (1, 1/n)$. Define

$$B = \overline{ad} \cup \overline{bc} \cup \bigcup\{\overline{aa_n} \cup \overline{a_n b_n} \cup \overline{b_n c_n} \cup \overline{c_n d_n} : n \in \mathbb{N}\} \tag{2.26}$$

(see Fig. 4; compare [25], Fig. 4, p. 78). Next put $r = c = (0, 0)$ and, for each $n \in \mathbb{N}$, let $r_n^1 = a_n$ and $r_n^2 = c_n$. Then the sequence r_n^1 does not dominate the sequence r_n^2 because taking $s = s_n^1 = a$ for each $n \in \mathbb{N}$ we see that the sequence $\overline{r_s^1 s_n^1}$ has the segment $\overline{rs}$ as its limit, while there is no sequence of points s_n^2 converging to s such that the sequence of arcs $r_n^2 s_n^2$ converges to $\overline{rs}$. On the other hand, putting $s = d$ and $s_n^2 = d_n$ we see that the points s_n^2 tend to s, the segments $\overline{r_n^2 s_n^2}$ tend to $\overline{rs}$, while there is no sequence of points s_n^1 converging to s such that the arcs $r_n^1 s_n^1$ converge to rs. Thus r_n^2 does not dominate r_n^1 either, and therefore B is not pairwise smooth. It is easy to observe that B is contractible.

Remark 2.27. Recall the dendroid A described in the Appendix of [33], Fig. 1, p. 89. Namely, in the Cartesian coordinates in the plane $\mathbb{R}^2$ put $a = (0, -1)$, $b = (0, 1)$, $c = (1, 0)$, $d = (0, 0)$ and, for each $n \in \mathbb{N}$ define the following points, where $\varepsilon_n = (1/n - 1/(n+1))/2$:

$$b_n^- = (-1/n, 1),\ b_n = (0, 1 + 1/n),\ b_n^+ = (1/n, 1),$$

$$e_n = (1/n, 1/n),\ c_n = (1, 1/n - \varepsilon_n),\ f_n = (1/n - \varepsilon_n, 1/n - \varepsilon_n),$$

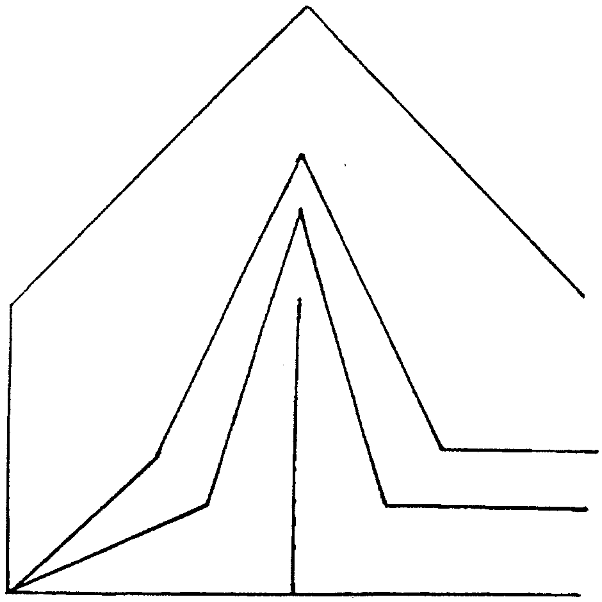

Fig. 4

$$b'_n = (1/n - \varepsilon_n, 1) \quad \text{and} \quad g_n = (0, 1 + 1/n - \varepsilon_n).$$

Further, let B_n denote the broken line in $\mathbb{R}^2$ of consecutive vertices a, b_n^-, b_n, b_n^+, e_n, c_n, f_n, b'_n and g_n, and put (see Fig. 5)

$$A = \overline{ab} \cup \overline{cd} \cup \bigcup \{B_n : n \in \mathbb{N}\}.$$

Note that A contains a copy of the dendroid B above (see (2.26)), and it also has all the properties mentioned in the conclusion of 2.25. In particular taking the point d as r in the definition of pairwise smoothness one can verify that A is not pairwise smooth.

Remark 2.28. To see that there exists a nonplanable contractible dendroid which is not pairwise smooth take the contractible dendroid D constructed in [43], p. 321 (compare also [34], Section 4, p. 70 for another description of the same dendroid). We recall its definition here, for the reader's convenience. In the Cartesian coordinates in the 3-space put $a = (0,1,0)$, $b = (0,0,0)$, $c = (0,1/2,0)$, and, for each $n \in \mathbb{N}$, let

$$a_n^+ = (1/n, 1, 0), \; b_n^+ = (1/n, 0, 0), \; c_n^+ = (1/n, 1/2, 0),$$

$$a_n^- = (-1/n, 1, 0), \; b_n^- = (-1/n, 0, 0), \; c_n^- = (-1/n, 1/2, 0)$$

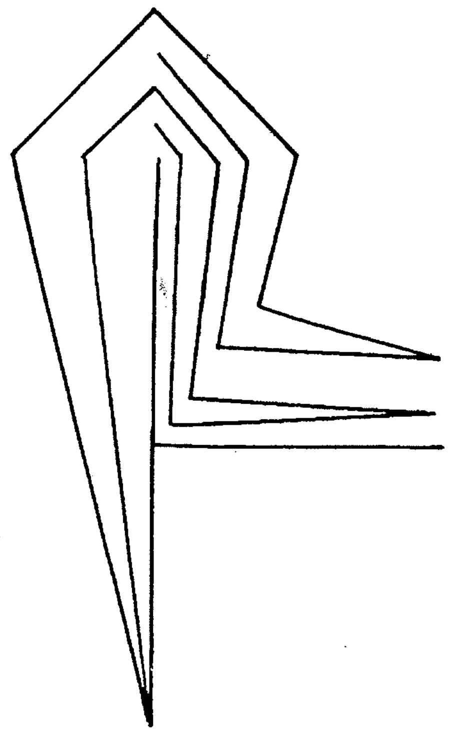

Fig. 5

Denote by C the cone with the vertex $(0, 1/2, 1)$ over the set $\{c\} \cup \{c_{2n}^{+} : n \in \mathbb{N}\} \cup \{c_{2n-1}^{-} : n \in \mathbb{N}\}$. Let pqr stand for the broken line $\overline{pq} \cup \overline{qr}$. Then the dendroid D is defined as

$$D = \overline{ab} \cup C \cup \bigcup \{a_{2n-1}^{+} b_{2n-1}^{+} c_{2n}^{+} \cup b_{2n-1}^{-} a_{2n-1}^{-} b_{2n}^{-} : n \in \mathbb{N}\}.$$

Now taking $r = c$ and putting for each $n \in \mathbb{N}$

$$r_n^1 = c_{2n}^{+} \text{ and } r_n^2 = (-(4n-1)/[4n(2n-1)], 1/2, 0)$$

(i.e., r_n^2 is the mid point of the segment joining a_{2n-1}^{-} with b_{2n}^{-}) we see that the sequence r_n^1 does not dominate the sequence r_n^2 because if $s = s_n^1 = (0, 1/2, 1)$, then the sequence $\overline{r_s^1 s_n^1}$ has the segment $\overline{rs}$ as its limit, while there is no sequence of points s_n^2 converging to s such that the sequence of arcs $r_n^2 s_n^2$ converges to $\overline{rs}$. On the other hand, putting $s = a$ and $s_n^2 = a_{2n-1}^{-}$ we see that the points s_n^2 tend to s, the segments $\overline{r_n^2 s_n^2}$ tend to $\overline{rs}$, while there is no sequence of points s_n^1 converging to s such that the arcs $r_n^1 s_n^1$ converge to $\overline{rs}$. Thus r_n^2 does not dominate r_n^1 either, and therefore D is not pairwise smooth. It is known that D is not planable (i.e., that it cannot be embedded into the plane): see [11], Observation 4, p. 28 for a detailed argumentation.

In connection with 2.25, 2.27 and 2.28 we have a problem.

PROBLEM 2.29. *Give an internal (i.e. structural) characterization of all dendroids which are contractible and not pairwise smooth.*

We close this section recalling an interesting connection between contractibility and planability of fans, which is due to L. G. Oversteegen (see [50] and [51]). Let us recall that the first example of a nonplanable fan has been constructed by K. Borsuk in [4].

Remark 2.30. A fan X is said to have *property* P provided that for each sequence of points $\{p_n\}$ in X converging to the top p of X we have Ls $pp_n = \{p\}$ (see [50], p. 498). It is evident from the definitions that if a fan has property p, then its top is not a Q-point. It is known that having property P is equivalent to local connectivity of the fan at its top ([50], Theorem 3.1, p. 498), which in turn implies that the fan can be embedded in the plane ([50], Theorem 5.2, p. 502). This result gives a solution to a problem raised by the author and Z. Rudy ([18], Problem 1015, p. 216). On the other hand, Theorem 6.1 of [51], p. 394 says that every contractible fan is locally connected at its top. Combining these two results we get the following important result which is due to L. G. Oversteegen [51], and which gives a solution to Problem 788 raised by the author and C. A. Eberhart in [15], p. 97.

THEOREM 2.31. (Oversteegen). *Every contractible fan is embeddable in the plane.*

An example below shows that the above result cannot be extended to arbitrary dendroids.

EXAMPLE 2.32. There is a nonplanable contractible dendroid having only two ramification points.

Proof. In fact, consider again the dendroid A of the Appendix of [33], p. 89 (recalled here in Remark 2.27) and note that it is contractible, it is located in the plane, and its point b is strongly inaccessible from its complement in the plane (see [18], p. 206 for the definition). Thus if bb' means the straight line segment in 3-space which is perpendicular to the plane containing the dendroid A, then the union $A \cup bb'$ is the needed dendroid. In particular, it is not embeddable into the plane by Proposition 2 of [18], p. 206.

In connection with Theorem 2.31 and Example 2.32 we have the following question.

Question 2.33. For which dendroids does contractibility imply planability?

3. Bend intersection property.

The following concept has been introduced in [42], p. 548. Let a continuum X and its subcontinuum A be given. A continuum $B \subset A$ is called a *bend set of* A provided there are two sequences $\{A_n\}$ and $\{A'_n\}$ of subcontinua of X satisfying the following conditions:

(3.1) $$A_n \cap A'_n \neq \emptyset \text{ for each } n \in \mathbb{N};$$

(3.2) $$A = \operatorname{Lim} A_n = \operatorname{Lim} A'_n;$$

(3.3) $$B = \operatorname{Lim}(A_n \cap A'_n).$$

We say that a continuum X has the *bend intersection property* provided for each continuum $A \subset X$ the intersection of all bend sets of A is nonempty. In [40] some relations between the two concepts introduced above and the concepts of a Q-point and of type N are studied for dendroids. In particular it is shown in Statements 1 and 2 of [40] that if a dendroid X contains a Q-point p, then there is a subcontinuum of X such that it has $\{p\}$ as its bend set, and that no dendroid having the bend intersection property is of type N. The main result of [40] says that if a fan contains no Q-point and is not of type N, then it has the bend intersection property. Thus by Theorem 2.9 one gets that every contractible fan has the bend intersection property. Moreover, the following characterization of contractible fans is an analog of Oversteegen's one of Theorem 2.9.

THEOREM 3.4. *A fan is contractible if and only if it contains no Q-point, has the bend intersection property and is pairwise smooth.*

Proof. Indeed, if a fan X is contractible, then it contains no Q-point and is pairwise smooth by Theorem 2.9, and it has the bend intersection property by Theorem 2 of [40]. Conversely, if a fan X has the property, then it is not of type N by Statement 2 of [40], and so its contractibility follows from Theorem 2.9.

Another result which is related to the discussed property has been obtained in [41]. It is proved that a dendroid X is not of type N if and only if for each arc A contained in X the intersection of all bend sets of A is nonempty. In particular, this condition is satisfied for all contractible dendroids (see [41], Corollary). However, we would like to attain a stronger result.

Question 3.5. (Lee). *Does every contractible dendroid have the bend intersection property?*

4. The set function T.

Some other conditions implying noncontractibility of dendroids are known which are expressed in terms of the set function T. They were discussed e.g. in [1], [3] and [9]. To formulate them, recall the needed definition. Given a compact space X and a set $A \subset X$, we define $T(A)$ as the set of all points x of X such that every subcontinuum of X which contains x in its interior must intersect A (see [32]. p. 113). It is known (see [2], Corollary 1, p. 373) that if X is a continuum and A is a subcontinuum of X, then $T(A)$ is a subcontinuum of X.

One of the conditions mentioned above can be formulated in the following way:

(4.1) the continuum X contains two closed subsets A and B such that $A \cap T(B) = \emptyset = B \cap T(A)$ and $T(A) \cap T(B) \neq \emptyset$.

Then we have the following result (see Corollary 1 in [1], p. 48 and in [9]. p. 273).

PROPOSITION 4.2. (Bellamy, Charatonik). *If a continuum X satisfies condition (4.1), then $T(A) \cap T(B)$ is a homotopically fixed proper subset of X, and thus X is not contractible.*

For dendroids condition (4.1) is equivalent (see [23], Lemma 5, p. 304) to the following one, which has been discussed e.g. in [1], p. 47, [3] and [9], p. 271:

(4.3) the dendroid X contains two points a and b having the property that

$$a \in X \setminus T(b), b \in X \setminus T(a) \text{ and } T(a) \cap T(b) \neq \emptyset.$$

Moreover, if A and B are subsets of X as in (4.1), then, according to the above quoted result, the points a and b of (4.3) can be chosen so that $a \in A$ and $b \in B$.

Note that the converse implication to that of 4.2 does not hold even for fans, as can easily be observed by various examples, e.g. by our Example 2.7. Furthermore, a stronger result can be shown: there is a fan having all three attributes of noncontractibility considered in Oversteegen's characterization 2.9, and still without (4.1). This can be seen by an example below.

EXAMPLE 4.4. There is a countable plane fan X which is of type N, contains a Q-point, is not pairwise smooth, and which does not satisfy condition (4.1).

Proof. Put (in the Cartesian coordinates in the plane) $a = (0, 0)$, $b = (0, -1)$, and, for each $n \in \mathbb{N}$, let

$$a_n = (-1/n, 0), \ b_n = (-1/n, -1), \ c_n = (1/n, -1).$$

Consider a semicircle

$$c_n b_n = \{(x, y) : x^2 + (y + 1)^2 = 1/n^2 \text{ and } y \leq 0\}$$

and put (see Fig. 6; compare [15], p. 95)

$$X_1 = \overline{ab} \cup \bigcup \{\overline{ac_n} \cup c_n b_n \cup \overline{b_n a_n} : n \in \mathbb{N}\}.$$

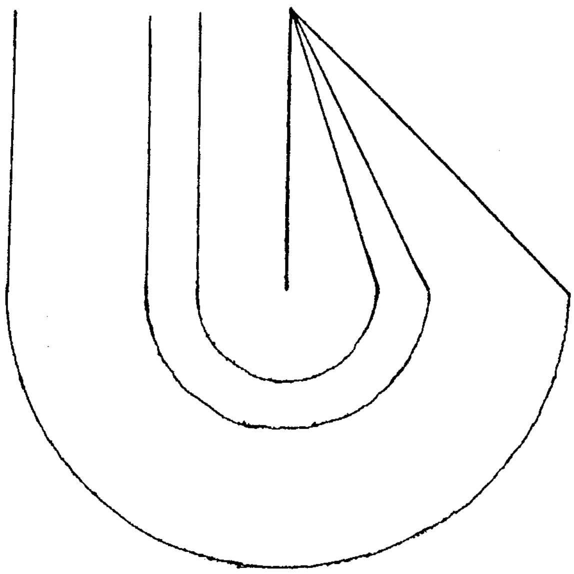

Fig. 6

Thus X_1 is a fan of type N whose top a is a Q-point. Further, put $d = (0, 2)$ and, for $n \in \mathbb{N}$,

$$d_n = (2/n, 2),\ e_n = ((2n+1)/[2n(n+1)], 1), f_n = (-1/n, 1).$$

Define

$$X_2 = \overline{ad} \cup \bigcup \{\overline{ad_n} \cup \overline{d_n e_n} \cup \overline{af_n} : n \in \mathbb{N}\}$$

(see Fig. 7; compare also [10], Proposition 4, p. 111 and Fig. 2, p. 112). Thus X_2 is a not pairwise smooth fan with the top a. Note that $X_1 \cap X_2 = \{a\}$. Thus the fan $X_1 \cup X_2$ meets all the required conditions.

On the other hand, we have the following observation.

Remark 4.5. If a fan X is noncontractible because of condition (4.1), then it must satisfy at least one of (2.12), (2.13) and (2.14), according to Theorem 2.9. The example below shows that X may satisfy all three conditions.

EXAMPLE 4.6. There exists a countable plane fan which satisfies (4.1), is of type N, contains a Q-point, and is not pairwise smooth.

Proof. In fact, let again X_1 have the same meaning as in the proof of 4.4, and let us take as X the one-point union of two copies of X_1 such that the fans under consideration have the common top

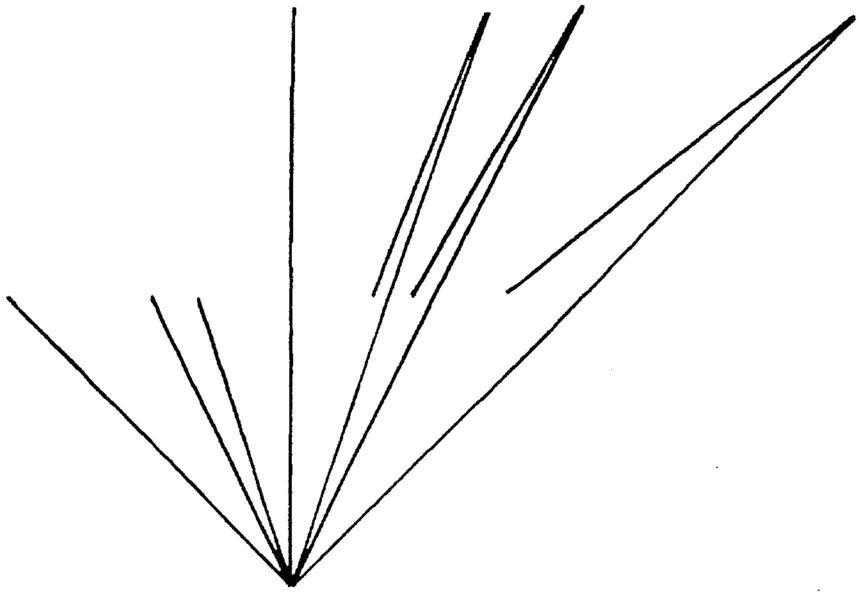

Fig. 7

as the only point of their intersection. Then X has all the needed properties.

Remark 4.7. Examples 4.4 and 4.6 show that all three conditions (2.13), (2.14) and (2.15) together imply neither condition (4.1) nor its negation.

In connection with the above discussion we have the following question on possible relations between (4.1) and the three conditions.

Question 4.8. Do there exist six (countable, plane) fans each of which satisfies condition (4.1) and either exactly one or exactly two of the three: (2.13), (2.14) and (2.15)?

The assumption demanding that the continuum X considered in Remark 4.5 is just a fan is necessary in the statement of 4.5 by an example below.

EXAMPLE 4.9. There exists a plane dendroid which satisfies (4.1), is not of type N, has no Q-point and is pairwise smooth.

Proof. Consider again the dendroid $D \cup \bigcup\{pp_n : n \in \mathbb{N}\}$ defined in the proof of Example 2.16 and drawn in Fig. 2. For each $n \in \mathbb{N}$ pick up a point $c_n \in pp_n$ so that for any point $z \in pc_n \setminus \{c_n\}$ its first coordinate is less than 2/3 and that $\lim c_n = (2/3, 0)$. Let $X_1 = D \cup \bigcup\{pc_n : n \in \mathbb{N}\}$ and denote by X_2 the image of X_1 under the central symmetry with centre (1/2,0). Finally put $X = X_1 \cup X_2$. Now taking in (4.1) $A = \{p\}$ and $B = \{q\}$ we see that $T(A) = D \cap \{(x,y) : x \le 2/3)\}$ and that $T(B)$ is the image of $T(A)$ under the central symmetry above considered. Thus $T(A) \cap T(B) = \{(x, 0) : 1/3 \le x \le 2/3\}$ and so (4.1) is satisfied. It is evident that X is not of type N and contains no Q-point. Finally, its pairwise smoothness is a consequence of Remark 2.8.

Given a compact space X and a set $A \subset X$, we define $K(A)$ as the set of all points x of X such that every subcontinuum of X which contains A in its interior must contain x (see [35], p. 404). It is known (see e.g. [52], Lemma 1, p. 374) that if X is a hereditarily unicoherent continuum and if A is connected, then $K(A)$ is a continuum. Further ([27], Lemma 6, p. 197), for any subcontinuum A of the continuum X we have $K(A) = \{x \in X : T(x) \cap A \neq \emptyset\}$.

Consider the following condition that a dendroid X may satisfy:

(4.10) there is a point p in X with $\{p\} \neq K(p) \subset T(p)$.

Question 4.11. Does noncontractibility of a dendroid X follow from condition (4.10)? For a partial result related to this question see Remark 6.9 below.

5. R-continua.

Another set of conditions that imply noncontractibility of dendroids is formed by ones considered in [16], [22], [23], [25] and [30]. Namely generalizing the concept of an R-arc and of an R-point defined in [16], p. 230 and 231 and exploited in [10], S. T. Czuba has introduced in [22], Definition 1, p. 300 the concept of an R-continuum, renamed later in [25] an R^1-continuum, and two other similar notions of R^2– and R^3-continua. Following [25], Definitions 1.1, 1.2 and 1.3, p. 75, a nonempty proper subcontinuum K of a dendroid X is called an *R^i-continuum* (where $i = 1, 2$, or 3) if there exist an open set U containing K and two sequences $\{C_n^1\}$ and $\{C_n^2\}$

of components of U such that

$$K = \begin{cases} \operatorname{Ls} C_n^1 \cap \operatorname{Ls} C_n^2 & \text{for } i = 1, \\ \operatorname{Lim} C_n^1 \cap \operatorname{Lim} C_n^2 & \text{for } i = 2, \\ \operatorname{Li} C_n^1 & \text{for } i = 3. \end{cases}$$

Theorem 9 of [25], p. 78 (see also [22], Theorem 3, p. 300) says that

(5.1) *if a subcontinuum K of a dendroid X is an R^i-continuum (where $i = 1,2$ or 3), then K is homotopically fixed, and so X is not contractible.*

A relation between (5.1) and (4.1) is known from [23], Theorem 7, p. 305 (see also [25], Proposition 12, p. 79 and the paragraph following it). It runs as follows.

(5.2) *If subcontinua A and B of a dendroid X satisfy the condition formulated in (4.1), then $T(A) \cap T(B)$ is an R^1-continuum, while it can be neither an R^2-continuum nor an R^3-continuum.*

Remark 5.3. The converse implication to that of (5.2) does not hold: the fan X_2 defined in the proof of Example 4.4 contains an R^1-continuum (being a singleton $\{(0,1)\}$) for which condition (4.1) is not satisfied.

Some interrelations between the concept of an R-arc and those of R^i-continua are studied in [25]. Namely it is proved (see [25], Proposition 2 and Examples 3 and 4, p. 75 and 76) that

(5.4) *a) each R-arc is an R^1-, an R^2- and an R^3-continuum;*

b) there exist dendroids containing R^i-continua (where $i = 1,2$ and 3) which are either arcs or points but not R-arcs.

Further, it is known ([25], Proposition 5 and the paragraph following it, p. 77; and Proposition 10 and Corollary 11, p. 78) that

(5.5) *a) each R^2-continuum is both an R^1- and an R^3-continuum;*

b) each R^1-continuum contains an R^2-continuum;

c) each R^1-continuum contains an R^3-continuum, and if the R^1-continuum is a singleton, then it is also both an R^2- and an R^3-continuum;

d) there is an R^1-continuum which is not an R^3-continuum;

e) there is an R^3-continuum which is not an R^1-continuum;

f) there is a continuum which is both an R^1- and an R^3-continuum, while it is not an R^2-continuum.

Remark 5.6. Example 6 of [25], p. 77 which shows assertion d) of (5.5) is invalid. Namely consider again the dendroid B defined by (2.26) (and pictured in Fig. 4), put $B_1 = B \cap \{(x,y) \in \mathbb{R}^2 : x \leq 1/2\}$, and denote by B_2 the image of B_1 under the central symmetry with center $c = (0,0)$. Finally put $Y = B_1 \cup B_2$. It is said in [25], Example 6, p. 77 that the segment

$$K = \{(x,0) \in \mathbb{R}^2 : -1/2 \leq x \leq 1/2\}$$

is an R^1-continuum but is not an R^3-continuum. However, taking $U = Y \cap \{(x,y) \in \mathbb{R}^2 - 3/4 < x < 3/4\}$ and considering the sequence of all components C_n of U we see that $K = \mathrm{Li}\, C_n$, so K is an R^3-continuum. A proper example, which is due to K. Omiljanowski, is presented below.

EXAMPLE 5.7. (Omiljanowski). There exists a plane dendroid having three ramification points and containing an arc which is an R^1- but not an R^3-continuum.

Proof. Put in the Cartesian coordinates in the plane:

$$a = (-2,0),\ b = (0,2),\ c = (2,0),\ d = (0,0)$$

and, for each $n \in \mathbb{N}$, let $a_n = (-1/n, 1/n)$, $c_n = (1/n, 1/n)$, $e_n = (-2, -1/n)$, $p_n = (-1, 1/n)$, $q_n = (1, 1/n)$, and define

$$X = \overline{ac} \cup \overline{bd} \cup \bigcup \{\overline{ba_n} \cup \overline{a_n p_n} \cup \overline{bc_n} \cup \overline{c_n q_n} \cup \overline{ce_n} : n \in \mathbb{N}\}$$

(see Fig. 8). Then X is a plane dendroid with b, c and d as the only ramification points. Put $p = (-1,0)$ and $q = (1,0)$. To see that the segment $\overline{pq}$ is an R^1-continuum consider as U the common part of X and of the square with vertices $(\pm 5/4, \pm 5/4)$. Putting for each $n \in \mathbb{N}$:

$$C^1_{2n-1} = U \cap \overline{bp_n},\ C^1_{2n} = U \cap \overline{bq_n},\ \text{and}\ C^2_n = U \cap \overline{ce_n}$$

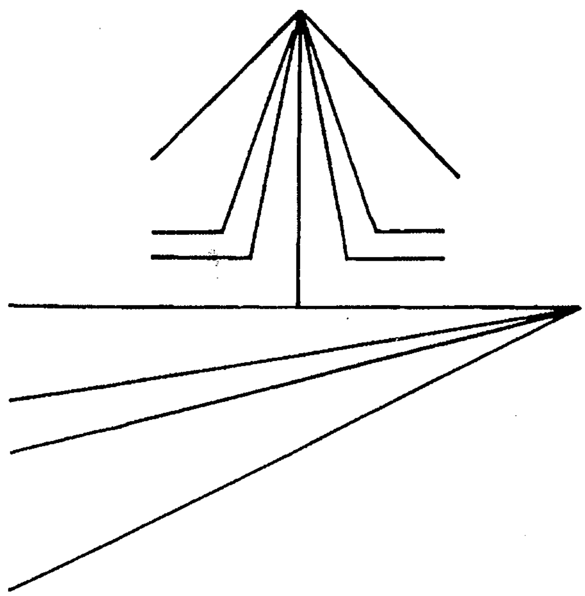

Fig. 8

we see that $pq = \mathrm{Ls}C_n^1 \cap \mathrm{Ls}C_n^2$.

To show that $\overline{pq}$ is not an R^3-continuum suppose the contrary. Thus there exist an open set U containing $\overline{pq}$ and a sequence of components C_n of U with

$$\overline{pq} = \mathrm{Li}\, C_n. \tag{5.8}$$

Let

$$X^+ = \{(x, y) \in X : y \geq 0\} = \overline{ac} \cup \overline{bd} \cup \bigcup\{\overline{p_n q_n} : n \in \mathbb{N}\},$$

and

$$X^- = \{(x, y) \in X : y \leq 0\} = \overline{ac} \cup \bigcup\{\overline{ce_n} : n \in \mathbb{N}\}.$$

We claim that infinitely many components C_n intersect X^+. Indeed, if not, almost all of them intersect $X^- \setminus \overline{ac}$, and since for each component of U its closure meets the boundary $\mathrm{Fr}U$ of U ([39], §47, III, Theorem 2, p. 172), we conclude that $\mathrm{Fr}U \cap \mathrm{Li}\ C_n \neq \emptyset$, i.e., $\mathrm{Fr}U \cap \overline{pq} \neq \emptyset$ by (5.8), contrary to $\overline{pq} \subset U$.

By the claim we have $\overline{p_i a_i} \subset C_{n(i)}$ and $\overline{q_j b_j} \subset C_{n(j)}$ for sufficiently large indices i and j, where $n(i)$ and $n(j)$ tend to infinity if i and j do. Since $\mathrm{Lim}\overline{p_i a_i} = \overline{pd}$ and $\mathrm{Lim}\overline{q_j b_j} = \overline{dq}$, we conclude from (5.8) that $b \in U$, and hence for sufficiently large i the arcs $\overline{p_i q_i}$ are all in the same fixed component C_{n_0}. So there is an index i_0 such that $\bigcup\{\overline{p_iq_i} : i > i_0\} \subset C_{n_0}$. Since $\overline{pq}$ and $\overline{bd}$ are in the closure of the above union and since the closure of a connected set is connected, we have $\overline{pq} \cup \overline{bd} \subset C_{n_0}$, and so

$$\overline{pq} \cup \overline{bd} \cup \bigcup\{\overline{p_iq_i} : i > i_0\} \subset C_{n_0}.$$

Denote by I the interior of the considered union and note that $I = \overline{bd}\setminus\{d\} \cup \bigcup\{\overline{p_iq_i} : i > i_0\}$. Since $I \subset \operatorname{Int} C_{n_0} \subset C_{n_0}$ and since components are pairwise disjoint, there is no subsequence C_{n_m} of the sequence C_n with $C_{n_m} \cap X^+ \neq \emptyset$, contrary to the claim. Therefore the proof is complete.

Combining Theorem 2.9 and (5.1) we see that if a fan X contains an R^i-continuum, where $i = 1, 2$ or 3, then at least one of the three conditions (2.13), (2.14) and (2.15) holds. It would be interesting to know if the existence of some particular R^i-continuum in X implies some special one of the three, and which one; and whether the implication is true for a wider class of dendroids.

Observe that the fan X_2 defined in the proof of Example 4.4 (see again Fig. 7; compare Remark 5.3) contains a singleton (viz. $\{(0, 1)\}$) which is an R^1-, R^2- and R^3-continuum, so that X_2 is not contractible, while it neither contains any Q-point nor is of type N. On the other hand, Example 2.25 (compare also Remarks 2.27 and 2.28) shows that the property of not being pairwise smooth does not imply the existence of an R^i-continuum for dendroids in general. Similarly, both the existence of a Q-point and being of type N do not imply containing any R^i-continuum, as it can be seen by the example of the fan X_1 discussed here in the beginning of the proof of Example 4.4 (and pictured in Fig. 6).

Remark 5.9. An example is shown (see [17], Example 5.5) of a plane dendroid X containing a dendrite Y with the property that for each point $r \in Y$ there is a sequence of points $r_n \in X\setminus Y$ converging to r such that for each point $s \in X\setminus\{r\}$ and for each sequence of points s_n of X converging to s, the limit $\operatorname{Lim} r_ns_n$ never equals the arc rs. Thus in the light of Remark 2.8 the dendroid X is pairwise smooth. Furthermore, X contains a singleton that is an R^i-continuum for each $i \in \{1, 2, 3\}$. Thus the presence of an R^i-continuum in a dendroid does not imply that the dendroid is not pairwise smooth. Some additional structural conditions under which the existence of an R^i-continuum in a dendroid X implies the property that X is not pairwise smooth are presented in Chapter 6 of [17].

Remark 5.10. The concept of an R^i-continuum, originally defined for dendroids only, can be extended into all continua without any change, so that its main property, (5.1), remains true (see [20], Theorem 2, p. 209). However some relations between various R^i-continua known for dendroids are not true for arbitrary continua. Such are e.g. the inclusions considered in b) and c) of (5.5), as it is stated in Theorem 1 of [30]. So in Proposition 1 of [20], p. 208, it should additionally be assumed, according to Theorem 2 of [30], that the continuum X under consideration is hereditarily unicoherent.

Remark 5.11. The concepts of R^1-continua, extended as indicated in 5.10, have been shown to be good tools in the study of contractibility of hyperspaces 2^X of all nonempty closed subsets or $C(X)$ of all subcontinua of a given continuum X (see [20]; for some other results in this topic see also [19]).

6. Hereditary contractibility.

A dendroid is said to be *hereditarily contractible* provided each of its subdendroids is contractible. The first example of a dendroid (even a fan) that is contractible but not hereditarily contractible has been constructed by F.B. Jones in [36] as a counterexample to a false result, viz. Theorem 1 of [15], p. 89. Namely the so called harmonic hooked fan is contractible (see [7], p. 31; compare also [16], Example 7, p. 232) and it contains a copy of a noncontractible fan of Proposition 4 and Fig. 2 of [10], p. 111 and 112. For a similar example see [45], Example (1.66), p. 116. Since then, the following problem remains open ([16], Question 13, p. 235).

PROBLEM 6.1. *Give an internal characterization of hereditarily contractible dendroids.*

It seems to me that S. T. Czuba and Z. Karno are very close to solving this problem (see their forthcoming paper [31]).

Another example of a contractible but not hereditarily contractible dendroid has been described by B. G. Graham in [33], Appendix, Fig. 1, p. 89. This dendroid A (recalled here in Remark 2.27) has a much stronger and surprising property: no matter which choice of a contraction is made, there must be a time at which the image is a noncontractible dendroid. More precisely,

(6.2) for every homotopy $H : A \times [0,1] \to A$ satisfying $H(x,0) = x$ and $H(x,1) = p$ for each x and some p in A, there exists $t \in [0,1]$ such that $H(A \times \{t\})$ is not contractible.

Note that the dendroid A above is not a fan: it has two ramification points. So, one can ask the following question.

Question 6.3. Is it true that condition (6.2) holds for no fan A?

PROBLEM 6.4. *Give an internal characterization of dendroids A satisfying condition (6.2).*

Recall that a dendroid X is said to be *smooth* ([14], p. 298) provided that there is a point p in X (called an *initial point* of X) such that for every point $a \in X$ and for every convergent sequence of points a_n of X the condition $\lim a_n = a$ implies that the sequence of arcs pa_n is convergent, and $\mathrm{Lim}\, pa_n = pa$. It is known that every smooth dendroid is contractible ([14], Corollary 12, p. 311). Since smoothness is a hereditary property ([14], Corollary 6, p. 299), we conclude (see [16], Proposition 14, p. 235) that

(6.5) *every smooth dendroid is hereditarily contractible,*

but not conversely ([16], p. 237). Recall that the converse implication is true for fans ([16], Theorem 16, p; 236; compare Theorem 6.8 below).

As a modification of smoothness, S.T. Czuba has introduced the following concept ([24], p. 169). A dendroid X is said to be *pointwise smooth* provided that for each point $a \in X$ there is a point $p(a) \in X$ (called an *initial point for a* in X) such that for every convergent sequence of points a_n of X the condition $\lim a_n = a$ implies that the sequence of arcs $p(a)a_n$ is convergent, and Lim $p(a)a_n = p(a)a$. It has been shown (see e.g. [19], [24], [26] and [28]) that this new concept plays an important role in study of hereditary contractibility of dendroids. Namely it is known ([24], Proposition 2, p. 170; [26], Corollary 3.10, p. 202) that

(6.6) *if a dendroid is hereditarily contractible, then it is pointwise smooth.*

We do not know whether the converse implication to (6.6) is true (compare [24], p. 170 and [26], (3.11), p. 202).

Question 6.7. (Czuba). Does pointwise smoothness of dendroids imply their hereditary contractibility?

The answer to 6.7 is known to be affirmative in the case when the dendroid is a fan ([8], Proposition 7, p. 74; [16], Corollary 17, p. 237; [24], Theorem 1 and Corollary 1, p. 170; [26], Proposition 2.4 and Corollary 2.5, p. 198). Namely the following result holds.

THEOREM 6.8. *For every fan smoothness, pointwise smoothness and hereditary contractibility are equivalent.*

Moreover, the above result has been extended in [28], Corollaries 4 and 9, and Theorem 5, p. 29 and 30) to a class of dendroids satisfying a condition (CS) expressed in terms of clumps of continua ([21], p. 91). In connection with question 6.7 it is not known whether this extension is proper or not (compare [28], Question 6, p. 29).

Remark 6.9. Recall that a dendroid is said to *have property* [T] provided for each two its distinct points x and y we have either $xy \cap T(x) \neq \{x\}$, or $xy \cap T(y) \neq \{y\}$, or else $T(x) \cap T(y) = \emptyset$ (see [26], §3, p. 198). A similar condition has been used in [14], Theorem 6, p. 302 to characterize smooth dendroids. Theorem 3.1 of [26]. p. 193 says that property [T] characterizes pointwise smooth dendroids. Some other characterizations of pointwise smooth dendroids, in particular ones employing the set functions T and K, are discussed in §3 of [26], p. 198-199, in Theorem 2 of [24], p. 171 and in [27], p. 197-199.

Remark 6.10. In connection with Question 4.11 let us note that condition (4.10) implies that the dendroid X is not pointwise smooth by Theorem 11 (iii) of [27], p. 198, and hence it is not hereditary contractible according to (6.6).

Remark 6.11. We say that a dendroid X *has property* $[R]$ provided there exist two subcontinua A and Y of X such that $A \subset Y \subset X$ and A is an R^3-continuum in Y. It is shown in Theorems 3.7 and 3.8 of [26], p. 201 that the negation of this property is intermediate between hereditary contractibllity and pointwise smoothness. Some other relations between R^3-continua, hereditary contractibility and pointwise smoothness are considered in Proposition 3.6 of [26], p. 201 and in Theorem 16 of [27], p. 199.

Let two spaces X and Y be given with $X \subset Y$. Then X is said to be *contractible in* Y provided that there is a mapping $H : X \times [0,1] \to Y$ satisfying $H(x,0) = x$ and $H(x,1) = p$ for each $x \in X$ and for some $p \in Y$. Then contractibility means contractibility in itself. A dendroid X is said to be *strongly noncontractible* provided it is contractible in no other dendroid. Observe the following statement.

Statement 6.12. *If a dendroid is strongly noncontractible, then it cannot be embedded into any contractible dendroid.*

Indeed, suppose on the contrary that there are a strongly noncontractible dendroid X, a contractible dendroid Y and an embedding $e : X \to e(X) \subset Y$, and let $H : Y \times [0,1] \in Y$ be a homotopy. Then the partial mapping

$$H|e(X) \times [0,1] : e(X) \times [0,1] \to Y$$

is obviously again a homotopy, i.e., $e(X)$ is contractible in the dendroid Y.

Question 6.13. For what dendroids is the converse to 6.12 true?

For example dendroids which are not uniformly arcwise connected are strongly noncontractible. To see this use Theorem 3 of [15], p. 94, saying that every contractible dendroid is uniformly arcwise connected and note that uniform arcwise connectivity is a hereditary property (cf. [8], Proposition 8, p. 74). On the other hand, the opposite class of noncontractible dendroids, i.e., of those which are embeddable into contractible ones, also is nonempty, as can easily be observed by any example of a dendroid which is not hereditary contractible. Thus the following problem seems to be natural.

PROBLEM 6.14. *Give a structural characterization of strongly noncontractible dendroids.*

A similar problem for fans is related to Oversteegen's characterization (Theorem 2.9). Namely let us observe that if a dendroid X of type N is a subset of a dendroid Y, then Y is also of type N. And similarly, if X contains a Q-point, then so does Y. But if X is not pairwise smooth, then Y can be pairwise smooth, as can easily be seen from the Jones' example of a harmonic hooked (contractible) fan that contains a noncontractible fan being not pairwise smooth, not of type N and without any Q-point (see the beginning of the present section). Hence we have the following result and question.

PROPOSITION 6.15. *If a noncontractible fan X can be embedded into a contractible fan Y, then X neither is of type N nor contains any Q-point.*

Question 6.16. For what fans X is the converse to 6.15 true?

7. Selections.

Given a continuum X, we denote by $C(X)$ the hyperspace of all nonempty subcontinua of X equipped with the Vietoris topology, or equivalently with the Hausdorff metric. A *continuous selection* on $C(X)$ means a mapping $\sigma : C(X) \to X$ such that $\sigma(A) \in A$ for each $A \in C(X)$. If $C(X)$ admits a continuous selection, then X is said to be *selectible*. It is known (see [46], Lemma 3, p. 370) that

(7.1) *each selectible continuum is a dendroid,*

and moreover (see [10], Proposition 2, p. 110) that

(7.2) *each selectible dendroid is a continuous image of the Cantor fan, and thus it is uniformly arcwise connected.*

As an easy consequence of (7.1) and of the definitions we see (cf. [46], Corollary, p. 371) that

(7.3) *a locally connected continuum is selectible if and only if it is a dendrite.*

It can easily be observed that contractible curves and selectible continua have some common properties: the reader is requested to compare (0.1), (0.2) and (0.3) with (7.1), (7.2) and (7.3). Moreover, condition (2.13), i.e. being of type N, implies both noncontractibility of dendroids ([47], Theorem 2.1, p. 838) as well as their nonselectibility ([42], p. 548). Further, it is conjectured that the bend intersection property is another common link between the class of contractible and the class of selectible dendroids. Namely selectible dendroids enjoy this property ([42], Corollary, p. 548), as well as contractible fans (see [40]), and for contractible dendroids we have a partial result saying that for each arc contained in such a dendroid the intersection of all its bend sets is nonempty (see [41] and recall Question 3.5).

On the other hand, Propositions 3 and 4 of [10], p. 110 and 111 contain examples of noncontractible and selectible dendroids (even with some extra properties). S. B. Nadler, Jr., asked in [44] if every contractible dendroid is selectible. The question, repeated in his book ([45], (5.11), p. 259) has been solved in the negative by T. Maćkowiak ([43], Example, p. 321) who constructed a contractible and nonselectible dendroid D (recalled here in Remark 2.28) by combining properties of some two examples, viz. his own Example 1 of [42], p. 548 and Graham's example A of the Appendix in [33], p. 89 (compare Remark 2.27). See [11] for a discussion concerning some further properties of dendroid D and problems related to this topic. Nevertheless, the question is still open if we require that the dendroid has some additional properties, e.g. is a fan. And though we know an internal characterization of contractible fans (see [49], Theorem 3.4, p. 393; cf. [33], Theorems 2.1, 2.3, 2.4 and 3.10, p. 81, 82 and 88; here Theorem 2.9), interrelations between contractibility and selectibility for these continua are not clear enough, and results in this topic seem to be rather far from being final ones. Thus, the following question is open ([11], Problem 7. p. 28).

Question 7.4. Does there exist a contractible and nonselectible fan?

To see other differences between contractibility and selectibility for dendroids recall that selectibility is a hereditary property (see [10], p. 113). In contrast to this, contractibility is not a hereditary property, even for countable plane fans, as it was recalled above, in the beginning of the previous section.

Hereditary contractibility of dendroids is strongly related to their selectibility. Namely L. E. Ward, Jr., has defined in [53] a special selection, called rigid, and has shown that a dendroid X admits a rigid continuous selection on $C(X)$ if and only if X is smooth ([53], Theorem 2, p. 1043). According to (6.5) each such dendroid is hereditarily contractible. The converse implication, i.e. from hereditary contractibility to the existence of a rigid selection, is not true in general because there is a hereditarily contractible and not smooth dendroid ([16], p. 237). However this implication is true for fans by Theorem 6.8 above and the quoted result of Ward:

(7.5) *a fan X is hereditarily contractible if and only if it admits a rigid selection on $C(X)$.*

In connection with (7.5) and with Maćkowiak's example of a contractible and nonselectible dendroid the question of Nadler can be modified as follows (compare [11], Problem 9, p. 29).

Question 7.6. Does hereditary contractibility of dendroids imply their selectibility?

8. Final remarks.

Some areas of continua theory which are related to contractibility problems were not discussed in this paper. One of them is contractibility of hyperspaces. For a large information on this topic the reader is referred to Chapter 16 of Nadler's book [45]. Some relations between the concept of an R^i-continuum and hyperspace contractibility can be found in [20]. See also Remark 5.11 above.

Another set of problems which was not even touched here concerns contractibility and mappings. To be more clear, let us formulate the following problem, which is rather a program of a future research than a particular question.

PROBLEM 8.1. *What are all mappings that preserve contractibility of dendroids?*

In connection with this recall that the property of being a dendroid is preserved under confluent mappings (and therefore under monotone or open ones, see [6], Corollaries 1 and 2, p. 219). But even monotone mappings do not preserve contractibility.

Examples are known showing that contractibility of dendroids is not preserved under the inverse limit operation even if all spaces of the inverse sequence are dendrites with finitely many end points only and all bonding mappings are either monotone relative to some points (see Example 2 of [13], p. 147, and a discussion following it, p. 148) or are retractions ([12], Example, p. 10). The following problem remains open.

PROBLEM 8.2. *Find some (necessary and / or sufficient) conditions under which the inverse limit of contractible continua (dendroids) is contractible.*

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