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REMARKS ON FORCED LAGRANGIAN SYSTEMS WITH PERIODIC POTENTIAL (*)

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0. Introduction.

Let us consider the Lagrangian function

$$\mathcal{L}(t,q,\xi) = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij}(t,q)\xi_i\xi_j - V(t,q) \quad q,\xi \in \mathbb{R}^N$$

where A_{ij} $(i,j = 1,..,N)$ and V are C^1 real functions defined in $\mathbb{R}^{N+1}$.

In this paper we look for periodic solutions $q = q(t)$ of the following forced Lagrangian system

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi}(t,q,\dot{q}) - \frac{\partial\mathcal{L}}{\partial q}(t,q,\dot{q}) = h(t) \tag{0.1}$$

where h is a T-periodic forcing term and $\mathcal{L}$ is the Lagrangian function periodic in the variables t and q.

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When h is a zero mean value function, the existence of multiple solutions of problem (0.1) has been already established (see [2], [3], [4], [5], [9]).

In the case when the mean value of h is not zero, it is reasonable to conjecture that $\left|\frac{1}{T}\int_0^T h(t)dt\right|$ must be small enough in order that (0.1) admits periodic solutions. Indeed it is possible to show that problem (0.1) may have no solutions if no assumptions on h are stated (see [10]).

In [10] it has been proved that problem (0.1) admits solutions if only one of the components of h has non-zero mean value (see also [11]).

The following sections are devoted to the study of problem (0.1) when more than one of the components of h have non-zero mean value.

In Theorem 1.2 it will be proved that, under suitable conditions, problem (0.1) has at least two T-periodic solutions.

Moreover some further information on the set of the forcing term h whose corresponding Lagrangian system (0.1) admits solutions will be given if $\frac{\partial \mathcal{L}}{\partial t} = 0$.

We introduce now some notations which will be used in the following sections.

- $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^N$ and $(\cdot|\cdot)$ its usual inner product;
- if $1 \leq p < \infty$ the space

$$L^p = L^p([0,T], \mathbb{R}^N) = \{q : \mathbb{R} \to \mathbb{R}^N | q\ T - \text{periodic}, \int_0^T |q(t)|^p dt\}$$

  is meant to be endowed with the usual L^p-norm here denoted by $|\cdot|_p$;
- $|\cdot|_\infty$ and $|\cdot|_{C^p}$ denote the standard norms of $C(\mathbb{R}, \mathbb{R}^N)$ and $C^p(\mathbb{R}, \mathbb{R}^N)$ respectively;
- $H = H^1([0,T], \mathbb{R}^N)$ represents the Sobolev space obtained by the closure of the C^∞ T-periodic $\mathbb{R}^N$-valued functions $q = q(t)$

endowed with the norm

$$||q|| = \left[\int_0^T (|\dot{q}|^2 + |q|^2)dt\right]^{1/2};$$

– $\tilde{H} = \{q \in H | \int_0^T q(t)dt = 0\}$.

1. The main result.

In this section we state the existence of periodic solutions of the periodic forced Lagrangian system (0.1) when the forcing term h has non-zero mean value.

In this case system (0.1) becomes

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(t,q,\dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(t,q,\dot{q}) = f(t) + c \tag{1.1$_c$}$$

where f has zero mean value and $c = (c_1, c_2, ..., c_N) \in \mathbb{R}^N$.

From now on the following hypotheses on A and V are assumed:

There esist T, T_1, $T_2, ..., T_N$ real positive constants such that

(A) $A(t,q) = \{A_{ij}(t,q)\}$ is a C^1 symmetric positive defined $N \times N$ matrix and $A(t + kT, q + (k_1T_1, \ldots, k_NT_N)) = A(t,q)$ for any $(t,q) \in \mathbb{R} \times \mathbb{R}^N$, k, $k_s \in \mathbb{Z}$, $s = 1, \ldots, N$;

(V) $V(t,q) \in C^1$ and $V(t + kT, q + (k_1T_1, ..., k_NT_N)) = V(t,q)$ for any $(t,q) \in \mathbb{R} \times \mathbb{R}^N$ and k, $k_s \in \mathbb{Z}$, $s = 1, ..., N$.

As $A(t,q)$ and $V(t,q)$ are periodic in the variable q, if $q = q(t)$ is a T-periodic solution of $(1.1)_c$, for any $k_s \in \mathbb{Z}$, $s = 1, ..., N$, $q(t) + (k_1T_1, ..., k_NT_N)$ is a solution too.

Thus, we need the following definition:

DEFINITION 1.1. *The solutions* $q_1 = q_1(t)$ *and* $q_2 = q_2(t)$ *are called distinct if there exist* $t \in [0,T]$ *such that* $q_1(t) - q_2(t) \neq (k_1T_1, ..., k_NT_N)$ *for any* $k_s \in \mathbb{Z}$, $s = 1, \ldots, N$.

The following theorem holds:

THEOREM 1.2. *Let* $A = A(t,q)$ *and* $V = V(t,q)$ *satisfy (A) and (V) and* $f = (f_1, \dots, f_N)$ *be a* T*-periodic continuous function with zero mean value.*

Then there exist $2N$ *real constants* $d_i \leq 0 \leq D_i$, $i = 1, \dots, N$ *such that*

i) if $d_i < D_i$ *for any* $i = 1, \dots, N$, *problem* $(1.1)_c$ *admits at least two distinct solutions for any* $c = (c_1, \dots, c_N) \in \mathbb{R}^N$ *such that* $d_i < c_i < D_i$ *for any* $i = 1, \dots, N$;

ii) if there exists $I \subset \{1, \dots, N\}$, $I \neq \emptyset$ *such that* $d_i = D_i$ *for any* $i \in I$ *and* $d_i < D_i$ *elsewhere, then problem* $(1.1)_c$ *admits infinitely many solutions for any* $c = (c_1, \dots, c_N) \in \mathbb{R}^N$ *such that* $c_i = 0$ *for any* $i \in I$ *and* $d_i < c_i < D_i$ *for any* $i \notin I$.

*Remark.*1.3. In [10] (Theorem 1) an analogous result has been stated when $c = (0, ., 0, c_i, 0, ., 0)$ and assumptions (V) and (A) hold.

Moreover Theorem 2 of [10] gives suitable additional conditions on V which estimate d_i and D_i, and assure $d_i \neq D_i$.

In particular that occurs in the case of the double pendulum.

2. Proof of Theorem 1.2.

The research of the T-periodic solutions of problem $(1.1)_c$ can be reduced to the research of the critical points of the following action functional

$$(2.1) \quad \begin{aligned} F_c(q) = \frac{1}{2}\int_0^T (A(t,q)\dot{q}|\dot{q})dt - \int_0^T V(t,q)dt + \int_0^T (f|q)dt - \\ - \sum_{i=1}^N c_i \int_0^T q_i dt = F_o(q) - \sum_{i=1}^N c_i \int_0^T q_i dt, \end{aligned}$$

where $q = (q_1, \dots, q_N) \in H$.

As V is bounded, in general nothing can be said about F_c satisfying the classical Palais-Smale condition. In fact a priori estimates on critical levels cannot be established.

In order to overcome this difficulty, a method introduced in [10] will be generalized.

We will prove theorem 1.2 by induction on the number n of the non zero components of the vector c.

We recall that if $c = (0, \dots, c_i, \dots 0)$ it has been proved that there exist two constants $\bar{d}_i \leq 0 \leq \bar{D}_i$, such that:

i) if $\bar{d}_i = 0 = \bar{D}_i$, for any $\xi \in \mathbb{R}$ problem $(1.1)_o$ admits a T-periodic solution $q = (q_1, \dots, q_N)$ with $\frac{1}{T}\int_0^T q_i dt = \xi$;

ii) if $\bar{d}_i < \bar{D}_i$, problem $(1.1)_c$ admits at least two distinct solutions for any $c = (0, \dots, c_i, \dots, 0)$, such that $\bar{d}_i < c_i < \bar{D}_i$.

Moreover the first solution is obtained minimizing the functional F_c on the set

$$\Lambda_{[\xi_1,\xi_2]} = \left\{ q \in H | \xi_1 \leq \frac{1}{T}\int_0^T q_i dt \leq \xi_2 \right\}$$

where ξ_1, ξ_2 are suitable real numbers, and proving that

$$\inf_{\Lambda_{[\xi_1,\xi_2]}} F_c(q)$$

is achieved at an interior point of $\Lambda_{[\xi_1,\xi_2]}$.

It follows that if $\bar{d}_i = 0 = \bar{D}_i$ for any $i \in \{1, ..., N\}$ the existence of solutions is assured only for problem $(1.1)_o$.

In order to prove theorem 1.2 let us suppose now that there exist $i \in \{1, ..., N\}$ such that $\bar{d}_i < \bar{D}_i$; without loss of generality assume $i = 1$ (see [10]).

Denote $d_1 = \bar{d}_1$ and $D_1 = \bar{D}_1$.

For sake of brevity, the proof by induction will be given when the vector c has only two non-vanishing components, for istance $c = (c_1, c_2, 0, \dots 0)$.

In this case the action functional becomes

$$F_c(q) = \frac{1}{2}\int_0^T (A(t,q)\dot{q}|\dot{q})dt - \int_0^T V(t,q)dt + \int_0^T (f|q)dt -$$
$$- c_1\int_0^T q_1 dt - c_2\int_0^T q_2 dt.$$

In the following we will assume $d_1 < c_1 < D_1$ and denote

$$F_{c_o}(q) = F_o(q) - c_1 \int_0^T q_1 dt$$

where $c_o = (c_1, 0, \ldots 0)$.

If $\eta \in \mathbb{R}$, as F_{c_o} is bounded from below in $\Lambda_{[\xi_1,\xi_2]}$, then F_c is bounded from below in

$$\Lambda_{[\xi_1,\xi_2],\eta} = \left\{ q \in H | \xi_1 \le \frac{1}{T}\int_0^T q_1 dt \le \xi_2,\ \frac{1}{T}\int_0^T q_2 dt = \eta \right\}$$

The following lemma holds:

LEMMA 2.1. *For any* $\eta \in \mathbb{R}$ *the functional* F_c *reaches its minimum in* $\Lambda_{[\xi_1,\xi_2],\eta}$.

Proof. As $\Lambda_{[\xi_1,\xi_2],\eta}$ is a closed subset of the following manifold of codimension one

$$\Lambda_\eta = \left\{ q \in H \left| \frac{1}{T}\int_0^T q_2 dt = \eta \right. \right\}$$

the thesis will be reached proving that the functional F_c satisfies a Palais-Smale - type condition.

Let $\{q_k\}$ be a sequence in $\Lambda_{[\xi_1,\xi_2],\eta}$ such that

$$\{F_c(q_k)\} \text{ is bounded} \tag{2.2}$$

and

$$\{(F_{c|\Lambda_\eta})'(q_k)\} \to 0 \ \text{ as } k \to \infty \tag{2.3}$$

As $\frac{1}{T}\int_0^T q_{1,k} dt \in [\xi_1, \xi_2]$, $\frac{1}{T}\int_0^T q_{2,k} dt = \eta$ and because of the periodicity of F_c in q_i, $i = 3, .., N$,

$$0 \le \frac{1}{T}\int_0^T q_{i,k} dt \le T_i \quad i = 3, \ldots, N,$$

it follows that $\{q_k^0\}$, $q_k^0 = \frac{1}{T}\int_0^T q_k dt$, is bounded.

Then, by (2.2),

$$\frac{1}{T}\int_0^T (A(t,q_k)\dot{q}_k|\dot{q}_k)dt + \int_0^T (f|q_k)dt$$

is bounded and hence $\{|\dot{q}_k|_2\}$ is bounded too.

Then $\{||q_k||\}$ is bounded and therefore it has a weakly convergent subsequence in H, still denoted $\{q_k\}$.

Using standard arguments (see [2]), it can be shown that $\{q_k\}$ strongly converges to an element of $\Lambda_{[\xi_1,\xi_2],\eta}$.

From now on $q_{\xi,\eta}$ will denote an element of H such that

$$\xi = \frac{1}{T}\int_0^T q_{1;\xi,\eta}\,dt, \quad \eta = \frac{1}{T}\int_0^T q_{2;\xi,\eta}\,dt.$$

In particular $q_{\xi(\eta),\eta}$ will denote an element of $\Lambda_{[\xi_1,\xi_2],\eta}$ such that

$$F_c(q_{\xi(\eta)},\eta) = \min_{\Lambda_{[\xi_1,\xi_2],\eta}} F_c(q) \tag{2.4}$$

and $\bar{q}_{\bar{\xi},\bar{\eta}}$ a minimum point of F_{c_o} in $\Lambda_{[\xi_1,\xi_2]}$.

Let us denote now L the Lipschitz constant of V and $\lambda_o \in \mathbb{R}_+$ such that

$$(A(t,q)\xi|\xi) \geq \lambda_0|\xi|^2 \quad \text{for any } t \in \mathbb{R}, q, \xi \in \mathbb{R}^N.$$

Moreover, set

$$\Gamma_{[\xi_1,\xi_2],\eta} = \{q \in \Lambda_{[\xi_1,\xi_2],\eta} | F_{c_o}(q) = \inf_{\Lambda_{[\xi_1,\xi_2],\eta}} F_{c_o}\}$$

LEMMA 2.2. *If* $q_{\xi(\eta),\eta} \in \Lambda_{[\xi_1,\xi_2],\eta}$ *satisfies (2.4), then*

$$|\dot{q}_{\xi(\eta),\eta}|_2 \leq T/(\pi\lambda_0)\,(|f|_2 + \sqrt{T}L)$$

Proof. See lemma 1.3 of [10].

LEMMA 2.3. *For every* $c = (c_1, c_2, 0, \ldots, 0) \in \mathbb{R}^N$*, there exists* $L_c > 0$ *such that*

$$|F_c(q_{\xi(\eta_1),\eta_1} + \sigma_1) - F_c(q_{\xi(\eta_2),\eta_2} + \sigma_2)| \leq L_c(||q_{\xi(\eta_1),\eta_1} - q_{\xi(\eta_2),\eta_2}|| + |\sigma_1 - \sigma_2|)$$

For all $\sigma_i \in \mathbb{R}^N$, $\eta_i \in \mathbb{R}$, $q_{\xi(\eta_i),\eta_i} \in \Gamma_{[\xi_1,\xi_2],\eta_i}$, $i = 1, 2$.
Proof. See lemma 1.4 of [10].

LEMMA 2.4. *There exists a neighbourhood* $I(\bar{n})$ *of* $\bar{\eta}$ *such that*

$$\xi(\eta) \in]\xi_1, \xi_2[\qquad \textit{for any } \eta \in I(\bar{\eta})$$

Proof. As $\bar{q}_{\bar{\xi},\bar{\eta}}$ is a minimum point for F_{c_0} in $\Lambda_{[\xi_1,\xi_2],\eta}$ and $\bar{\xi} \in]\xi_1, \xi_2[$, then, for any $\eta \in \mathbb{R}$,

$$F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) < F_{c_0}(q_{\xi_1,\eta}), \quad F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) < F_{c_0}(q_{\xi_2,\eta})$$

The functional F_{c_0} achieves its minimum in Λ_{ξ_1} and Λ_{ξ_2} (see lemma 1.2 of [10]), where

$$\Lambda_{\xi_i} = \left\{ q \in H | \frac{1}{T} \int_0^T q_i dt = \xi_i \right\}$$

Then, if

$$\alpha = \min_{\Lambda_{\xi_1} \cup \Lambda_{\xi_2}} F_{c_0}(q)$$

it results that

$$F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) < \alpha \leq F_{c_0}(q_{\xi_1,\eta}), \quad F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) < \alpha \leq F_{c_0}(q_{\xi_2,\eta})$$

for any $\eta \in \mathbb{R}$.

Let $\varepsilon > 0$ be such that

$$\alpha > F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) + \varepsilon.$$

Set

$$\bar{q}_{\bar{\xi},\bar{\eta}} = q^0_{\bar{\xi},\bar{\eta}} + \tilde{q}_{\bar{\xi},\bar{\eta}}$$

where $\tilde{q}_{\bar{\xi},\bar{\eta}} \in \tilde{H}$ and

$$q^0_{\bar{\xi},\bar{\eta}} = (\bar{\xi}, \bar{\eta}, \bar{\sigma}_3, \bar{\sigma}_4, \dots, \bar{\sigma}_N) \in \mathbb{R}^N.$$

As F_{c_0} is continuous, then there exists $\delta > 0$ such that

$$F_{c_0}((\xi, \eta, \bar{\sigma}_3, \dots, \bar{\sigma}_N) + \tilde{q}_{\bar{\xi},\bar{\eta}}) < F_{c_0}(\bar{q}_{\bar{\xi},\bar{\eta}}) + \varepsilon < \alpha$$

for any $(\xi,\eta) \in]\bar{\xi}-\delta,\bar{\xi}+\delta[\times]\bar{\eta}-\delta,\bar{\eta}+\delta[$.

It follows that, for any $(\xi,n) \in]\bar{\xi}-\delta,\bar{\xi}+\delta[\times]\bar{\eta}-\delta,\bar{\eta}+\delta[$, $\sigma_i \in \mathbb{R}$, $i=3,\dots,N$, $\tilde{q} \in \tilde{H}$,

$$F_{c_o}((\xi,\eta,\bar{\sigma}_3,\dots,\bar{\sigma}_N)+\tilde{q}_{\bar{\xi},\bar{\eta}}) < F_{c_o}((\xi_1,\eta,\sigma_3,\dots,\sigma_N)+\tilde{q})$$

and

$$F_{c_o}((\xi,\eta,\bar{\sigma}_3,\dots,\bar{\sigma}_N)+\tilde{q}_{\bar{\xi},\bar{\eta}}) < F_{c_o}((\xi_2,\eta,\sigma_3,\dots,\sigma_N)+\tilde{q})$$

Then for any $\eta \in I(\bar{\eta}) =]\bar{\eta}-\delta,\bar{\eta}+\delta[$ the minimum of F_{c_o} in $\Lambda_{[\xi_1,\xi_2],\eta}$ is achieved at a point $q_{\xi(\eta),\eta}$ with $\xi(\eta) \in]\xi_1,\xi_2[$.

Proof of the Theorem Given $q \in H$, define

$$\psi_2(q) = -\frac{1}{T}F_o'(q)(0,1,0,\dots,0) = -\frac{1}{T}\left[\frac{1}{2}\int_0^T (\frac{\partial A}{\partial q_2}(t,q)\dot{q}|\dot{q})dt - \int_0^T \frac{\partial V}{\partial q_2}(t,q)dt\right].$$

Denote

$$d_2 = \inf_{\eta\in I(\bar{\eta})} \inf_{\Gamma_{[\xi_1,\xi_2],\eta}} \psi_2(q_{\xi(\eta),\eta})$$

$$D_2 = \sup_{\eta\in I(\bar{\eta})} \sup_{\Gamma_{[\xi_1,\xi_2],\eta}} \psi_2(q_{\xi(\eta),\eta})$$

By lemma 2.2 it follows that $-\infty \le d_2 \le D_2 \le +\infty$.

Moreover, as $\bar{q}_{\bar{\xi},\bar{\eta}}$ is a minimum point of F_{c_o} in $\Lambda_{[\xi_1,\xi_2]}$, then $\psi_2(\bar{q}_{\bar{\xi},\bar{\eta}}) = 0$ and therefore $d_2 \le 0 \le D_2$.

Remark that for any $\eta \in I(\bar{\eta})$ and $q_{\xi(\eta),\eta} \in \Gamma_{[\xi_1,\xi_2]}$ it results:

$$F_c'(q_{\xi(\eta),\eta}) = -T(0,\psi_2(q_{\xi(\eta),\eta}),0,\dots,0) \in \mathbb{R}^N. \tag{2.5}$$

Then if $d_2 = D_2 = 0$, $\psi_2(q_{\xi(\eta),\eta}) = 0$ for any $\eta \in I(\bar{\eta})$ and for any $q_{\xi(\eta),\eta} \in \Gamma_{[\xi_1,\xi_2],\eta}$ and, by (2.5) $q_{\xi(\eta),\eta}$ is a critical point for F_{c_o}.

Hence, if $d_2 = D_2 = 0$ for any $c_1 \in]d_1,D_1[$ and for any $\eta \in I(\bar{\eta})$, problem $(1\ 1)_{c_o}$ admits a T-periodic solution $q_{\xi(\eta),\eta}$ such that

$$\eta = \frac{1}{T}\int_0^T q_{2;\xi(\eta),\eta}\,dt.$$

Suppose now $d_2 < D_2$ and $c_2 \in]d_2,D_2[$

Then there exist η_1, $\eta_2 \in I(\bar{\eta})$ such that

$$\psi_2(q_{\xi(\eta_2),\eta_2}) < c_2 < \psi_2(q_{\xi(\eta_1),\eta_1})$$

As $\bar{q}_{\bar{\xi},\bar{\eta}}$ is a minimum point for F_c, we can assume that $0 < \eta_2 - \eta_1 < T_2$.

Let us consider now

$$\Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]} = \{q \in H | \xi_1 \leq \frac{1}{T}\int_0^T q_1 dt \leq \xi_2, \quad \eta_1 \leq \frac{1}{T}\int_0^T q_2 dt \leq \eta_2\}.$$

The functional F_c is bounded from below in $\Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]}$, then denote

$$m = \inf_{\Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]}} F_c(q)$$

Let us prove that F_c achieves m at an interior point of $\Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]}$.

Let us consider $q_k = (q_{1,k}, \ldots, q_{N,k}) \in \Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]}$ such that

$$\lim_k F_c(q_k) = m.$$

If $\eta_k = \frac{1}{T}\int_0^T q_{2,k} dt \in [\eta_1, \eta_2]$, without loss of generality we can assume

$$q_k = q_{\xi(\eta_k),\eta_k} \in \Gamma_{[\xi_1,\xi_2],\eta_k}$$

$\lim_k \eta_k = \eta_o$ with $\eta_1 \leq \eta_o \leq \eta_2$ and $\lim_k \xi(\eta_k) = \xi_0$, with $\xi_1 \leq \xi_0 \leq \xi_2$.

By lemma (2.3) it follows that

$$m \leq F_c(q_{\xi(\eta_o),\eta_o}) \leq F_c(q_{\xi(\eta_k),\eta_k} + (\xi_o - \xi(\eta_k), \eta_o - \eta_k, 0, \ldots, 0)) -$$

$$- F_c(q_{\xi(\eta_k),\eta_k}) + F_c(q_{\xi(\eta_k),\eta_k}) \leq L_c(|\xi_0 - \xi(\eta_k)| + |\eta_0 - \eta_k|) +$$

$$+ F_c(q_{\xi(\eta_k),\eta_k}) \to m \text{ as } k \to +\infty.$$

Then

$$F_c(q_{\xi(\eta_0),\eta_0}) = m.$$

By lemma 2.4 it follows that

$$\xi_1 < \xi(\eta_o) < \xi_2.$$

Moreover it can be shown that $\eta_1 < \eta_0 < \eta_2$.
Indeed, denote

$$S_2(s) = F_c(q_{\xi(\eta_1),\eta_1} + s(0,1,0,\dots,0)).$$

Then

$$S_2'(0) = -T(\psi_2(q_{\xi(\eta_1),\eta_1}) - c_2) < 0$$

If $\varepsilon > 0$ is small enough, then

$$F_c(q_{\xi(\eta_0),\eta_0})) \leq F_c(q_{\xi(\eta_1),\eta_1}) + (0,\varepsilon,\dots,0)) < F_c(q_{\xi(\eta_1),\eta_1})) \leq F_c(q_{\xi(\eta_0),\eta_1})$$

and therefore $\eta_1 < \eta_0$.

Arguing similarly it can be shown that $\eta_0 < \eta_2$.

Finally $q_{\xi(\eta_0),\eta_0}$ is an interior local minimum point for F_c in $\Lambda_{[\xi_1,\xi_2],[\eta_1,\eta_2]}$ and thus there exists a solution of problem $(1.1)_c$ in the case $c = (c_1, c_2, 0, .., 0)$.

Arguing by induction, it is possible to find $2N$ real constants d_i, D_i such that if $d_i < D_i$ for any $i = 1,\dots,N$, then F_c admits a local minimum point q_0 when c is small enough.

Thus problem $(1.1)_c$ admits at least one T-periodic solution.

In order to find a second distinct solution of problem $(1.1)_c$, a generalized version of the mountain-pass theorem due to Guo-Sun-Qi will be used (see [7] and [10]).

Indeed, let $j \in \{1,...,N\}$ such that $T_j = \min\{T_1,...,T_N\}$ then there exists $q^* \in H$,

$$q^* = q_0 + (0,...,T_j,...,0) \quad \text{if } c_j \leq 0$$

$$(q^* = q_0 - (0,...,T_j,...,0) \quad \text{if } c_j > 0)$$

satisfying

$$F_c(q^*) \leq F_c(q_0)$$

Moreover, as q_0 is a local minimum point, there exists $\rho > 0$, $\rho < T_j\sqrt{T}$, such that

$$F_c(q) \geq F_c(q_0) \qquad \text{for any } q \in H,\ ||q - q_0|| = \rho$$

Although the functional F_c doesn't satisfy the (P.S) condition, a deformation lemma still holds (see theorem 1.10 of [9], properties $1° - 3° - 4°$) because F_c' is periodic.

Then applying theorem 1 of [7], there exists a solution of $(1.1)_c$, different from $q_0 + (k_1T_1, \dots, k_NT_n)$ for any $(k_1, \dots, k_N) \in \mathbb{Z}^N$.

3. Further results in the autonomous case.

In this section we want now to give further information about the forcing terms f with zero mean value whose corresponding problems $(1.1)_c$ $c = 0$ admit at least one solution.

Let us denote

$$E = \{f \in C(\mathbb{R}, \mathbb{R}^N) | f\ T - \text{periodic}, \int_0^T f(t)dt = 0\}$$

and

$$R(f) = \{c \in \mathbb{R}^N | (1.1)_c \text{ has at least a } T\text{-periodic solution}\}$$

Remark that $0 \in R(f)$ for any $f \in E$ (see [2]), thus we can consider the set

$$S = \{f \in E | R(f) \neq \{0\}\}$$

Suppose that the case ii) of theorem 1.2 holds; then S contains a small C-ball centered in 0.

The following theorem deals with the structure of the set S; analogous results have been established in [6] and [8].

THEOREM 3.1. *Suppose the assumptions of theorem 1.2 hold, case ii); if* $\frac{\partial \mathcal{L}}{\partial t}(t, q) = 0$ *and the zeros of* $\frac{\partial \mathcal{L}}{\partial q}$ *are isolated, then the set* S *is dense in* E.

Proof. We shall argue here as in section 5 of [8].

Arguing by contradiction, suppose S is not dense in E. Then there exist $f \in E$, $f \neq 0$ and $r > 0$ such that

$$R(g) = \{0\} \qquad \text{for any } g \in E, |f - g|_\infty < r \tag{3.1}$$

As $0 \in R(f)$, problem $(1.1)_0$ admits at least one T-periodic solution that is there exists $q_0 \in C^2([0,T], \mathbb{R}^N)$ satisfying

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q_0, \dot{q}_0) - \frac{\partial \mathcal{L}}{\partial q}(q_0, \dot{q}_0) = f(t) \tag{3.2}$$

Let $R > 0$ be such that

$$\begin{cases} \left| \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) - f \right|_\infty < \frac{r}{2\sqrt{N}} \\ \text{for any } q \in C^2(\mathbb{R}, \mathbb{R}^N) \ T\text{-periodic such that } |q - q_0|_{C^2} \leq R \end{cases} \tag{3.3}$$

Denote

$$B = \{q \in C^2(\mathbb{R}, \mathbb{R}^N) | \, |q - q_0|_{C^2} \leq R\}$$

and $\Phi : B \to \mathbb{R}^N$ such that

$$\Phi(q) = \frac{1}{T}\int_0^T \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) \right] dt.$$

We want to show that

$$\Phi(q) = 0 \qquad \text{for any } q \in B$$

We argue by contradiction and suppose that there exist $a \in \mathbb{R}^N$ and $q \in B$ such that

$$\Phi(q) = a \neq 0. \tag{3.4}$$

Denote

$$g(t) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) - \Phi(q)$$

then, for any $j = 1, \ldots, N$,

$$\left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) \right]_j = a_j + g_j(t) \tag{3.5}$$

By (3.3) and (3.5) it follows that

$$|a_j + g_j(t) - f_j(t)| < \frac{r}{2\sqrt{N}}$$

and then, as $g - f \in E$,

$$|a_j| < \frac{r}{2\sqrt{N}} \qquad \text{for each } j = 1, \dots, N$$

that is $|a| < r/2$.

That implies that

$$|f(t) - g(t)| \leq |f(t) - g(t) - a| + |a| < r$$

then

$$|f - g|_\infty < r$$

and therefore $R(g) = \{0\}$ which contradicts (3.4).

Let us consider now $q \in C^2$ and $s \in \mathbb{R}$ small enough such that

$$0 = \Phi(q_0 + sq) = -\frac{1}{T}\int_0^T \frac{\partial \mathcal{L}}{\partial q}(q_0 + sq, \dot{q}_0 + s\dot{q})dt.$$

Then, as for any $j = 1, ..., N$ and for any $q \in C^2$,

$$\left[\frac{d}{ds}\int_0^T \frac{\partial \mathcal{L}}{\partial q_j}(q_0 + sq, \dot{q}_0 + s\dot{q})dt\right]_{s=0} = 0$$

it follows that

$$\frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j}(q_0, \dot{q}_0) = 0 \qquad \text{for any } i, j = 1, \dots, N$$

and therefore $\frac{\partial \mathcal{L}}{\partial q_j}(q_0, \dot{q}_0)$ is constant for any $j = 1, ..., N$.

Moreover, by (3.2)

$$\frac{\partial \mathcal{L}}{\partial q_j}(q_0, \dot{q}_0) = 0$$

As the zeros of $\frac{\partial \mathcal{L}}{\partial q_j}$ are isolated, then q_0 is constant, which contradicts (3.2).

Hence the claim follows.

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