LE MATEMATICHE
Vol. XLVI (1991) - Fasc. II, pp. 609-624

SEMICONTINUOUS SET VALUED MAPPINGS AND CONNECTED SETS (*)

IMMACOLATA DEL PRETE (Napoli)
MARGHERITA DI IORIO (Napoli)

In this paper is studied a condition, called P-continuity, which can be viewed as mimimal in order that a set valued mapping be connected. Moreover for families of functions are considered notions of convergence which preserve P-semicontinuity. Other results connected with the notion of semiconnected multifunction are obtained.

1. Introduction.

Recently several authors have considered *connected multifunctions*, that is set-valued functions which preserve connected sets (see Valadier [24], Correa, Hiriart-Urruty and Penot [2], Ewert and Lipiński [5]). In particular Hiriart-Urruty [6], after recalling the classical result on this topic (ensuring that every upper or lower semicontinuous connected valued multifunction is connected), gathers several examples of various areas where the above mentioned properties are involved, as the mean value theorem in Nonsmooth Analysis, existence theorems in problems of Calculus of Variations

(*) Entrato in Redazione l'8 maggio 1991.

and others. On the other hand Correa, Hiriart-Urruty and Penot try to weaken the assumptions of semicontinuity while preserving the connectedness property. For this purpose they introduce two new notions of continuity for set valued mappings, *inf-d-continuity* and *semicontinuity* (P-*semicontinuity* in our terminology) and prove that these conditions can be viewed as "minimal" in order that set valued mappings be connected.

Our aim in this paper is to study P-semicontinuity property and some related problems. More precisely at first (see section 3) we exhibit sufficient conditions for a set valued mapping to be P-semicontinuous and relations between P-semicontinuity and upper or lower semicontinuity. Next in section 4 we introduce two notions of convergence on the space of multifunctions, the first of which preserves connected set valued functions, the second P−semicontinuous ones. The section 5 is devoted to upper and lower *semiconnected* multifunctions [21]; we prove that these are topological continuity properties, in the sense that each of them can be reduced to continuity by a suitable change of the topology on the range. The last two sections are concerned with an unifying approach to connected preserving functions which allows us to obtain results contained in [16], [17]. We deduce sufficient conditions, weaker than Penot's, in order that the multifunction image of a connected topological space X be connected.

2. Terminology and notations.

In this section we recall some results and definitions that we shall use throughout the paper.

Let Y be a non empty set and let $\mathcal{A}(Y)$ be the family of all nonempty subsets of Y. A mapping F from a set X to $\mathcal{A}(Y)$ will be called a multifunction and denoted by $F : X \to Y$. If $A \subseteq X$ and $B \subseteq Y$, we put $F(A) = \bigcup_{x\in A} F(x)$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, $F^{+}(B) = \{x \in X : F(x) \subseteq B\}$.

If X and Y are topological spaces, $\mathcal{N}(x)$ and $\mathcal{N}(y)$ will denote respectively the neighborhood filters of x in X and of y in Y. A multifunction $F : X \to Y$ is *upper semicontinuous* (usc) at $x \in X$ if for every open set $G \subseteq Y$, $G \supseteq F(x)$ there exists a neighborhood $U \in \mathcal{N}(x)$ such that $F(U) \subseteq G$. F is usc if it is usc at each x in X or, equivalently if $F^-(B)$ is a closed subset of X for every closed set $B \subseteq Y$. The multifunction F is *lower semicontinuous* (lsc) at $x \in X$ if for every open set $G \subseteq Y, G \cap F(x) \neq \emptyset$ there is a neighborhood $U \in \mathcal{N}(x)$ such that $F(t) \cap B \neq \emptyset$ for each t in U. F is lsc if it is lsc at each x in X or, equivalently if $F^-(V)$ is an open subset of X for each open set $V \subseteq Y$. A multifunction F is continuous if it is both lsc and usc.

The family $\{A \in \mathcal{A}(Y) : A \cap W \neq \emptyset, W \subseteq Y$ and open $\}$ is a subbase for the lower Vietoris topology V^- on $\mathcal{A}(Y)$, while the family $\{A \in \mathcal{A}(Y) : A \subseteq W, W \subseteq Y$ and open $\}$ is a base for the upper Vietoris topology V^+ on $\mathcal{A}(Y)$. The supremum topology $V = V^- \vee V^+$ is the Vietoris topology on $\mathcal{A}(Y)$. These topologies were studied in detail by Michael [14]. It is clear that a multifunction $F : X \to Y$ is lsc (resp. usc, continuous) at $x \in X$ if, and only if, the function $F : X \to \mathcal{A}(Y)$ is continuous at $x \in X$ with respect to the topology V^- (resp. V^+, V) on $\mathcal{A}(Y)$.

A multifunction $F : A \to Y$ defined on a dense subset A of X is *subcontinuous* at $x \in X$ if for every net $(x_i) \subseteq A$ convergent to x, every net $(y_i) \subseteq Y$, with $y_i \in F(x_i)$ for each i, has a convergent subnet.

Let $(A_i)_{i \in I}$ be a family of subsets of Y and let G be a filter on I. The lower and upper limits of $(A_i)_{i \in I}$ on G are defined as follows:

$$\lim{}' A_i = \bigcap_{E \in G\#} \overline{\bigcup_{i \in E} A_i} \quad \text{and} \quad \lim{}'' A_i = \bigcap_{H \in G} \overline{\bigcup_{i \in E} A_i}$$

where $G\#$ is the grill of G, defined by

$$G\# = \{E \subseteq I : E \cap H \neq \emptyset \text{ for every } H \in G\}.$$

If $(A_i)_{i \in I}$ is a net of subsets of Y and G is the filter generated by the family $\{\{j \in I : j \geq i\} i \in I\}$ then

$$\lim{}' A_i = \{y \in Y : \forall V \in \mathcal{N}(y)\ \exists i \in I : \forall j \geq i\ A_j \cap V \neq \emptyset\}$$

$$\lim{}'' A_i = \bigcap_{i \in I} \overline{\bigcup_{j \geq i} A_j}$$

If $(A_n)_{n \in N}$ is a sequence of subsets of Y $\lim' A_n$ and $\lim'' A_n$ are the usual Kuratowski limits (see [10]).

Let $F : X \to Y$ be a multifunction. Consider the family $(F(x))_{x\in X}$ and the filter $\mathcal{N}(x_0)$ where x_0 is a point of X. The corresponding lower and upper limits will be denoted by $\lim_{x\to x_0}{}' F(x)$ and $\lim_{x\to x_0}{}'' F(x)$.

A net $(A_i)_{i\in I}$ of non empty subsets of Y is V^--convergent (resp V^+-convergent, V-convergent) to a subset $A \in \mathcal{A}(Y)$ of Y if it is convergent to A in the topological space $(\mathcal{A}(Y), V^-)$ (resp $(\mathcal{A}(Y), V^+)$, $(\mathcal{A}(Y), V)$). Accordingly a net $(A_i)_{i\in I}$ is V^--convergent to $A \in \mathcal{A}(Y)$ if and only if, for every open subset G of Y which intersects A, there is $\bar{i} \in I$ such that $A_i \cap G \neq \emptyset$ for $i \geq \bar{i}$. It is well known that $(A_i)_{i\in I}$ is V^--convergent to A if and only if $A \subseteq lim'\ A_i$ [11].

We assume for the rest of this paper that X and Y are topological spaces.

A Hausdorff space Y is *completely normal* [4] if every pair of sets A, B, satisfying $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ (where $\bar{A}$ is the closure of A in the topological space) can be separated (that is there exist A' and B' open and disjoint such that $A' \supseteq A$ and $B' \supseteq B$).

3. P-semicontinuous multifunctions.

Let X and Y be topological spaces and F a multifunction from X to Y.

DEFINITION 3.1. [2] *F is said to be P-semicontinuous at $x_0 \in X$ if for every open subset V of Y containing $F(x_0)$ there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for every $x \in U$. F is said to be P-semicontinuous in X if it is P-semicontinuous at x for every $x \in X$.*

Obviously F is P-semicontinuous whenever F is lsc or usc, but there exist P-semicontinuous multifunctions which are neither upper nor lower semicontinuous. To see this consider, for instance, the multifunction F from $\mathbf{R}$ to $\mathbf{R}$ defined as

$$F(x) = \begin{cases} [-1, 0] & \text{if } x = 0 \\ [0, 1/|x|] & \text{if } x \neq 0. \end{cases}$$

DEFINITION 3.2. [2] *If (Y, d) is a metric space, F is said to be* *inf-d-continuous at $x_0 \in X$ if*

$$\lim_{x \to x_0} d(F(x),\ F(x_0)) = 0$$

where $d(A, B) = \inf\{d(a, b),\ a \in A, b \in B\}$.

Correa, Hiriart-Urruty and Penot [2] prove that P-semicontinuous multifunctions are inf-d-continuous. They also prove that connected sets are preserved both by P-semicontinuous connected valued multifunctions and by inf-d-continuous connected compact valued ones.

The following lemma summarizes some basic results about P-semicontinuity:

LEMMA 3.1.

a) *If F is compact valued, F is inf-d-continuous iff F is P-semicontinuous.*

b) *If F is P-semicontinuous then $F \cup G$ is P-semicontinuous for each multifunction G from X to Y.*

c) *If $F(x_0)$ is closed and $\lim_{x \to x_0}{}' F(x) \neq \emptyset$ then F is P-semicontinuous at x_0.*

Let us observe that the above result c) can fail if $F(x_0)$ is not closed. To illustrate such a situation it suffices to consider the multifunction from $\mathbf{R}$ to $\mathbf{R}$ defined by

$$F(x) = \begin{cases}]0, 1[& \text{if } x = 0 \\ ]1, 2[& \text{if } x \neq 0 \end{cases}$$

One may wonder whether under the assumption of P-semicontinuity for F the graph multifunction, that is $\Gamma_g : x \in X \to \{x\} \times F(x)$, is P-semicontinuous too. The answer is the following statement which extends the analogous result for usc and lsc multifunctions.

THEOREM 3.2. *If F is P-semicontinuous and compact valued, then Γ_g is P-semicontinuous.*

Let us observe that if F is not compact valued the above result can fail, see for example the multifunction

$$F(x) = \begin{cases} \{1/|x|\} & \text{if } x\neq 0 \\ \mathbf{R} & \text{if } x = 0 \end{cases}$$

Let us introduce the following definition similar to one due to Whyburn [25]:

DEFINITION 3.3. *If F is a set valued function from X to Y, Y is said peripherally F normal if for every point image $F(x)$ and every closed subset C satisfying $F(x) \cap C = \emptyset$ there exist two disjoint open sets A and B such that $A \supseteq F(x)$, $B \supseteq C$ and for a neighborhood U of x it results $F(U) \subseteq A \cap B$.*

We prove the following result:

THEOREM 3.3. *Let F be a multifunction from X to Y and let Y be peripherally F normal. If F is P-semicontinuous and connected valued, then F is usc.*

Proof. Let V be an open set $V \supseteq F(x)$; if A and B are open sets satisfying $A \supseteq F(x)$, $B \supseteq -V$, $F(U) \subseteq A \cap B$ for a neighborhood U of x, there exists a neighborhood U' of x for which $F(z) \cap A\neq\emptyset$ for every $z \in U'$. If $z \in U \cap U'$, since $F(z)$ is connected, it results $F(z) \subseteq V$.

Now we look at P-semicontinuity of the upper limit of a multifunction. Let us recall that, if F is a set valued function defined on a dense subset A of X, the upper and lower limits of F at $x_O \in X$ are defined in [11] by

$$\lim_{x\to x_0}{}'' \ F(x) = \{y \in Y : \forall W \in \mathcal{N}(y) \ \forall U \in \mathcal{N}(x_0) \exists t \in A \cap U : F(t) \cap W\neq\emptyset\}$$

$$\lim_{x\to x_0}{}' \ F(x) = \{y \in Y : \forall W \in \mathcal{N}(y) \ \exists U \in \mathcal{N}(x_0) \forall t \in A \cap U : F(t) \cap W\neq\emptyset\}$$

It is well known that:

a) for each $x_0 \in A$, $F(x_0) \subseteq \lim\limits_{x\to x_0}{}'' \ F(x)$ and equality holds if and only if F is graph-closed at x_0.

b) $\lim\limits_{x\to x_0}{}' \ F(x) \subseteq \lim\limits_{x\to x_0}{}'' \ F(x)$ for each $x_0 \in X$.

c) $\lim\limits_{x\to x_0}{}' \ F(x) \subseteq \overline{F(x_0)}$ for every $x_0 \in A$ and $\lim\limits_{x\to x_0}{}' \ F(x) = F(x_0)$ if and only if F is lsc at x_0 and $F(x_0)$ is closed.

THEOREM 3.4. *If $x_0 \in X$ and F is a multifunction from X to Y for which $\lim\limits_{x \to x_0}{}' F(x) \neq \emptyset$, then the multifunction $F'' : z \in X \to \lim\limits_{x \to z}{}'' F(x)$ is P-semicontinuous at x_0.*

Proof. It suffices to observe that for an open set V containing $\lim\limits_{x \to x_0}{}'' F(x)$ there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for every $x \in U$. Now we only need to recall property a).

The following results are similar to those obtained in [11] and related to upper semicontinuous extensions.

THEOREM 3.5. *Let Y be a regular and locally compact topological space, A a dense subset of X and $F : A \to Y$ P-semicontinuous and compact valued. Then the multifunction*

$$\tilde{F} : x \in X \to \begin{cases} F(x) & \text{if } x \in A \\ \lim\limits_{t \to x}{}'' F(t) & \text{if } x \notin A \end{cases}$$

is P-semicontinuous in A.

Proof. Let a be an element of A and V an open set containing $\tilde{F}(a)$. There exists an open set W with compact closure for which $F(a) \subseteq W \subseteq \overline{W} \subseteq V$. By P-semicontinuity of F there is a neighborhood U of a such that $F(t) \cap W \neq \emptyset$ for every $t \in U \cap A$. If $t \in U - A$, for every net $(t_i)_{i \in I}$ $t_i \in A \cap U$ convergent to t, and $y_i \in F(t_i)$, clearly there exists $y \in \overline{W} \cap \tilde{F}(t)$. Thus $\tilde{F}(t) \cap V \neq \emptyset$.

With a similar technique we prove an extension theorem for P-semicontinuous multifunctions satisfying the condition

$$(\alpha) \ \lim_{t \to x}{}' F(t) \neq \emptyset$$

THEOREM 3.6. *Let $F : A \to Y$ be P-semicontinuous in a dense subset A of X, subcontinuous in $X - A$. If Y is normal and $\lim\limits_{t \to x}{}' F(t) \neq \emptyset$ for every $x \in X - A$, then $\tilde{F}$ is P-semicontinuous in $X - A$.*

Proof. If $x \in X - A$ and $V \supseteq \tilde{F}(x)$ is open, there exists an open subset W of Y for which $\tilde{F}(x) \subseteq W \subseteq \overline{W} \subseteq V$. Since $\lim\limits_{t \to x}{}' F(t) \neq \emptyset$ there is a neighborhood U of x such that $\tilde{F}(t) \cap W \neq \emptyset$ for every $t \in U \cap A$. If $t \in U - A$ since F is subcontinuous the proof is the same as in 3.5.

Hrycay proves the following theorem:

THEOREM 3.7. [7] *Let* X *be locally connected and* Y *normal. If* $F : X \to Y$ *is connected and subcontinuous, then* $\lim\limits_{t \to x}{}'' \, F(t)$ *is a connected set for every* x *in* X.

By using it we obtain an extension theorem for P-semicontinuous and connected valued multifunctions satisfying condition (α).

THEOREM 3.8. *Let* X *be locally connected,* Y *normal and* A *a dense subset of* X. *If* F *is connected valued,* P*-semicontinuous in* A, *subcontinuous in* $X - A$ *and satisfies* (α), *then the extension* $\tilde{F}$ *of* F *is* P*-semicontinuous and connected valued.*

4. Topologies and convergences on connected multifunction spaces.

Let $\mathcal{C}$ (resp. $\mathcal{P}(S)$) be the space of all connected (resp. P-semicontinuous) multifunctions from X to Y. Obviously $\mathcal{C} \supseteq \mathcal{P}(S)$ if the multifunctions are connected valued. Our aim in this section is to introduce a convergence on $\mathcal{C}$ (resp. on $\mathcal{P}(S)$) which preserves connected (resp. P-semicontinuous) multifunctions.

Before proceeding it may be worth while mentioning the following results about nets in $\mathcal{A}(Y)$.

LEMMA 4.1. *If* Y *is completely normal and* $(A_i)_{i \in I}$ *is a net of closed and connected sets* $A_i \in \mathcal{A}(Y)$ *which is* V*-convergent to* A, *then* A *is connected.*

Proof. We only need to observe that if A were not connected then there would be an open cover of A consisting of two disjoint subsets each of them meeting A.

LEMMA 4.2. *If* Y *is a compact Hausdorff space and* $(A_i)_{i \in I}$ *is a net of closed connected sets in* $\mathcal{A}(Y)$ *which is* V*-convergent to a closed subset* A, *then* A *is connected.*

LEMMA 4.3. [15] *If* Y *is a connected compact metric space,*

$(A_i)_{i\in I}$ is a net of connected compact sets in $\mathcal{A}(Y)$ and $\lim' A_i \neq \emptyset$, then $\lim'' A_i$ is connected.

DEFINITION 4.1. *Let $\mathcal{F}$ be the space of all multifunctions from X to Y, and $(F_i)_{i\in I}$ a net with $F_i \in \mathcal{F}$. The net $(F_i)_{i\in I}$ is said to be C-convergent to $F \in \mathcal{F}$ if F_i converges V-uniformly to F on the connected subsets of X (i.e. for each connected set $K \subseteq X$ and for each open set $V \subseteq Y$ if $K \subseteq F^+(V)$ ($K \subseteq F^-(V)$) there exists $\bar{i} \in I$ such that $K \subseteq F_i^+(V)$ ($K \subseteq F_i^-(V)$) for every $i \geq \bar{i}$).*

This convergence is topological and the associated topology is generated by the family $\{F \in \mathcal{F} : F(K) \subseteq \langle U, V\rangle\}$ where K is a connected subset of X, U, V are open subsets of Y, and $\langle U, V\rangle = \{E \subseteq Y : E \subseteq U \cup V,\ E \cap V \neq \emptyset,\ E \cap U \neq \emptyset\}$, which is open in the Vietoris topology V on $\mathcal{A}(Y)$.

We shall call it the *connected open topology* with respect to V, in analogy with that studied for the space of connected functions from X to Y in [8].

It easy to prove that:

THEOREM 4.4. *If Y is completely normal and $(F_i)_{i\in I}$ is a net of connected multifunctions C-convergent to F, then F is connected.*

Proof. It suffices to observe that for every connected set K the net $(F_i(K))_{i\in I}$ V-converges to $F(K)$ and to recall the lemma 4.1.

A classical topology on the space $\mathcal{F}$ is the *compact open topology* (with respect to V^+) [23] generated by the family $\{F \in \mathcal{F} : F(K) \subseteq V\}$ where $K \subseteq X$ is compact and $V \subseteq Y$ is open.

The links between the connected open and the compact open topologies are given by:

THEOREM 4.5. *If X is locally connected then the compact open topology with respect to V^+ is smaller than the connected open topology with respect to V^+ (generated by $\{F \in \mathcal{F} : F(K) \subseteq V\}$ where $K \subseteq X$ is connected and $V \subseteq Y$ is open) on the space of upper semicontinuous multifunctions.*

It should be remarked that the set $\mathcal{P}(S)$ of all P-semicontinuous multifunctions from X to Y is not closed in the connected open topology. Indeed for each n

$$F_n : x \in \mathbf{R} \to F_n(x) = \begin{cases}]0,1[& \text{if } x = 0 \\ ]-1, 1/n[& \text{if } x \neq 0 \end{cases}$$

is P-semicontinuous at 0 and the sequence (F_n) C-converges to

$$F : x \in \mathbf{R} \to F(x) = \begin{cases}]0,1[& \text{if } x = 0 \\ ]-1, 0[& \text{if } x \neq 0 \end{cases}$$

which is not P-semicontinuous at 0.

DEFINITION 4.2. *A net $(F_i)_{i\in I}$ is said to be uniformly V^+-convergent to F if for every open set $V \subseteq Y$ such that $F^+(V) \neq \emptyset$ there exists an $\bar{i} \in I$ for which $F^+(V) \subseteq F_i^+(V)$ whenever $i \geq \bar{i}$.*

Let us observe that uniform V^+-convergence is topological and the collection $I(F) = \{G \in \mathcal{F} : F^+(V) \subseteq G^+(V),\ V \subseteq Y \text{ and open }\}$ is a subbase for the neighborhood system of F in the associated topology. Obviously this topology is stronger than the compact open topology and the connected open topology with respect to V^+.

We prove the following result.

THEOREM 4.6. *Let Y be regular and $(F_i)_{i\in I}$ a net of multifunctions uniformly V^+-convergent to F. If F_i is P-semicontinuous at x_0 for each $i \in I$ and $F(x_0)$ is compact, then F is P-semicontinuous at x_0.*

Proof. If F were not P-semicontinuous at x_0, then it would exist an open set V containing $F(x_0)$ and, for each $U \in \mathcal{N}(x_0)$, $x_U \in U$ such that $F(x_U) \cap V = \emptyset$. Since Y is regular and $F(x_O)$ is compact there exist two open disjoint sets $G_1 \supseteq F(x_0)$ and $G_2 \supseteq -V$. Hence for some i_0 in I, $F_i(x_0) \subseteq G_1$ and $F_i(x_U) \cap G_1 = \emptyset$ for every $i \geq i_0$ which contradicts the hypothesis.

The assumption that $F(x_0)$ is compact, is essential in the previous theorem. Let us consider, as an example, the multifunctions

$$F_n : x \in \mathbf{R} \to F_n(x) = \begin{cases}]0,1[& \text{if } x = 0 \\ ]1 - 1/n, 2] & \text{if } x \neq 0 \end{cases}$$

$$F : x \in \mathbf{R} \to F(x) = \begin{cases}]0,1[& \text{if } x = 0 \\ [1, 2] & \text{if } x \neq 0 \end{cases}$$

The sequence (F_n) is uniformly V^+-convergent to F, F_n is P-semicontinuous at $x_0 = 0$ for every n, but F is not.

Let us observe, furthermore, that in Theorem 4.6 the statement may not hold if the net (F_i) is not uniformly V^+-convergent to F as shown by the following example

$$G_n : x \in [0,1] \to G_n(x) = \begin{cases} \{1\} & \text{if } x = 1 \\ [0, x^n] & \text{if } x \neq 1 \end{cases}$$

Each multifunction G_n is P-semicontinuous at 0, the sequence (G_n) is pointwise convergent to

$$G : x \in [0,1] \to G(x) = \begin{cases} \{1\} & \text{if } x = 1 \\ \{0\} & \text{if } x \neq 1 \end{cases}$$

but G is not P-semicontinuous at 1.

5. Upper and lower semiconnected multifunctions.

Smithson gives the following definition in [21]:

DEFINITION 5.1. *A multifunction $F : X \to Y$ is said to be*

i) upper semiconnected in X if for each closed and connected set $K \subseteq Y$ the set $F^-(K)$ is closed.

ii) lower semiconnected in X if for each open and connected set $K \subseteq Y$, $F^-(K)$ is open.

He also gives a sufficient condition in order that a lower or upper semiconnected multifunction be continuous.

We shall prove that upper and lower semiconnectedness can be reduced to continuity by an appropriate change of the topology on the range Y of F. In order to do this, on the family $\mathcal{A}(Y)$ of all nonempty subsets of Y, let us consider the collection $\mathcal{B}^+ = \{E \subseteq Y : E \subseteq V,\ V$ open in Y and $Y - V$ connected$\}$. Let V^{*+} denote the topology on $\mathcal{A}(Y)$ generated by taking $\mathcal{B}^+$ as a subbase. Obviously $V^{*+} \subseteq V^+$, moreover it results

THEOREM 5.1. *The multifunction F is upper semiconnected if and only if F is continuous from X to the topological space $(\mathcal{A}(Y), V^{*+})$.*

At the same time, let us consider on Y the topology generated by the collection $\{V \subseteq Y, V$ open and connected$\}$ and the related lower Vietoris V^{*-} on $\mathcal{A}(Y)$, that is the topology generated by $\mathcal{B}^- = \{E \subseteq Y : E \cap V \neq \emptyset,\ V$ open and connected in $Y\}$. Then $V^{*-} \subseteq V^-$ and it is also evident that:

THEOREM 5.2. *The multifunction F is lower semiconnected if and only if F is continuous from X to the topological space $(\mathcal{A}(Y), V^{*-})$.*

THEOREM 5.3. *If Y is a locally connected space, each lower semiconnected multifunction is lower semicontinuous.*

Proof. We only need to observe that $V^{*-} = V$, since the family of the open connected subsets of Y is a base for its topology.

Whyburn gives the following definition in [25].

DEFINITION 5.3. *A topological space Y is said to be semilocally connected about a subset A provided that for every open set U containing A, there exists an open set V with $A \subseteq V \subseteq U$ such that $Y - V$ has only a finite number of components.*

We shall say that:

DEFINITION 5.4. *Y is semilocally connected about a multifunction F if Y is semilocally connected about $F(x)$, for each $x \in X$.*

Now we prove the following result.

THEOREM 5.4. *If F is upper semiconnected and Y is semilocally connected about F, then F is upper semicontinuous.*

Proof. Let x_0 be an element of X and V an open set containing $F(x_0)$. There exists an open set W such that $F(x_0) \subseteq W \subseteq V$ and $Y - W = \bigcup_{i=1}^{n} C_i$ where each C_i is closed and connected. The set $C = \bigcup_{i=1}^{n} F^-(C_i)$ is closed and $x_0 \notin C$. Thus $X - C$ is a neighborhood of

x_0 mapped by F into V.

Since a *continuum* Y, that is a compact connected Hausdorff space, is locally connected if and only if it is semilocally connected about each of its compact subsets [25], we have:

THEOREM 5.5. *Let Y be continuum and locally connected. If F is compact valued and upper semiconnected multifunction, then F is upper semicontinuous.*

Moreover we can prove the following result.

THEOREM 5.6. *Let Y be locally connected and regular. If F is closed valued and upper semiconnected, then F is graph closed.*

Proof. If $(x,y) \notin \mathrm{graph}F$ there exist two disjoint open sets A, B such that $y \in A$ and $B \supset F(x)$; since Y is locally connected and regular there exists also an open connected set V_1 such that $y \in V_1 \subseteq \overline{V_1} \subseteq A$. Moreover it results $F(x) \cap \overline{V_1} = \emptyset$ and, since F is upper semiconnected, there is a neighborhood U of x such that $F(U) \subseteq -\overline{V_1}$. Thus $U \times V_1$ is a neighborhood of (x,y) which does not meet the graph of F.

6. An unifying approach to the study of functions preserving connected sets.

Several authors proved that some continuous-like functions preserve connected sets. In this section we provide a unified method to obtain these results, based on the approach to the study of continuity-like properties developed in [3]. In this way we obtain also some results in [16], [17].

Let us begin by recalling some definitions: given a non empty set X, a family $\xi(x)$ of subsets of X is said to be a *generalized local sieve* (g.l.s.) at $x \in X$ when

i) if $B \in \xi(x)$ and $A \supseteq B$ then $A \in \xi(x)$

ii) the intersection of any two members of $\xi(x)$ is non empty

iii) $x \in A$ for every $A \in \xi(x)$.

For $A \subseteq X$ we define

$$\xi\text{-}clA = \{y \in X : \forall I \in \xi(y) \quad I \cap A \neq \emptyset\}.$$

A subset C of X is said ξ-connected if C is not the union of two non empty subsets B_1 and B_2 of X such that $B_j \cap \xi\text{-}clB_i = \emptyset$ $i, j \in \{1,2\}$.

If $(\xi(x))_{x\in X}$ and $(\xi'(y))_{y\in Y}$ are families of g.l.s. on X and Y respectively, a function $f : X \to Y$ is (ξ,ξ')-*continuous* at $x \in X$ if $f^{-1}(V) \in \xi(x)$ for every $V \in \xi'(f(x))$. A function $f : X \to Y$ is said to be (ξ,ξ')-continuous in X if it is (ξ,ξ')-continuous at x, for each $x \in X$.

There is no difficulty in showing that a (ξ,ξ')-continuous function maps ξ-connected subsets of X onto ξ'-connected subsets of Y.

Let us sketch various possibilities:

a) In [17] *strongly semicontinuous* functions from X to Y are studied. These are functions f for which the inverse image $f^{-1}(V)$ of any open set $V \subseteq Y$, is an α-set (A is an α-set if $A \subseteq int(\overline{intA})$, where $intA$ is the interior of A). It is also proved that if f is strongly semicontinuous from X onto Y and X is connected, then Y is connected too. Since the family T^α of all α-sets in X is a topology on X and a strongly semicontinuous function is simply a continuous one between the the topological spaces (X, T^α) and Y [20], the above result is an immediate consequence of the fact that the topological space X is connected if, and only if, it is T^α-connected [20].

b) The following definition is given in [16]: a function f from X to Y is said to be *weakly continuous* if for each point $x \in X$ and each open set $V \ni f(x)$ there exists a neighborhood U of x such that $f(U) \subseteq \overline{V}$. Such a continuity notion is a (ξ,ξ')-continuity for the g.l.sieves of the neighborhood filters on X and of the θ-neighborhoods on Y (A is a θ-neighborhood of $y \in Y$ if there is an open set O such that $y \in O \subseteq \overline{O} \subseteq A$). Since every θ-connected space is connected, it follows that if f is a weakly continuous mapping of a connected space X onto a space Y, then Y is also connected.

With the same technique used before we obtain similar results about *semiweakly continuous* functions [18]. In fact they are (ξ,ξ')-continuous functions with respect to the following sieves on X and Y respectively (see [3])

$$\xi'(y) = \{A \subseteq Y : \exists V \text{ open} : y \in V \subseteq sclV \subseteq A\}$$

$$\xi(x) = \{A \subseteq X : A \text{ is a semi-neighborhood of } x\}.$$

Let us remember that a semi-neighborhood of x is a subset containing a semi-open set A to which x belongs (A is semi-open if there is an open set O such that $O \subseteq A \subseteq \overline{O}$ [12], and $sclV$ is the semiclosure of V [3]).

Keeping in mind the terminology from [3] and the above results, it follows that the image Y of a point connected multifunction from a connected space X onto Y is connected if F satisfies one of the following conditions:

1) F is *weakly P-semicontinuous*, that is for each $x \in X$ and for every open subset V containing $F(x)$, there exists a neighborhood U of x such that $F(z) \cap \overline{V} \neq \emptyset$ for every $z \in U$.

2) F is almost *P-semicontinuous*, that is for each $x \in X$ and for every open subset V containing $F(x)$, there is a neighborhood U of x such that $F(z) \cap int(\overline{V}) \neq \emptyset$ for every $z \in U$.

3) F is *quasi P-semicontinuous*, that is for each $x \in X$ and for every open set V containing $F(x)$, there exists a semi-neighborhood U of x for which $F(z) \cap V \neq \emptyset$ for every $z \in U$.

BIBLIOGRAFIA

[1] Banzaru T., Crivat N, *On the upper semicontinuity of upper topological limits for multifunction nets*, Inst. Politehn. "Traian Vuia" Timisoara Lucrar. Sem. Mat. Fiz. 1983, May, 59-64.

[2] Correa R., Hiriart Urruty J.B., Penot J.P., *A note on connected set-valued mappings*, Boll. Un. Mat. Ital. C, **1** (1986), 357-366.

[3] Del Prete I., Di Iorio M., *An unifying approach to continuity-like notions for functions and multifunctions*, Rend. Acc. Sc. Fis. Mat., Serie IV, Vol. LVII, CXXIX, 67-82.

[4] Dugundji J., *Topology*, Allyn and Bacon 1970.

[5] Ewert J., Lipiński J.S., *Weakly connected multivalued maps*, Rad. Mat., vol. 5, (1989), 193-200.

[6] Hiriart Urruty J.B., *Images of connected sets by semicontinuous multifunctions*, J. Math. Anal. Appl. lll, (1985), 407-422.

[7] Hrycay R., *Non continuous multifunctions*, Pacific J. Math. **35** (1970), 141-154.

[8] Irudayanathan A., Naimpally S., *Connected open topology for function spaces*, Indag. Math. **28** (1966), 22-24.

[9] Kohli J.K., *A unified approach to continuous and certain non-continuous functions*, J. Austral. Math. Soc. (Series A) **48** (1990), 347-358.

[10] Kuratowski K, *Topology*, Vol. 1 Academic Press New York 1966.

[11] Lechicki A., Levi S., *Extensions of semicontinuous multifunctions,* Quaderno 28. Dipart. Mat "Enriques" Milano (1988).

[12] Levine N.,*Semiopen sets and semicontinuity in topological spaces,* Amer. Math. Montly, **70** (1963), 36-41.

[13] Long P.E., *Connected mappings*, Duke Math. J. **35** (1968), 677-682.

[14] Michael E., *Topologies on spaces of subsets,* Trans. Amer. Math. Soc. **71** (1951), 152-183.

[15] Nadler S.B., *Hyperspaces of sets*, Pure and applied Mathematics. M. Dekker Inc. 1978.

[16] Noiri T., *On weakly continuous mappings*, Proc. Amer. Math. Soc. **46** (1974), 120-124.

[17] Noiri T., *A function which preserves connected spaces,* Cas. Pest. Mat. **107** (1982), 393-396.

[18] Noiri T., Ahmad B., *On semi-weakly continuous mappings,* Kyungpook Math. J. **25** (1985), 123-126.

[19] Pervin W.J., *On the connected open topology*, Indag. Math. **29** (1967), 126-127.

[20] Reilly I., Vamanamurthy M.K., *Connectedness and strong semicontinuity,* Čas. Pěst. Mat. **109** (1984), 261-265.

[21] Smithson R.E., *A note on the continuity of multifunctions*, J. Nat. Sci. and Math.,(Lahore) **7**, (1967), 197-202.

[22] Smithson R.E., *Almost and weak continuity for multifunctions,* Bull. Cal. Math. Soc. **70** (1978), 383-390.

[23] Smithson R.E., *Topologies on sets of relations,* J. Nat. Sci. and Math. (Lahore) **11** (1971), 43-50.

[24] Valadier M., *Deus propriètés des trajectoires d'une différentielle multivoque,* Séminaire d'Analyse convexe de Montpellier (1975), Exposé No. 12.

[25] Whyburn G.T., *Continuity of multifunctions,* Proc. Nat. Acad. Sci. **6** (54),(1965), 1494-1501.

Dipartimento di Matematica e Applicazioni
Università degli Studi di Napoli
via Claudio 21, (NA)