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ON SOME LINE CONGRUENCES IN $\mathbb{P}^4$

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Consider a line congruence in $\mathbb{P}^4(\mathbb{C})$ or, equivalently, a smooth threefold V in the Grassmannian $G(1,4)$; we say that the congruence has type (m,n) if V is numerically equivalent to $m\Omega(0,4)+n\Omega(1,3)$.

We prove that there are no general, non-degenerate line congruences of type $(m,1)$, for any m, and $(m,2)$, for $m \leq 5$.

Further, we give an explicit example of a general line congruence in $\mathbb{P}^4(\mathbb{C})$, which is a generalization of the classical Reye congruence in $\mathbb{P}^3(\mathbb{C})$, and we show that its type is (15,10).

Introduction.

By line congruence C_V in $\mathbb{P}^4 = \mathbb{P}^4(\mathbb{C})$ we mean a set of lines in $\mathbb{P}^4$ parametrized by the points of a threefold V in the Grassmannian $G(1,4)$ of lines in $\mathbb{P}^4$. Line congruences C_V with the property that the set of points in $\mathbb{P}^4$ through which infinitely many lines of C_V pass is finite are classically called general. In the cohomology ring of the Grassmannian $G(1,4)$, V can be written as

$$V \approx m\Omega(0,4) + n\Omega(1,3),$$

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where the pair (m, n) is called the type of V.

In the spirit of classifying projective varieties with small projective invariants, we showed in [7] that there are no general line congruences of type $(2, n)$, for any n. Here we give non-existence results for line congruences of type $(m, 1)$, for any m, and $(m, 2)$, for $m \leq 5$ (Theorems 1.1 and 1.3).

In Section 2, in order to give an explicit example of a general line congruence in $\mathbb{P}^4$, we exhibit a line congruence which is a generalization of the classical Reye congruence in $\mathbb{P}^3$ (see also [6] and [11]) and, in Proposition 2.3, we show that it has type (15,10). We are grateful to the Referee for having suggested us a shorter proof of this fact.

1. Line congruences with small characters.

Throughout this work we denote by:

- $\mathbb{P}^4$ the four-dimensional complex projective space;
- $G(1,4) \subseteq \mathbb{P}^9$ the Grassmannian of lines in $\mathbb{P}^4$, which has dimension 6;
- $\approx$ homological equivalence;
- $A \cdot B$, or AB if no confusion arises, both the intersection cycle and its degree;
- pencil, net, web a linear system of dimension one, two, three respectively;
- $\langle A, B \rangle$ the linear span of A and B.

Let $A_0 \subset A_1 \subseteq \mathbb{P}^4$ be a flag of linear subspaces of dimension $a_i = \dim A_i$ $(i = 0, 1)$ and denote by $\Omega(A_0, A_1)$ the set of all lines ℓ in $\mathbb{P}^4$ such that $\dim(\ell \cap A_i) \geq i$. Using the same notation as in [10], we will denote by $\Omega(a_0, a_1)$ the Schubert cycle of type (a_0, a_1), i.e. the corresponding cohomology class $\Omega(a_0, a_1) \in H^*(G(1,4), \mathbb{Z})$.

Let V be a smooth threefold in $G(1,4)$ and denote again by V the cohomology class corresponding to V in $H^*(G(1,4), \mathbb{Z})$. Since the Schubert cycles of dimension k form a basis of the cohomology group

$H^{2(3-k)}(G(1,4),\mathbb{Z})$ and dim $\Omega(A_0,A_1)=a_0+a_1-1$, for suitable integers m and n we have

$$V \approx m\Omega(0,4)+n\Omega(1,3).$$

As

$$m=V\cdot\Omega(0,4) \quad \text{and} \quad n=V\cdot\Omega(1,3),$$

m is the number of lines of C_V passing through a generic point of $\mathbb{P}^4$ and n is the number of lines of C_V intersecting a generic line and contained in a given generic hyperplane through it; the integers m, n and the pair (m,n) are respectively called *order, class* and *type* of V. The degree of V is

$$d=m+2n.$$

For any point $L\in G(1,4)$, we will denote by ℓ the corresponding line in $\mathbb{P}^4$ and viceversa. The family C_V of lines ℓ parametrized by the points $L\in V$ will be called *line congruence* associated to V. Conversely, for a given algebraic three-dimensional family C of lines in $\mathbb{P}^4$, we will denote by V_C, the corresponding subvariety of $G(1,4)$, which will also be called line congruence, provided that it is smooth.

If $V\subseteq\mathbb{P}^9$ is non-degenerate, we will say that the corresponding line congruence C_V is non-degenerate as well.

A point $P\in\mathbb{P}^4$ is said to be *fundamental* for V if infinitely many lines $\ell\in C_V$ pass through P. The set of fundamental points for V is called the *fundamental locus* of V. Line congruences with finite fundamental locus are said to be *general.*

From now on, we will suppose that V is general and that $m\geq 2$. For a fixed generic two-dimensional linear subspace $\pi\subset\mathbb{P}^4$, consider the intersection V_π of V with the Schubert variety $\Omega(\pi,\mathbb{P}^4)$, hyperplane section of $G(1,4)$. Since the singular locus of $\Omega(\pi,\mathbb{P}^4)$ is $\Omega(\lambda,\pi)$, where λ denotes a line in π, and $V\cdot\Omega(1,2)=0$, it follows, by transversality and by generality of π, that V_π is a non-singular hyperplane section of V. We can define a morphism

$$\psi: V_\pi\to\mathbb{P}^1,$$

as follows: identify $\mathbb{P}^1$ with the pencil of hyperplanes containing π and set $\psi(L)=\langle\ell,\pi\rangle$. Clearly ψ is everywhere defined and its fibers are curves of degree n, since

$$\psi^{-1}(\langle \ell, \pi \rangle) = V \cdot \Omega(\pi, \langle \ell, \pi \rangle) \approx n\Omega(P, \pi).$$

THEOREM 1.1. *In* $\mathbb{P}^4$ *there are no general line congruences of type* $(m, 1)$, $m \geq 2$.

Proof. Since $n = 1$, ψ is the ruling of a rational scroll of degree $d > 2$, so that V is a scroll of planes over $\mathbb{P}^1$ (cf. [9, Proposition 2.3]). The classification of 3-scrolls in $G(1,4)$ (cf. [2]) implies $m \leq 2$, but, if $m = 2$, V cannot be general (cf. [7, Theorem 3.2]). ■

Let V be non-degenerate and g_s its sectional genus, i.e. the genus of the curve C_s, section of V with two generic hyperplanes in $\mathbb{P}^9$. Since C_s is a non-degenerate curve of degree $d = m + 2n$ in $\mathbb{P}^7$, Castelnuovo inequality gives

$$g_s \leq 3a(a-1) + a\varepsilon, \tag{1.1}$$

where $a = [(d-1)/6]$ and $d - 1 = 6a + \varepsilon$.

LEMMA 1.2. *In* $\mathbb{P}^4$ *there are no general non-degenerate line congruences with class* $n = 2$, *order* $m \geq 2$ *and sectional genus* $g_s \leq 2$.

Proof. Since $n = 2$, V_π is a conic bundle over $\mathbb{P}^1$, hence, by adjunction as in [9, Proposition 1.11, ii], V is a quadric bundle over $\mathbb{P}^1$. Let k be the number of singular fibers of V; the Betti numbers of V, which can be computed via the Mayer-Vietoris argument, are

$$b_0 = b_6 = 1, \; b_1 = b_5 = 0, \; b_2 = b_4 = 3, \; b_3 = k.$$

We proceed by checking the list of threefolds with small sectional genus in [9].

Case $g_s = 0$. By [9, Proposition 2.3], V is one of the following:

case 0.1) a hyperplane in $\mathbb{P}^4$;

case 0.2) a hyperquadric in $\mathbb{P}^4$;

case 0.3) a scroll of planes over a rational curve.

Case $g_s = 1$. By [9, Proposition 2.6], V is one of the following:

case 1.1) a Fano threefold;

case 1.2) a scroll of planes over an elliptic curve.

Case $g_s = 2$. By (1.1) $d \geq 9$, so [9, Corollary 3.3 and Theorem 3.4] imply that V is one of the following:

case 2.1) the Segre embedding of $\mathbb{P}^1 \times \mathbb{F}_1$, where $\mathbb{F}_1$ denotes the blow-up of $\mathbb{P}^2$ at one point;

case 2.2) a scroll of planes over a curve of genus 2.

Cases 0.1 and 0.2 are obviously ruled out.

Cases 0.3, 1.2 and 2.2 do not occur since all these threefolds have $b_2 = 2$.

In case 1.1, since by (1.1) we have $d \geq 8$, by [8] the only possibility is $d = 8$ and V is $\mathbb{P}^3$ embedded into $\mathbb{P}^9$ via the Veronese embedding. This gives a contradiction since $b_2(\mathbb{P}^3) = 1$.

Also case 2.1 is ruled out since there are no Segre embeddings of $\mathbb{P}^1 \times \mathbb{F}_e$ in $G(1.4)$ for $e > 0$ (cf. [3]). ∎

THEOREM 1.3. *In* $\mathbb{P}^4$ *there are no general non-degenerate line congruences of type* $(m, 2)$ *with* $2 \leq m \leq 5$.

Proof. In our hypotheses, (1.1) implies $g_s \leq 2$. Thesis follows from Lemma 1.2. ∎

2. Reye congruence in $\mathbb{P}^4$.

Let S be a 4-dimensional linear system of quadrics in $\mathbb{P}^4$ such that:

1) S is base-point-free;

2) if a line $\ell \subset \mathbb{P}^4$ is singular for a quadric of S, then in S there are no pencils of quadrics containing ℓ.

The family C of lines $\ell \subset \mathbb{P}^4$ contained in a net in S will be called *Reye congruence* in $\mathbb{P}^4$ and the corresponding subvariety $V = V_C$, in $G(1,4)$ can be described as follows.

Two points P, P' in $\mathbb{P}^4$ are said to be conjugated with respect to a quadric Q if $Q(\mathbf{p}, \mathbf{p}') = 0$, where, here and in the following, we denote by the same symbol a quadric and a symmetric bilinear form defining it and by $\mathbf{p}$ a representative vector in $\mathbf{C}^5$ for $P \in \mathbb{P}^4$. Define

$$X = \{(P, P') \in \mathbb{P}^4 \times \mathbb{P}^4 \mid P, P' \text{ are conjugated with respect to all the quadrics in } S\};$$

notice that X is three-dimensional, since it is the complete intersection of five sufficiently general hypersurfaces of bidegree (1,1) in $\mathbb{P}^4 \times \mathbb{P}^4$. Indeed the equations for X are

$$Q_i(\mathbf{p}, \mathbf{p}') = 0, \quad i = 1, \ldots, 5, \tag{2.1}$$

where $Q_1, \ldots, Q_5$ form a basis for S. If $(P, P') \in X$, then $P \neq P'$, for otherwise P would be a base point of S; hence the involution

$$\imath : \mathbb{P}^4 \times \mathbb{P}^4 \to \mathbb{P}^4 \times \mathbb{P}^4$$

defined by $\imath(P, P') = (P', P)$ is fixed-point-free on X.

Let U be the complement of the diagonal in $\mathbb{P}^4 \times \mathbb{P}^4$ and consider the map

$$\phi : U \to G(1, 4) \tag{2.2}$$

defined by $\phi((P, P')) = L$, where L denotes the point corresponding to the line $\ell = \langle P, P' \rangle$. As in the case of classical Reye congruences in $\mathbb{P}^3$ (cf. [4]), its restriction φ to X induces an isomorphism between the quotient of X under the involution $\imath$ and V.

PROPOSITION 2.1. *X and V are smooth threefolds*(¹).

Proof. Since $\imath$ is fixed-point-free on X, it suffices to prove the smoothness of X. Denote by $\mathbf{e}_i$ $(i = 1, \ldots, 5)$ the standard basis of $\mathbf{C}^5$; the Jacobian matrix of X is

$$J(X)(\mathbf{p}, \mathbf{p}') = \begin{bmatrix} Q_1(\mathbf{e}_1, \mathbf{p}') & \ldots & Q_1(\mathbf{e}_5, \mathbf{p}') & Q_1(\mathbf{p}, \mathbf{e}_1) & \ldots & Q_1(\mathbf{p}, \mathbf{e}_5) \\ \cdots\cdots\cdots & \ldots & \cdots\cdots\cdots & \cdots\cdots\cdots & \ldots & \cdots\cdots\cdots \\ Q_5(\mathbf{e}_1, \mathbf{p}') & \ldots & Q_5(\mathbf{e}_5, \mathbf{p}') & Q_5(\mathbf{p}, \mathbf{e}_1) & \ldots & Q_5(\mathbf{p}, \mathbf{e}_5) \end{bmatrix}.$$

(¹) Since V is a threefold, this generalization of the Reye construction is completely different from that given in [5].

Its rank at a point (P, P') is less than 5 if and only if the line $\ell = \langle P, P'\rangle$ is singular for a quadric $Q \in \mathcal{S}$. But a line in $\mathcal{C}$ cannot be singular for any quadric $Q \in \mathcal{S}$, due to assumption 2. ■

PROPOSITION 2.2. *V is fundamental-point-free, hence general.*

Proof. By assumption 1, for any $P \in \mathbb{P}^4$, the quadrics of $\mathcal{S}$ through P form a web $\mathcal{W}_P$ which has a finite number of base points, including P itself. Let $\ell \in \mathcal{C}$ be a line through P; the quadrics of $\mathcal{S}$ containing ℓ form a net $\mathcal{N}_\ell \subset \mathcal{W}_P$. If now Q^* is a quadric in $\mathcal{W}_P \setminus \mathcal{N}_\ell$, ℓ intersects Q^* in P and in another base point of $\mathcal{W}_P$. Hence ℓ is a line joining P with one of the other base points of $\mathcal{W}_P$. ■

PROPOSITION 2.3. *V has type* (15,10).

Proof. Since for generic P the web $\mathcal{W}_P$ has 16 distinct base points, it follows from the proof of Proposition 2.2 that $m = 15$.

In order to compute n, consider the natural homomorphisms of cohomology rings

$$j^* : H^*(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Z}) \to H^*(U, \mathbb{Z})$$

and

$$\phi^* : H^*(G(1,4), \mathbb{Z}) \to H^*(U, \mathbb{Z})$$

induced by the inclusion $j : U \to \mathbb{P}^4 \times \mathbb{P}^4$ and by the map ϕ defined in (2.2) respectively. Denote by A and B the generators of $H^*(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Z})$ and again by X the class of X; by (2.1) we have

$$X = (A + B)^5. \tag{2.3}$$

Observe that

$$\phi^*(\Omega(1,3)) = j^*(A^2B + AB^2). \tag{2.4}$$

In order to show it, notice that the Zariski closure Z of $\phi^*(\Omega(1,3))$ in $\mathbb{P}^4 \times \mathbb{P}^4$ is five-dimensional, hence

$$Z \approx xA^3 + yA^2B + zAB^2 + tB^3,$$

where

$$x = Z \cdot AB^4 = Z \cdot A^4B = t, \quad y = Z \cdot A^2B^3 = Z \cdot A^3B^2 = z.$$

An easy geometrical argument shows that $x = t = 0$ and $y = z = 1$.

Projection formula yields

$$\phi_*(\phi^*(\Omega(1,3)) \cdot j^*(X)) = \Omega(1,3) \cdot \phi_*(j^*(X)).$$

Recalling that φ is a double cover and $X \subseteq U$, by (2.3) and (2.4) we obtain

$$\phi_*(j^*((A^2B + AB^2)(A+B)^5)) = 2\Omega(1,3) \cdot V;$$

computing degrees we get $20 = 2n$. ■

Remark 2.4. The same technique used to compute n could be used to compute m as well. Indeed, substituting

$$\phi^*(\Omega(0,4)) = j^*(A^3 + A^2B + AB^2 + B^3)$$

for (2.4), we get $30 = 2m$.

Moreover, the description of X as a complete intersection in $\mathbb{P}^4 \times \mathbb{P}^4$ allows to compute its Chern classes, from which the Chern classes of V can be easily deduced. Thus it can be checked that our computations for m and n agree with the general formula for threefolds in $G(1,4)$ (cf. [1]).

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