ON SOME LINE CONGRUENCES IN $\mathbb{P}^4$

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Consider a line congruence in $\mathbb{P}^4(\mathbb{C})$ or, equivalently, a smooth threefold $V$ in the Grassmannian $G(1,4)$; we say that the congruence has type $(m,n)$ if $V$ is numerically equivalent to $m\Omega(0,4) + n\Omega(1,3)$.

We prove that there are no general, non-degenerate line congruences of type $(m,1)$, for any $m$, and $(m,2)$, for $m \leq 5$.

Further, we give an explicit example of a general line congruence in $\mathbb{P}^4(\mathbb{C})$, which is a generalization of the classical Reye congruence in $\mathbb{P}^3(\mathbb{C})$, and we show that its type is $(15,10)$.

**Introduction.**

By line congruence $C_V$ in $\mathbb{P}^4 = \mathbb{P}^4(\mathbb{C})$ we mean a set of lines in $\mathbb{P}^4$ parametrized by the points of a threefold $V$ in the Grassmannian $G(1,4)$ of lines in $\mathbb{P}^4$. Line congruences $C_V$ with the property that the set of points in $\mathbb{P}^4$ through which infinitely many lines of $C_V$ pass is finite are classically called general. In the cohomology ring of the Grassmannian $G(1,4)$, $V$ can be written as

$$V \cong m\Omega(0,4) + n\Omega(1,3),$$

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where the pair \((m, n)\) is called the type of \(V\).

In the spirit of classifying projective varieties with small projective invariants, we showed in [7] that there are no general line congruences of type \((2, n)\), for any \(n\). Here we give non-existence results for line congruences of type \((m, 1)\), for any \(m\), and \((m, 2)\), for \(m \leq 5\) (Theorems 1.1 and 1.3).

In Section 2, in order to give an explicit example of a general line congruence in \(\mathbb{P}^4\), we exhibit a line congruence which is a generalization of the classical Reye congruence in \(\mathbb{P}^3\) (see also [6] and [11]) and, in Proposition 2.3, we show that it has type \((15, 10)\).

We are grateful to the Referee for having suggested us a shorter proof of this fact.

1. Line congruences with small characters.

Throughout this work we denote by:

- \(\mathbb{P}^4\) the four-dimensional complex projective space;
- \(G(1, 4) \subseteq \mathbb{P}^9\) the Grassmannian of lines in \(\mathbb{P}^4\), which has dimension 6;
- \(\approx\) homological equivalence;
- \(A \cdot B\), or \(AB\) if no confusion arises, both the intersection cycle and its degree;
- pencil, net, web a linear system of dimension one, two, three respectively;
- \(\langle A, B \rangle\) the linear span of \(A\) and \(B\).

Let \(A_0 \subset A_1 \subset \mathbb{P}^4\) be a flag of linear subspaces of dimension \(a_i = \dim A_i\) (\(i = 0, 1\)) and denote by \(\Omega(A_0, A_1)\) the set of all lines \(\ell\) in \(\mathbb{P}^4\) such that \(\dim(\ell \cap A_i) \geq i\). Using the same notation as in [10], we will denote by \(\Omega(a_0, a_1)\) the Schubert cycle of type \((a_0, a_1)\), i.e. the corresponding cohomology class \(\Omega(a_0, a_1) \in H^*(G(1, 4), \mathbb{Z})\).

Let \(V\) be a smooth threefold in \(G(1, 4)\) and denote again by \(V\) the cohomology class corresponding to \(V\) in \(H^*(G(1, 4), \mathbb{Z})\). Since the Schubert cycles of dimension \(k\) form a basis of the cohomology group
$H^{2(3-k)}(G(1,4), \mathbb{Z})$ and \( \dim \Omega(A_0, A_1) = a_0 + a_1 - 1 \), for suitable integers \( m \) and \( n \) we have

\[
V \cong m\Omega(0, 4) + n\Omega(1, 3).
\]

As

\[
m = V \cdot \Omega(0, 4) \quad \text{and} \quad n = V \cdot \Omega(1, 3),
\]

\( m \) is the number of lines of \( C_V \) passing through a generic point of \( \mathbb{P}^4 \) and \( n \) is the number of lines of \( C_V \) intersecting a generic line and contained in a given generic hyperplane through it; the integers \( m \), \( n \) and the pair \((m, n)\) are respectively called order, class and type of \( V \). The degree of \( V \) is

\[
d = m + 2n.
\]

For any point \( L \in G(1,4) \), we will denote by \( \ell \) the corresponding line in \( \mathbb{P}^4 \) and vice versa. The family \( C_V \) of lines \( \ell \) parametrized by the points \( L \in V \) will be called line congruence associated to \( V \). Conversely, for a given algebraic three-dimensional family \( C \) of lines in \( \mathbb{P}^4 \), we will denote by \( V_C \), the corresponding subvariety of \( G(1,4) \), which will also be called line congruence, provided that it is smooth.

If \( V \subseteq \mathbb{P}^5 \) is non-degenerate, we will say that the corresponding line congruence \( C_V \) is non-degenerate as well.

A point \( P \in \mathbb{P}^4 \) is said to be fundamental for \( V \) if infinitely many lines \( \ell \in C_V \) pass through \( P \). The set of fundamental points for \( V \) is called the fundamental locus of \( V \). Line congruences with finite fundamental locus are said to be general.

From now on, we will suppose that \( V \) is general and that \( m \geq 2 \). For a fixed generic two-dimensional linear subspace \( \pi \subset \mathbb{P}^4 \), consider the intersection \( V_\pi \) of \( V \) with the Schubert variety \( \Omega(\pi, \mathbb{P}^4) \), hyperplane section of \( G(1,4) \). Since the singular locus of \( \Omega(\pi, \mathbb{P}^4) \) is \( \Omega(\lambda, \pi) \), where \( \lambda \) denotes a line in \( \pi \), and \( V \cdot \Omega (1, 2) = 0 \), it follows, by transversality and by generality of \( \pi \), that \( V_\pi \) is a non-singular hyperplane section of \( V \). We can define a morphism

\[
\psi : V_\pi \to \mathbb{P}^1,
\]

as follows: identify \( \mathbb{P}^1 \) with the pencil of hyperplanes containing \( \pi \) and set \( \psi(L) = \langle \ell, \pi \rangle \). Clearly \( \psi \) is everywhere defined and its fibers
are curves of degree \( n \), since

\[
\psi^{-1}(\langle \ell, \pi \rangle) = V \cdot \Omega(\pi, (\ell, \pi)) \approx n\Omega(P, \pi).
\]

**Theorem 1.1.** In \( \mathbb{P}^4 \) there are no general line congruences of type \((m, 1)\), \( m \geq 2 \).

**Proof.** Since \( n = 1 \), \( \psi \) is the ruling of a rational scroll of degree \( d > 2 \), so that \( V \) is a scroll of planes over \( \mathbb{P}^1 \) (cf. [9, Proposition 2.3]). The classification of 3-scrolls in \( G(1, 4) \) (cf. [2]) implies \( m \leq 2 \), but, if \( m = 2 \), \( V \) cannot be general (cf. [7, Theorem 3.2]).

Let \( V \) be non-degenerate and \( g_s \) its sectional genus, i.e. the genus of the curve \( C_s \), section of \( V \) with two generic hyperplanes in \( \mathbb{P}^9 \). Since \( C_s \) is a non-degenerate curve of degree \( d = m + 2n \) in \( \mathbb{P}^7 \), Castelnuovo inequality gives

\[
(1.1) \quad g_s \leq 3a(a - 1) + a\varepsilon,
\]

where \( a = [(d - 1)/6] \) and \( d - 1 = 6a + \varepsilon \).

**Lemma 1.2.** In \( \mathbb{P}^4 \) there are no general non-degenerate line congruences with class \( n = 2 \), order \( m \geq 2 \) and sectional genus \( g_s \leq 2 \).

**Proof.** Since \( n = 2 \), \( V_x \) is a conic bundle over \( \mathbb{P}^1 \), hence, by adjunction as in [9, Proposition 1.11, ii], \( V \) is a quadric bundle over \( \mathbb{P}^1 \). Let \( k \) be the number of singular fibers of \( V \); the Betti numbers of \( V \), which can be computed via the Mayer-Vietoris argument, are

\[
b_0 = b_6 = 1, \quad b_1 = b_5 = 0, \quad b_2 = b_4 = 3, \quad b_3 = k.
\]

We proceed by checking the list of threefolds with small sectional genus in [9].

**Case** \( g_s = 0 \). By [9, Proposition 2.3], \( V \) is one of the following:

- case 0.1) a hyperplane in \( \mathbb{P}^4 \);
- case 0.2) a hyperquadric in \( \mathbb{P}^4 \);
- case 0.3) a scroll of planes over a rational curve.

**Case** \( g_s = 1 \). By [9, Proposition 2.6], \( V \) is one of the following:
case 1.1) a Fano threefold;

case 1.2) a scroll of planes over an elliptic curve.

Case \( g_s = 2 \). By (1.1) \( d \geq 9 \), so [9, Corollary 3.3 and Theorem 3.4] imply that \( V \) is one of the following:

case 2.1) the Segre embedding of \( \mathbb{P}^1 \times \mathcal{F}_1 \), where \( \mathcal{F}_1 \) denotes the blow-up of \( \mathbb{P}^2 \) at one point;

case 2.2) a scroll of planes over a curve of genus 2.

Cases 0.1 and 0.2 are obviously ruled out.

Cases 0.3, 1.2 and 2.2 do not occur since all these threefolds have \( b_2 = 2 \).

In case 1.1, since by (1.1) we have \( d \geq 8 \), by [8] the only possibility is \( d = 8 \) and \( V \) is \( \mathbb{P}^3 \) embedded into \( \mathbb{P}^9 \) via the Veronese embedding. This gives a contradiction since \( b_2(\mathbb{P}^3) = 1 \).

Also case 2.1 is ruled out since there are no Segre embeddings of \( \mathbb{P}^1 \times \mathcal{F}_e \) in \( G(1,4) \) for \( e > 0 \) (cf. [3]).

\[ \square \]

**THEOREM 1.3.** In \( \mathbb{P}^4 \) there are no general non-degenerate line congruences of type \((m,2)\) with \(2 \leq m \leq 5\).

\[ \text{Proof.} \text{ In our hypotheses, (1.1) implies } g_s \leq 2. \text{ Thesis follows from Lemma 1.2.} \]

\[ \square \]

2. **Reye congruence in \( \mathbb{P}^4 \).**

Let \( S \) be a 4-dimensional linear system of quadrics in \( \mathbb{P}^4 \) such that:

1) \( S \) is base-point-free;

2) if a line \( \ell \subset \mathbb{P}^4 \) is singular for a quadric of \( S \), then in \( S \) there are no pencils of quadrics containing \( \ell \).

The family \( \mathcal{C} \) of lines \( \ell \subset \mathbb{P}^4 \) contained in a net in \( S \) will be called **Reye congruence** in \( \mathbb{P}^4 \) and the corresponding subvariety \( V = V_\mathcal{C} \), in \( G(1,4) \) can be described as follows.
Two points $P, P'$ in $\mathbb{P}^4$ are said to be conjugated with respect to a quadric $Q$ if $Q(p, p') = 0$, where, here and in the following, we denote by the same symbol a quadric and a symmetric bilinear form defining it and by $p$ a representative vector in $\mathbb{C}^5$ for $P \in \mathbb{P}^4$. Define

$$X = \{(P, P') \in \mathbb{P}^4 \times \mathbb{P}^4 | \text{ } P, P' \text{ are conjugated with respect to all the quadrics in } S\};$$

notice that $X$ is three-dimensional, since it is the complete intersection of five sufficiently general hypersurfaces of bidegree $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$. Indeed the equations for $X$ are

(2.1) $$Q_i(p, p') = 0, \quad i = 1, \ldots, 5,$$

where $Q_1, \ldots, Q_5$ form a basis for $S$. If $(P, P') \in X$, then $P \neq P'$, for otherwise $P$ would be a base point of $S$; hence the involution

$$\iota : \mathbb{P}^4 \times \mathbb{P}^4 \rightarrow \mathbb{P}^4 \times \mathbb{P}^4$$

defined by $\iota(P, P') = (P', P)$ is fixed-point-free on $X$.

Let $U$ be the complement of the diagonal in $\mathbb{P}^4 \times \mathbb{P}^4$ and consider the map

(2.2) $$\phi : U \rightarrow G(1, 4)$$

defined by $\phi((P, P')) = L$, where $L$ denotes the point corresponding to the line $\ell = \langle P, P' \rangle$. As in the case of classical Reye congruences in $\mathbb{P}^3$ (cf. [4]), its restriction $\varphi$ to $X$ induces an isomorphism between the quotient of $X$ under the involution $\iota$ and $\mathbb{V}$.

**PROPOSITION 2.1.** $X$ and $\mathbb{V}$ are smooth threefolds$^{(1)}$.

**Proof.** Since $\iota$ is fixed-point-free on $X$, it suffices to prove the smoothness of $X$. Denote by $e_i$ ($i = 1, \ldots, 5$) the standard basis of $\mathbb{C}^5$; the Jacobian matrix of $X$ is

$$J(X)(p, p') = \begin{bmatrix}
Q_1(e_1, p') & \ldots & Q_1(e_5, p') & Q_1(p, e_1) & \ldots & Q_1(p, e_5) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
Q_5(e_1, p') & \ldots & Q_5(e_5, p') & Q_5(p, e_1) & \ldots & Q_5(p, e_5)
\end{bmatrix}.$$ 

$^{(1)}$ Since $\mathbb{V}$ is a threefold, this generalization of the Reye construction is completely different from that given in [5].
Its rank at a point \((P, P')\) is less than 5 if and only if the line \(\ell = \langle P, P' \rangle\) is singular for a quadric \(Q \in S\). But a line in \(C\) cannot be singular for any quadric \(Q \in S\), due to assumption 2.

**Proposition 2.2.** \(V\) is fundamental-point-free, hence general.

**Proof.** By assumption 1, for any \(P \in \mathbb{P}^4\), the quadrics of \(S\) through \(P\) form a web \(\mathcal{W}_P\) which has a finite number of base points, including \(P\) itself. Let \(\ell \in C\) be a line through \(P\); the quadrics of \(S\) containing \(\ell\) form a net \(\mathcal{N}_\ell \subset \mathcal{W}_P\). If now \(Q^*\) is a quadric in \(\mathcal{W}_P \setminus \mathcal{N}_\ell\), \(\ell\) intersects \(Q^*\) in \(P\) and in another base point of \(\mathcal{W}_P\). Hence \(\ell\) is a line joining \(P\) with one of the other base points of \(\mathcal{W}_P\).

**Proposition 2.3.** \(V\) has type \((15, 10)\).

**Proof.** Since for generic \(P\) the web \(\mathcal{W}_P\) has 16 distinct base points, it follows from the proof of Proposition 2.2 that \(m = 15\).

In order to compute \(n\), consider the natural homomorphisms of cohomology rings

\[
j^* : H^*(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Z}) \to H^*(U, \mathbb{Z})
\]

and

\[
\phi^* : H^*(G(1, 4), \mathbb{Z}) \to H^*(U, \mathbb{Z})
\]

induced by the inclusion \(j : U \to \mathbb{P}^4 \times \mathbb{P}^4\) and by the map \(\phi\) defined in (2.2) respectively. Denote by \(A\) and \(B\) the generators of \(H^*(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Z})\) and again by \(X\) the class of \(X\); by (2.1) we have

\[
(2.3) \quad X = (A + B)^5.
\]

Observe that

\[
(2.4) \quad \phi^*(\Omega(1, 3)) = j^*(A^2B + AB^2).
\]

In order to show it, notice that the Zariski closure \(Z\) of \(\phi^*(\Omega(1, 3))\) in \(\mathbb{P}^4 \times \mathbb{P}^4\) is five-dimensional, hence

\[
Z \approx xA^3 + yA^2B + zAB^2 + tB^3,
\]
where
\[ x = Z \cdot AB^4 = Z \cdot A^4B = t, \quad y = Z \cdot A^2B^3 = Z \cdot A^3B^2 = z. \]

An easy geometrical argument shows that \( x = t = 0 \) and \( y = z = 1 \).

Projection formula yields
\[ \phi_*(\phi^*(\Omega(1, 3)) \cdot j^*(X)) = \Omega(1, 3) \cdot \phi_*(j^*(X)). \]

Recalling that \( \varphi \) is a double cover and \( X \subseteq U \), by (2.3) and (2.4) we obtain
\[ \phi_*(j^*((A^2B + AB^2)(A + B)^5)) = 2\Omega(1, 3) \cdot V; \]
computing degrees we get \( 20 = 2n \). \( \blacksquare \)

Remark 2.4. The same technique used to compute \( n \) could be used to compute \( m \) as well. Indeed, substituting
\[ \phi^*(\Omega(0, 4)) = j^*(A^3 + A^2B + AB^2 + B^3) \]
for (2.4), we get \( 30 = 2m \).

Moreover, the description of \( X \) as a complete intersection in \( \mathbb{P}^4 \times \mathbb{P}^4 \) allows to compute its Chern classes, from which the Chern classes of \( V \) can be easily deduced. Thus it can be checked that our computations for \( m \) and \( n \) agree with the general formula for threefolds in \( G(1, 4) \) (cf. [1]).

REFERENCES


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