LE MATEMATICHE
Vol. XLVI (1991) - Fasc. II, pp. 625-644

PARTIAL REGULARITY FOR QUASILINEAR NONUNIFORMLY ELLIPTIC SYSTEMS

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We prove the partial regularity of the weak solutions of the quasilinear nonuniformely elliptic system div$(A(\nabla u) = 0$ under an ellipticity condition wich lies between strong ellipticity and Legendre-Hadamard condition.

1. Introduction.

The partial regularity of the weak solutions $u : \Omega \to \mathbb{R}^N$ of second order quasilinear systems in divergence form

$$\frac{\partial}{\partial x_\alpha} A^i_\alpha(x, u, u_x) = B^i(x, u, u_x), \quad i = 1, \dots, N$$

where $N \geq 1$, Ω is an open set of $\mathbb{R}^n$, $n \geq 2$, $u_x = (u^i_{x_\alpha})$, $i = 1, \dots, N$, $\alpha = 1, \dots, n$, has been studied by many authors (in particular by Morrey, Campanato, Giusti, Giaquinta, Modica, Ivert). We refer the reader to the Giaquinta's book [5] for a complete survey on the subject.

(*) Entrato in Redazione l'8 maggio 1991.

Up to recent years the regularity proofs required the condition of uniformly strong ellipticity.

Consider in particular the system

$$\frac{\partial}{\partial x_\alpha} A^i_\alpha(u_x) = 0, \quad i = 1, \dots, N \tag{1.1}$$

and suppose that functions $A^i_\alpha(p)$ satisfy the conditions

(S) *A^i_α belong to $C^1(\mathbb{R}^{nN})$;*

(G) *there exists a constant $c_0 > 0$ such that*

$$|A^i_\alpha(p)| \leq c_0(1 + |p|^{m-1}) \quad \forall p \in \mathbb{R}^{nN}, \ m \geq 2.$$

In this paper m will always denote a real number, $m \geq 2$.

The condition of strong ellipticity for system (1.1) means that

$$(SE) \qquad \nu_0(1 + |p|^{m-2})\, |\xi|^2 \leq \frac{\partial A^i_\alpha(p)}{\partial p^j_\beta} \xi^i_\alpha \xi^j_\beta, \quad \forall \xi, p \in \mathbb{R}^{nN}, \ \nu_0 > 0,$$

and the condition of uniformly strong ellipticity means that (SE) holds and moreover

$$\frac{\partial A^i_\alpha(p)}{\partial p^j_\beta} \xi^i_\alpha \xi^j_\beta \leq \mu_0(1 + |p|^{m-2})|\xi|^2, \quad \forall \xi, \ p \in \mathbb{R}^{nN}. \tag{1.2}$$

However there exist quasilinear systems interesting for applications witch are not uniformly strong elliptic.

Consider for example the Euler equations

$$\frac{\partial}{\partial x_\alpha} F_{p^i_\alpha}(u_x) = 0, \quad i = 1, \dots, N \tag{1.3}$$

of the variational problem of minimizing an integral with an integrand of the type

$$F(p) = a|p|^2 + b|p|^n + \sqrt{1 + (\det p)^2}, \tag{1.4}$$

$a > 0$, $b > 0$ $(N = n)$. It is easy to see that for the system (1.3), (1.4) the condition (SE) with $m = n$ holds if b is large enough, but condition (1.2) does not hold (see sect. 5).

In 1984 Evans [2] proved the partial regularity of minimizers of variational integrals $\int_\Omega F(p)dx$ under the assumption of quasi-convexity for the integrand $F(p)$, i.e.

(QC) *there exists a constant* $\bar{\nu} > 0$ *such that*

$$\int_\Omega [F(p_0 + \varphi_x) - F(p_0)]dx \geq$$

$$\geq \bar{\nu} \int_\Omega (|\varphi_x|^2 + |\varphi_x|^m)dx, \quad \forall p_0 \in \mathbb{R}^{nN}, \quad \forall \varphi \in C_0^1(\Omega, \mathbb{R}^N).$$

The Evans's result was generalized independently by Giaquinta-Modica [6] and Fusco-Hutchinson [3] to quasiconvex integrands $F(x, u, p)$. In [2], [3] and [6] there are also assumptions like

$$|F_{pp}| \leq c(1 + |p|^{m-2}), \quad \forall p \in \mathbb{R}^{nN}. \tag{1.5}$$

Later Acerbi-Fusco [1] and Giaquinta [7] proved the partial regularity of minimizers without assuming any control on the growth of the second derivatives of F with respect to p.

In particular, if $F = F(p)$, they proved the partial $C^{1,\alpha}$-regularity of minimizers assuming that $F \in C^2$, $|F(p)| \leq c(1 + |p|^m)$ and quasiconvexity condition (QC). From this result it follows that minimizers (but not stationary points) of variational integral with integrand like (1.4) are partial regular (for any $a > 0$, $b > 0$).

We shall prove later (see Lemma 2.1) that condition (SE) implies the following condition

(E) *there exists a constant* $\nu > 0$ *such that*

$$\int_\Omega A_\alpha^i(p_0 + \varphi_x) - \varphi_{x_\alpha}^i dx \geq$$

$$\geq \nu \int_\Omega (|\varphi_x|^2 + |\varphi_x|^m)dx, \quad \forall p_0 \in \mathbb{R}^{nN}, \quad \forall \varphi \in C_0^1(\Omega, \mathbb{R}^N).$$

In the case $m = 2$ the condition (E) was firstly introduced by Fuchs [3] who proved the partial $C^{1,\alpha}$-regularity of weak solutions of the system (1.1) assuming that conditions (S), (E) and

$$\left|\frac{\partial A^i_\alpha(p)}{\partial p^j_\beta}\right| \leq c, \quad \forall p \in \mathbb{R}^{nN}$$

hold.

Unfortunately we know no quasiconvex (but not convex) function $F(p)$ satisfying all the conditions of [3] with $A^i_\alpha = F_{p^i_\alpha}$. Moreover we know no nonuniformly strong elliptic system for which these conditions are valid.

For instance any system of type (1.3), (1.4) does not satisfy these conditions.

The aim of this paper is to establish the partial $C^{1,\alpha}$-regularity of weak solutions of system (1.1) assuming only the conditions (S), (G) and (E).

More precisely, recalling that a weak solution of system (1.1) is a function $u \in H^{1,m}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega A^i_\alpha(u_x)\varphi^i_{x_\alpha} dx = 0, \quad \forall \varphi \in C^1_0(\Omega, \mathbb{R}^N), \tag{1.6}$$

we will prove the following (see sect. 4)

THEOREM 1.1. *Suppose that conditions (S), (G) and (E) hold. Let u be a weak solution of system (1.1). Then there exists an open set $\Omega_0 \subset \Omega$ where the first derivatives of u are Hölder continuous; moreover the Lebesgue measure of the possible singular set $\Omega \backslash \Omega_0$ is zero.*

The proof of theorem 1.1 is achieved by a suitable adaptation of the techniques used by Giaquinta in [7].

We explicitly remark that the partial $C^{1,\alpha}$-regularity of weak solutions of a nonuniformly elliptic system under assumptions (S), (G), (SE) and of the stationary points of variational integrals with integrands like (1.4) (b large enough) follows from theorem 1.1 (see lemma 2.1).

As last remark let us point out that in the case $A^i_\alpha(p) = F_{p^i_\alpha}(p)$ the condition (QC) is a consequence of the condition (E). In fact for all $\varphi \in C^1_0(\Omega, \mathbb{R}^{nN})$ and $p_0 \in \mathbb{R}^{nN}$ we have

$$\int_\Omega [F(p_0 + \varphi_x) - F(p_0)]dx = \int_\Omega \int_0^1 \tau^{-1} F_{p^i_\alpha}(p_0 + \tau\varphi_x)(\tau\varphi)^i_{x_\alpha} d\tau dx \geq$$
$$\geq \int_0^1 \tau^{-1} \int_\Omega \nu(|\tau\varphi_x|^2 + |\tau\varphi_x|^m) dx d\tau =$$
$$= \nu\left(\frac{1}{2} + \frac{1}{m}\right) \int_\Omega (|\varphi_x|^2 + |\varphi_x|^m) dx.$$

2. Connections between the conditions (SE), (E) and Legendre-Hadamard condition.

LEMMA 2.1. *Assume (SE). Then the condition (E) holds.*

Proof. From the condition (SE) we have for all $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$

$$\int_\Omega A^i_\alpha(p_0 + \varphi_x)\varphi^i_{x_\alpha} dx = \int_\Omega \int_0^1 (A^i_\alpha)_{p^j_\beta}(p_0 + \tau\varphi_x)\varphi^i_{x_\alpha}\varphi^j_{x_\beta} d\tau dx \geq \tag{2.1}$$
$$\geq \int_\Omega \int_0^1 \nu_0[1 + |p_0 + \tau\varphi_x|^{m-2}]|\varphi_x|^2 d\tau dx \geq$$
$$\geq \nu_0 \int_\Omega |\varphi_x|^2 dx + \nu_0 \int_\Omega |\varphi_x|^2 \int_0^1 (1-\tau)|p_0 + \tau\varphi_x|^{m-2} d\tau dx.$$

Because of Lemma 8.1 of [2], there exists a constant $\chi \in (0,1)$ depending only on m such that

$$\int_0^1 (1-\tau)|p_0 + \tau\varphi_x|^{m-2} d\tau \geq \chi|\varphi_x|^{m-2}. \tag{2.2}$$

Hence from (2.1) and (2.2) we have

$$\int_\Omega A^i_\alpha(p_0 + \varphi_x)\varphi^i_{x_\alpha} dx \geq \nu_0\chi \int_\Omega (|\varphi_x|^2 + |\varphi_x|^m) dx,$$

and lemma is proved.

LEMMA 2.2. *Assume conditions (S), (G) and (E). Then it follows that for all* $p_0 \in \mathbb{R}^{nN}$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$ *the Legendre-Hadamard inequality*

$$\frac{\partial A^i_\alpha(p_o)}{\partial p^j_\beta}\xi^i\xi^j\eta_\alpha\eta_\beta \geq \nu_1|\xi|^2|\eta|^2, \tag{2.3}$$

holds, where $\nu_1 = \nu_1(\nu, n)$.

Proof. From the condition (E) it follows that for all $p_0 \in \mathbb{R}^{nN}$ and $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$ we have (with some $\vartheta, (0,1)$)

$$\begin{aligned} \nu\int_\Omega(|\varphi_x|^2 + |\varphi_x|^m)dx &\leq \\ &\leq \int_\Omega [A^i_\alpha(p_0 + \varphi_x) - A^i_\alpha(p_0)]\varphi^i_{x_\alpha}dx = \\ &= \int_\Omega (A^i_\alpha)_{p^j_\beta}(p_0 + \vartheta\varphi_x)\varphi^i_{x_\alpha}\varphi^j_{x_\beta}dx. \end{aligned} \tag{2.4}$$

Let us select some point $x_0 \in \Omega$ and a unit vector $\eta \in \mathbb{R}^n$, $\eta = (\eta_\alpha)$, and choose new coordinates y related to x by the transformation

$$y^\gamma = \sum_{\alpha=1}^n d_{\gamma\alpha}(x^\alpha - x^\alpha_0), \quad x^\alpha - x^\alpha_0 = \sum_{\gamma=1}^n d_{\gamma\alpha}y^\gamma, \quad (d_{1\alpha}) = (\eta_\alpha) \tag{2.5}$$

where $d = (d_{\gamma\alpha})$ is a constant orthogonal matrix so that η is the unit vector in the y^1 direction. Let $\Phi \in C^1_0(\Omega)$ and define

$$\begin{cases} \varphi^i(x) = \xi^i\Phi(x), \quad \xi^i \in \mathbb{R}, \quad \omega(y) = \Phi(x(y)), \\ A^{ij}_{\alpha\beta}(p) = (A^i_\alpha)_{p^j_\beta}(p) \\ a_{\alpha\beta}(p) = A^{ij}_{\alpha\beta}(p)\xi^i\xi^j, \quad {}'a_{\gamma\delta}(p) = a_{\alpha\beta}d_{\gamma\alpha}d_{\delta\beta} \end{cases} \tag{2.6}$$

Making the change of variables (2.5), (2.6) we get from (2.4) the inequality

$$\nu|\xi|^2\int_{\Omega'}|\nabla\omega|^2dy \leq \int_{\Omega'} {}'a_{\gamma\delta}(p_0 + \vartheta\varphi_x)\frac{\partial\omega}{\partial y^\gamma}\frac{\partial\omega}{\partial y^\delta}dy, \tag{2.7}$$

where Ω' denotes the image of Ω. Now choose

$$\omega(y)=\begin{cases}\epsilon(h-|y^1|)(1-r/H) & \text{if } |y^1|\le h,\ 0\le r\le H,\\ 0 & \text{outside}\end{cases}$$
$$(y^2)^2+\ldots+(y^n)^2=r^2,\quad \epsilon>0,$$

with $0<h<H$ so small that supp$\omega\subset\Omega'$. Then if we divide both sides of (2.7) by ϵ^2 times the measure of the support of ω and let $\varepsilon\to 0$ and then h and $H\to 0$ so that $h/H\to 0$ we conclude that

$$\nu_1|\xi|^2\le {}'a_{11}(p_0)=a_{\alpha\beta}(p_0)d_{1\alpha}d_{1\beta}=(A^i_\alpha)_{p^j_\beta}\xi^i\xi^j\eta_\alpha\eta_\beta$$

and lemma is proved.

Remark. 2.1. It is obvious that

$$\frac{\partial A^i_\alpha(p_0)}{\partial p^j_\beta}\xi^i\xi^j\eta_\alpha\eta_\beta\le c(p_0)|\xi|^2\,|\eta|^2 \tag{2.8}$$

3. Caccioppoli's inequalities and higher integrability.

We denote with $B_r=B_r(x_0)$ the ball of $\mathbb{R}^n$ with center x_0 and radius r and with $v_{x_0,r}$ the average $\displaystyle -\!\!\!\!\!\!\int_{B_r} v(x)dx$.

LEMMA 3.1. *Let u be a weak solution of the system (1.1). Suppose that conditions (S), (E) are satisfied. Moreover suppose that*

$$|A^i_\alpha(p)|\le k_0(|p|+|p|^{m-1}),\quad \forall p\in\mathbb{R}^{nN} \tag{G_0}$$

holds. Then there exists a constant $c=c(n,m,\nu,k_0)$ such that

$$\int_{B_{R/2}}(|u_x|^2+|u_x|^m)dx\le \tag{3.1}$$
$$\le c\left\{\frac{1}{R^2}\int_{B_R}|u-u_{x_0,R}|^2dx+\frac{1}{R^m}\int_{B_R}|u-u_{x_0,R}|^m dx\right\}$$

for all $B_R \subset\subset \Omega$.

Proof. Let $B_R \subset\subset \Omega$, $R/2 \leq r < s \leq R$ and $\eta \in C_0^1(B_s)$ with $\eta = 1$ in B_r, $|\eta_x| \leq c/(s-r)$. We set

$$\varphi = \eta(u-u_0), \quad \psi = (1-\eta)(u-u_0), \tag{3.2}$$

$u_0 = u_{x_0,R}$. Then we have $u - u_0 = \varphi + \psi$, $u_x = \varphi_x + \psi_x$ in B_s, and $\psi = \psi_x = 0$ in B_r.

From (E) (with $p_0 = 0$), (1.6), (G_0) and (3.2) we have

$$\begin{aligned} \nu\int_{B_s}(|\varphi_x|^2+|\varphi_x|^m)dx &\leq \int_{B_s}[A_\alpha^i(\varphi_x)-A_\alpha^i(u_x)]\varphi_{x_\alpha}^i dx = \\ &= \int_{B_s\setminus B_r}[A_\alpha^i(u_x-\psi_x)-A_\alpha^i(u_x)]\varphi_{x_\alpha}^i dx \leq \\ &\leq c\int_{B_s\setminus B_r}(|u_x|+|\psi_x|+|u_x|^{m-1}+|\psi_x|^{m-1})|\varphi_x|dx \leq \\ &\leq \frac{\nu}{2}\int_{B_s}(|\varphi_x|^2+|\varphi_x|^m)dx+ \\ &+c\int_{B_s\setminus B_r}(|u_x|^2+|u_x|^m+|\psi_x|^2+|\psi_x|^m)dx \end{aligned} \tag{3.3}$$

Since $\varphi = u - u_0$ in B_r and

$$|\psi_x| \leq (1-\eta)|u_x| + |\eta_x|\,|u-u_0|$$

in B_s we conclude from (3.3) that

$$\int_{B_r}(|u_x|^2+|u_x|^m)dx \leq c\left\{\int_{B_s\setminus B_r}(|u_x|^2+|u_x|^m)dx + \frac{1}{(s-r)^2}\int_{B_R}|u-u_0|^2dx+\frac{1}{(s-r)^m}\int_{B_R}|u-u_0|^m dx\right\},$$

and hence

$$\int_{B_r}(|u_x|^2+|u_x|^m)dx \leq \frac{c}{1+c}\left\{\int_{B_s}(|u_x|^2+|u_x|^m)dx+\right. \tag{3.4}$$

$$+\frac{1}{(s-r)^2}\int_{B_R}|u-u_0|^2dx+\frac{1}{(s-r)^m}\int_{B_R}|u-u_0|^m dx\Bigg\},$$

By means of the algebraic lemma 2.3 of [6] (3.4) gives at once the thesis.

Remark. 3.1. Let us explicitly observe that the proof of lemma 3.1 requires the condition (E) to hold only for $p_0 = 0$

Let $p \in \mathbb{R}^{nN}$ and define

$$\bar{A}^i_\alpha(p) = A^i_\alpha(p_0 + p) - A^i_\alpha(p_0). \tag{3.5}$$

LEMMA 3.2. *Assume (S) and (G). Then there exists a constant* $c = c(m, c_0, |p_0|)$ *such that*

$$|\bar{A}^i_1(p)| \le c(|p| + |p|^{m-1}), \quad \forall p \in \mathbb{R}^{nN}. \tag{3.6}$$

Proof. Set

$$K(|p_0|) = \max\{|(A^i_\alpha)_p(p)| : |p| \le 1 + |p_0|\}.$$

Then we have for $|p| \le 1$ and a suitable $\vartheta \in (0,1)$

$$|\bar{A}^i_\alpha(p)| = |(A^i_\alpha)_p(p_0 + \vartheta p)\cdot p| \le K(|p_0|)\,|p|,$$

while for $|p| > 1$ from (G) it follows that

$$\begin{aligned}|\bar{A}^i_\alpha(p)| &\le c(1 + |p_0| + |p_0|^{m-1} + |p| + |p|^{m-1}) \le \\ &\le c(1 + |p_0| + |p_0|^{m-1})(1 + |p| + |p|^{m-1}) \le \\ &\le c(1 + |p_0| + |p_0|^{m-1})(|p| + |p|^{m-1})\end{aligned}$$

and (3.6) is proved.

LEMMA 3.3. *Assume (S), (G) and (E). Let u be a weak solution of the system (1.1). Then for all $p_0 \in \mathbb{R}^{nN}$ there exists a constant*

$c = c(n, m, \nu, c_0, |p_0|)$ *such that*

$$\int_{B_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)dx \leq \tag{3.7}$$

$$\leq c\left\{\frac{1}{R^2}\int_{B_R} |u - u_{x_0,R} - p_0(x - x_0)|^2 dx + \right.$$

$$\left. + \frac{1}{R^m}\int_{B_R} |u - u_{x_0,R} - p_0(x - x_0)|^m dx\right\}.$$

for all $B_R \subset\subset \Omega$.

Proof. Denote $\bar{u} = u - p_0(x - x_0)$. In view of (3.5) we have

$$\int_\Omega \bar{A}^i_\alpha(\bar{u}_x)\varphi^i_{x_\alpha} dx = \int_\Omega A^i_\alpha(u_x)\varphi^i_{x_\alpha} dx = 0, \quad \forall\varphi \in C^1_0(\Omega, \mathbb{R}^N).$$

Taking into account that for all $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$

$$\int_\Omega \bar{A}^i_\alpha(\varphi_x)\varphi^i_{x_\alpha} dx = \int_\Omega A^i_\alpha(p_0 + \varphi_x)\varphi^i_{x_\alpha} dx \geq \nu\int_\Omega (|\varphi_x|^2 + |\varphi_x|^m)dx,$$

so that $\bar{A}^i_\alpha$ verify the condition (E) (with $p_0 = 0$), we can use lemma 3.1 and lemma 3.2 and obtain the inequality

$$\int_{B_{R/2}} (|\bar{u}_x|^2 + |\bar{u}_x|^m)dx \leq$$

$$\leq c\left\{\frac{1}{R^2}\int_{B_R} |\bar{u} - \bar{u}_{x_0,R}|^2 dx + \frac{1}{R^m}\int_{B_R} |\bar{u} - \bar{u}_{x_0,R}|^m dx\right\}$$

and hence (3.7).

Remark. 3.2. The constant c in lemma 3.3 is increasing in $|p_0|$.

We now recall a well-known theorem by Giaquinta-Modica (see [5], cap. V).

THEOREM 3.1. [Reverse Hölder]. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$. *Suppose we have have*

$$\left(\fint_{B_{R/2}} |g|^q dx\right)^{1/q} \leq b\left(\fint_{B_R} |g|^r dx\right)^{1/r} + \left(\fint_{B_R} |f|^q dx\right)^{1/q}$$

for all $B_R \subset\subset \Omega$, *where* $g \in L^q(\Omega)$, $f \in L^s(\Omega)$, $0 < r < q < s < +\infty$.

Then there exists a positive $\delta = \delta(n, q, r, s, b)$ *so that* $g \in L^{q+\delta}_{\text{loc}}(\Omega)$. *Moreover for every* $\Omega' \subset\subset \Omega$

$$\left(\fint_{\Omega'} |g|^{q+\delta} dx\right)^{\frac{1}{q+\delta}} \leq c\left(\fint_{\Omega} |g|^q dx\right)^{\frac{1}{q}} + c\left(\fint_{\Omega} |f|^{q+\delta} dx\right)^{\frac{1}{q+\delta}}$$

where c *is a constant depending on* n, q, r, s, b, $|\Omega|/|\Omega'|$ *and* $|\Omega|/(dist(\Omega', \partial\Omega))^n$

We are now in a position to state the following theorem on higher integrability of the gradient.

THEOREM 3.2. *Suppose that conditions (S) (G), and (E) are satisfied. Let* u *be a weak solution of (1.1). Then there exist a positive number* δ *and a constant* c, *depending on the same quantities on which depend the constant in lemma 3.3, such that*

$$\left(\fint_{B_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)^{1+\delta} dx\right)^{\frac{1}{1+\delta}} \leq \tag{3.8}$$
$$\leq c\left(\fint_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m\right) dx$$

for all $B_R \subset\subset \Omega$ *and for all* $p_0 \in \mathbb{R}^{nN}$.

Proof. Fix a point $x_0 \in B_R$, a ball $B_{x_0,r} \subset\subset B_R$ and apply lemma 3.3. Using the Sobolev-Poincaré inequality, we estimate the right hand side of the corresponding (3.7) by

$$c\left\{\frac{1}{r^2}\left(\int_{B_r} |u_x - p_0|^{2_*} dx\right)^{\frac{2}{2_*}} + \frac{1}{r^m}\left(\int_{B_r} |u_x - p_0|^{m_*} dx\right)^{\frac{m}{m_*}}\right\}; \tag{3.9}$$

we have set $q_* = \dfrac{nq}{n+q}$.

Note that $2m_* \leq 2_* m$ so that, using the Hölder inequality in

(3.9) from (3.7) we have

$$\fint_{B_{r/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)dx \leq$$

$$\leq c\left(\fint_{B_r} (|u_x - p_0|^2 + |u_x - p_0|^m)^{2_*/2} dx\right)^{\frac{2}{2_*}}.$$

Applying theorem 3.1 with

$$g = (|u_x - p_0|^2 + |u_x - p_0|^m)^{2_*/2}, \quad q = 2/2_*, \; r = 1, \; f \equiv 0$$

we have at once (3.8).

4. Partial regularity.

In this section we prove the theorem 1.1.

Fix a point $x_0 \in \Omega$ and a radius $R <$dist$(x_0, \partial\Omega)$ and set $p_0 = (u_x)_{x_0,R}$.

Let v be the solution of the Dirichlet problem

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial}{\partial x_\alpha}\left((A^i_\alpha)_{p^j_\beta}(p_0)v^j_{x_\beta}\right) = 0 \quad i = 1, \dots, N & \text{in } B_{R/2} \\ v - u = 0 & \text{on } \partial B_{R/2} \end{cases};$$

B_r is the ball $B_r(x_0)$. Recall that the coefficients of system (4.1) satisfy the conditions (2.3) and (2.8). Then it is well-known (see [5] cap. III) that for all $\pi \in \mathbb{R}^{nN}$ and e $\rho < R/2$

$$\text{(4.2)} \qquad \int_{B_{R/2}} |v_x - \pi|^2 dx \leq c\int_{B_{R/2}} |u_x - \pi|^2 dx,$$

$$\text{(4.3)} \qquad \fint_{B_\rho} |v_x - (v_x)_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^2 \fint_{B_{R/2}} |v_x - (v_x)_{x_0,R/2}|^2 dx.$$

Moreover from L^p-theory for elliptic systems we deduce that if $u \in H^{1,p}(B_{R/2}, \mathbb{R}^N)$, $p > 2$, then $v \in H^{1,p}(B_{R/2}, \mathbb{R}^N)$ and

$$\text{(4.4)} \qquad \int_{B_{R/2}} |v_x - \pi|^p dx \leq c\int_{B_{R/2}} |u_x - \pi|^p dx,$$

$$\fint_{B_\rho} |v_x - (v_x)_{x_0,\rho}|^p dx \le c\left(\frac{\rho}{R}\right)^p \fint_{B_{R/2}} |v_x - (v_x)_{x_0,R/2}|^p dx. \tag{4.5}$$

Notice that constants appearing in (4.2)...(4.5) depend on $|p_0|$ and it is possible to assume that they are increasing in such quantity. From (4.3), (4.5) we deduce for $\rho < R/2$ and hence for $\rho < R$

$$\rho^n \Phi(x_0,\rho) \le c(|p_0|)\left\{\left(\frac{\rho}{R}\right)^{n+2} R^n \Phi(x_0,R) + \right. \tag{4.6}$$

$$\left. + \int_{B_{R/2}} (|u_x - v_x|^2 + |u_x - v_x|^m) dx \right\},$$

where

$$\Phi(x_0,R) = \fint_{B_\rho} (|u_x - (u_x)_{x_0,\rho}|^2 + |u_x - (u_x)_{x_0,\rho}|^m) dx. \tag{4.7}$$

We only need to estimate the integral

$$\int_{B_{R/2}} (|w_x|^2 + |w_x|^m) dx,$$

where $w = u - v$; obviously $w \in H_0^{1,m}(B_{R/2}, \mathbb{R}^N)$.

Using the condition (E) with $p_0 = 0$ we have

$$\nu \int_{B_{R/2}} (|w_x|^2 + |w_x|^m) dx \le \int_{B_{R/2}} [A_\alpha^i(w_x) - A_\alpha^i(0)] w_{x_\alpha}^i dx = I. \tag{4.8}$$

Set

$$D_{R/2} = \{x \in B_{R/2} : |w_x| \ge \chi_1\},$$

for some χ_1 to be fixed later. Then

$$I = \int_{B_{R/2} \setminus D_{R/2}} \{[A_\alpha^i(w_x) - A_\alpha^i(0) - (A_\alpha^i)_{p_\beta^j}(0) w_{x_\beta}^j] w_{x_\alpha}^i + \tag{4.9}$$

$$+ (A_\alpha^i)_{p_\beta^j}(0) w_{x_\alpha}^i w_{x_\beta}^j\} dx + \int_{D_{R/2}} [A_\alpha^i(w_x) - A_\alpha^i(0)] w_{x_\alpha}^i dx =$$

$$= \int_{B_{R/2}\setminus D_{R/2}} \left\{ \int_0^1 [(A^i_\alpha)_{p^j_\beta}(\tau w_x) - (A^i_\alpha)_{p^j_\beta}(0)] d\tau w^i_{x_\alpha} w^j_{x_\alpha} + \right.$$

$$\left. + (A^i_\alpha)_{p^j_\beta}(0) w^i_{x_\alpha} w^j_{x_\beta} \right\} dx + \int_{D_{R/2}} [A^i_\alpha(w_x) - A^i_\alpha(0)] w^i_{x_\alpha} dx = J_1 + J_2.$$

Notice that for $|p| \leq \chi_1$ we have

$$\int_0^1 |(A^i_\alpha)_{p^j_\beta}(\tau p) - (A^i_\alpha)_{p^j_\beta}(0)| d\tau \leq \omega(\chi_1; |p|) \tag{4.10}$$

where $\omega(t; s)$ is the modulus of continuity of $(A^i_\alpha)_p(p)$: we may assume ω to be concave in s and

$$\omega(t; s) \leq 2 \max\{|(A^i_\alpha)_p(p)| : |p| \leq t\}$$

Taking into account (4.10) we have

$$J_1 \leq c' \int_{B_{R/2}} \omega(\chi_1; |w_x|) |w_x|^2 dx + c'' \int_{B_{R/2}} |w_x|^2 dx. \tag{4.11}$$

Using Hölder inequality, (4.2), (4.4), L^p-estimates in theorem 3.2, and Jensen inequality we estimate the first integral on the right hand side of (4.11) by

$$\begin{aligned} c' \int_{B_{R/2}} \omega(\chi_1; |w_x|) |w_x|^2 dx &\leq c(\chi_1) R^n \left(\fint_{B_{R/2}} |w_x|^{2(1+\delta)} dx \right)^{\frac{1}{1+\delta}} \cdot \\ &\cdot \left(\fint_{B_{R/2}} \omega(\chi_1; |w_x|) dx \right)^{\frac{\delta}{1+\delta}} \leq c(\chi_1) R^n \cdot \\ &\cdot \left(\fint_{B_{R/2}} |u_x - p_0|^{2(1+\delta)} dx + \fint_{B_{R/2}} |v_x - p_0|^{2(1+\delta)} dx \right)^{\frac{1}{1+\delta}} \cdot \\ &\cdot \omega \left(\chi_1; \fint_{B_{R/2}} |u_x - p_0| dx + \fint_{B_{R/2}} |v_x - p_0| dx \right) \leq \\ &\leq c(\chi_1, |p_0|) \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m) dx \cdot \\ &\cdot \omega \left(\chi_1; c(|p_0|) \left(\fint_{B_R} |u_x - p_0|^2 dx \right)^{1/2} \right) = \end{aligned} \tag{4.12}$$

$$= \eta_1(x_0, R)\int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx.$$

On the other hand we have

$$(4.13) \qquad |D_{R/2}| \le (\chi_1^2 + \chi_1^m)^{-1}\int_{B_{R/2}} (|w_x|^2 + |w_x|^m)dx \le$$
$$\le c(|p_0|)(\chi_1^2 + \chi_1^m)^{-1}\int_{B_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)dx;$$

and taking into account the condition (G)

$$(4.14) \qquad J_2 \le \int_{D_{R/2}} |A_\alpha^i(w_x) - A_\alpha^i(0)|\,|w_x|dx \le$$
$$\le c\int_{D_{R/2}} |(1 + |w_x|^2 + |w_x|^m)dx \le$$
$$\le c(|p_0|)\left\{|D_{R/2}| + \int_{D_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)dx\right\}.$$

Using L^p-estimates in theorem 3.2 we get from (4.13), (4.14)

$$(4.15)\; J_2 \le c(|p_0|)\left\{|D_{R/2}| + \right.$$
$$\left. + \left(\int_{D_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} |D_{R/2}|^{\frac{\delta}{1/\delta}}\right\} \le$$
$$\le c\left\{|D_{R/2}| + \right.$$
$$\left. + |B_{R/2}|^{\frac{1}{1/\delta}}\left(⨍_{B_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} |D_{R/2}|^{\frac{\delta}{1/\delta}}\right\} \le$$
$$\le c_1(|p_0|)\left\{(\chi_1^2 + \chi_1^m)^{-1}\int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx + \right.$$
$$\left. + (\chi_1^2 + \chi_1^m)^{-\frac{\delta}{1+\delta}}\left(⨍_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx\right)^{\frac{\delta}{1+\delta}}\right. .$$

$$\cdot \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx \Big\} .$$

Fix now an arbitrary $\epsilon \in (0, 1)$ and set

$$\chi_1 = \frac{1 + c_1}{\epsilon}$$

$c_1 = c_1(|p_0|)$ being the constant in (4.15). Then from (4.15) it follows

$$J_2 \leq [\epsilon + \eta_2(x_0, R)] \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx \tag{4.16}$$

where

$$\eta_2(x_0, R) = c_1 \left(\fint_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx \right)^{\frac{\delta}{1+\delta}} .$$

From (4.8), (4.9)-(4.12), (4.16) we get

$$\nu \int_{B_{R/2}} (|w_x|^2 + |w_x|^m)dx \leq [\epsilon + \eta_1(x_0, R) + \tag{4.17}$$
$$+ \eta_2(x_0, R)] \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx + c'' \int_{B_{R/2}} |w_x|^2 dx.$$

Now we estimate the integral

$$\int_{B_{R/2}} |w_x|^2 dx.$$

Set

$$f_\alpha^i = A_\alpha^i(u_x) - A_\alpha^i(p_0) - (A_\alpha^i)_{p_\beta^j}(p_0)(u_{x_\beta}^j - p_0{}_\beta^j) =$$
$$= \int_0^1 [(A_\alpha^i)_{p_\beta^j}(p_0 + \tau(u_x - p_0)) - (A_\alpha^i)_{p_\beta^j}(p_0)](u_{x_\beta}^j - p_0{}_\beta^j)d\tau.$$

Using the Fourier transform and Legendre-Hadamard inequality (2.3), we can write

$$\nu_1 \int_{B_{R/2}} |w_x|^2 dx \leq \int_{B_{R/2}} (A_\alpha^i)_{p_\beta^j}(p_0)w_{x_\alpha}^i w_{x_\beta}^j dx = \int_{B_{R/2(x_0)}} (-f_\alpha^i)w_{x_\alpha}^i dx. \tag{4.18}$$

Set

$$E_{R/2} = \{x \in B_{R/2} : |u_x - p_0| \geq \chi_2\}$$

for some χ_2 to be fixed later. Then

$$\int_{B_{R/2}} (-f^i_\alpha) w^i_\alpha dx = \int_{B_{R/2}\setminus E_{R/2}} \ldots + \int_{E_{R/2}} \ldots = J_3 + J_4. \tag{4.19}$$

For $|p - p_0| \leq \chi_2$ we have

$$\int_0^1 |(A^i_\alpha)_{p^j_\beta}(p_0 + \tau(p - p_0)) - (A^i_\alpha)_{p^j_\beta}(p_0)| d\tau \leq \omega(|p_0| + \chi_2; |p - p_0|) \tag{4.20}$$

where $\omega(t;s)$ has the same meaning as in (4.10). Taking into account (4.20) we obtain

$$J_3 \leq \frac{\nu_1}{4} \int_{B_{R/2}} |w_x|^2 dx + c \int_{B_{R/2}} \omega^2(|p_0| + \chi_2; |u_x - p_0|)\, |u_x - p_0|^2 dx. \tag{4.21}$$

Using L^p-estimates in theorem 3.2 and Jensen inequality we get as before

$$\begin{aligned} c\int_{B_{R/2}} \omega^2(|p_0| + \chi_2; |u_x - p_0|)|u_x - p_0|^2 dx \leq \\ \leq c(\chi_2, |p_0|) \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m) dx \cdot \\ \cdot \omega \left(|p_0| + \chi_2; \left(\fint_{B_R} |u_x - p_0|^2 dx \right)^{1/2} \right) = \\ = \eta_3(x_0, R) \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m) dx. \end{aligned} \tag{4.22}$$

On the other hand we have

$$\int_{E_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m) dx \geq (\chi_2^2 + \chi_2^m)|E_{R/2}|. \tag{4.23}$$

Recalling the condition (G) we have

$$J_4 \leq \int_{E_{R/2}} |A^i_\alpha(u_x) - A^i_\alpha(p_0) - (A^i_\alpha)_{p^j_\beta}(p_0)(u^j_{x_\beta} - p_0{}^j_\beta)|\, |w_x| dx \leq \tag{4.24}$$

$$\leq c(|p_0|)\left\{|E_{R/2}| + \int_{E_{R/2}} (|u_x - p_0|^2 + |u_x - p_0|^m)dx\right\} + \\ + \frac{\nu_1}{4}\int_{B_{R/2}} (|w_x|^2 + \frac{\nu}{c''}|w_x|^m)dx;$$

c'' is the constant in (4.11). From (4.23), (4.24) we get as before

$$J_4 \leq c(|p_0|)\left\{(\chi_2^2 + \chi_2^m)^{-1}\int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx + \right. \\ + (\chi_2^2 + \chi_2^m)^{-\frac{\delta}{1+\delta}}\left(\fint_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx\right)^{\frac{\delta}{1+\delta}} \cdot \\ \left. \cdot \int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx\right\} + \\ + \frac{\nu_1}{4}\int_{B_{R/2}} (|w_x|^2 + \frac{\nu}{c''}|w_x|^m)dx. \tag{4.25}$$

Fix now an arbitrary $\epsilon \in (0, 1)$ and set

$$\chi_2 = \frac{1 + c_2(|p_0|)}{\epsilon}$$

From (4.18), (4.19), (4.21), (4.22) and (4.25) we get at once the estimate

$$\int_{B_{R/2}} |w_x|^2 dx \leq [\epsilon + \eta_4(x_0, R)]\int_{B_R} (|u_x - p_0|^2 + |u_x - p_0|^m)dx + \\ + \frac{\nu}{2c''}\int_{B_{R/2}} |w_x|^m dx, \tag{4.26}$$

where

$$\eta_4(x_0, R) = \eta_3(x_0, R) + c_2\left(\fint_{B_R} |u_x - p_0|^2 + |u_x - p_0|^m)dx\right)^{\frac{\delta}{1+\delta}}.$$

Finally from (4.6), (4.17) and (4.25) it follows that

$$\Phi(x_0, \rho) \leq c(|p_0|)\left[\left(\frac{\rho}{R}\right)^2 + \left(\frac{R}{\rho}\right)^n (\epsilon + \eta(x_0, R))\right]\Phi(x_0, R); \tag{4.27}$$

we have set

$$\eta(x_0, R) = \eta_1(x_0, R) + \eta_2(x_0, R) + \eta_4(x_0, R).$$

Theorem 1.1 follows in a standard way from (4.27); see [7] proposition 3.1 for a similar argument.

5. An example.

Consider the Euler equations of variational integral with integrand

$$F(p) = a|p|^2 + b|p|^n + \sqrt{1 + (\det p)^2}. \tag{5.1}$$

$a > 0$, $b > 0$. We have

$$F_{p^i_\alpha} = 2ap^i_\alpha + nb|p|^{n-2}p^i_\alpha + \frac{\Delta}{\sqrt{1+\Delta^2}}\Delta^i_\alpha,$$

where

$$\Delta = \det p, \quad \Delta^i_\alpha = \frac{\partial \Delta}{\partial p^i_\alpha} \quad \left(\Delta = \sum_{\alpha=1}^{n} p^i_\alpha \Delta^i_\alpha\right),$$

$$F_{p^i_\alpha p^j_\beta} = 2a\delta^j_i\delta^\beta_\alpha + nb(n-2)|p|^{n-4}p^i_\alpha p^j_\beta + nb|p|^{n-2}\delta^j_i\delta^\beta_\alpha + \\ + \frac{\Delta^i_\alpha \Delta^j_\beta}{(1+\Delta^2)^{3/2}} + \frac{\Delta}{\sqrt{1+\Delta^2}}\Delta^{ij}_{\alpha\beta},$$

where

$$\Delta^{ij}_{\alpha\beta} = \frac{\partial^2 \Delta}{\partial p^i_\alpha \partial p^j_\beta}.$$

We have

$$F_{p^i_\alpha p^j_\beta}\xi^i_\alpha\xi^j_\beta = 2a|\xi|^2 + nb|p|^{n-2}|\xi|^2 + n(n-2)b(p\cdot\xi)^2|p|^{n-4} + \\ + \frac{(\Delta^i_\alpha\xi^i_\alpha)\Delta^j_\beta\xi^j_\beta)}{(1+\Delta^2)^{3/2}} + \frac{\Delta}{\sqrt{1+\Delta^2}}\Delta^{ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta \geq 2a|\xi|^2 + \\ + nb|p|^{n-2}|\xi|^2 - c(n)|p|^{n-2}|\xi|^2 \geq \left(2a + \frac{nb}{2}|p|^{n-2}\right)|\xi|^2, \tag{5.2}$$

if b is taken so that $nb \geq 2c(n)$. In particular in the case $n = 2$ we can take $b \geq 4$. In view of (5.2) condition (SE) holds but it is obvious that condition (1.2) (with $m = n$) does not hold because we have only

$$F_{p^i_\alpha p^j_\beta}\xi^i_\alpha\xi^j_\beta \leq c(1+|p|^{2n-2})|\xi|^2, \quad \forall \xi, p \in \mathbb{R}^{nN}.$$

Therefore the considered system is not uniformly elliptic. It is easy to see that this system satisfies all the hypothesis of theorem 1.1 with $m = n$. Hence every stationary point of the integral $\int F(u_x)dx$ with integrand of type (5.1) is partial $C^{1,\alpha}$-regular.

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