LE MATEMATICHE
Vol. LXIII (2008) – Fasc. II, pp. 101–112

EXISTENCE OF SOLUTIONS FOR SOME DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

ALBO CARLOS CAVALHEIRO

In this paper we are interested in the existence of solutions for Dirichlet problem associated to the degenerate quasilinear elliptic equations

$$-\sum_{j=1}^{n} D_j[\omega(x)\mathscr{A}_j(x,u,\nabla u)] + \omega(x)g(x,u(x),\nabla u(x)) = f_0 - \sum_{j=1}^{n} D_j f_j,$$

on Ω, in the setting of the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$ (see Definition 2.3) for the Dirichlet problem

$$(P)\begin{cases} Lu(x) & = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \ \text{ on } \ \Omega \\ u(x) & = 0, \ \text{ on } \ \partial\Omega \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\sum_{j=1}^{n} D_j\left[\omega(x)\mathscr{A}_j(x,u(x),\nabla u(x))\right] + \omega(x)g(x,u(x)\nabla u(x)) \qquad (1)$$

Entrato in redazione: 11 novembre 2008

AMS 2000 Subject Classification: 37J70, 35J60.

Keywords: degenerate quasilinear equations, weighted Sobolev spaces.

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω is a weight function, and the functions $\mathscr{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1,...,n)$ and $g : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfying the following assumptions:

(H1) $x\mapsto\mathscr{A}_j(x,\eta,\xi)$ is measurable in Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$
$(\eta,\xi)\mapsto\mathscr{A}_j(x,\eta,\xi)$ is continuous in $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H2) $[\mathscr{A}(x,\eta,\xi)-\mathscr{A}(x,\eta',\xi')].(\xi-\xi')\geq 0$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where

$$\mathscr{A}(x,\eta,\xi) = (\mathscr{A}_1(x,\eta,\xi),...,\mathscr{A}_n(x,\eta,\xi)).$$

(H3) $\mathscr{A}(x,\eta,\xi).\xi\geq\lambda|\xi|^p+\Lambda|\eta|^p-h_1(x)|\eta|-h_2(x)$, with $1<p<\infty$, where λ and Λ are positive constants, $h_1\in L^{p'}(\Omega,\omega)$ and $h_2\in L^1(\Omega,\omega)$ (we denote by p' the real number such that $1/p+1/p'=1$).
(H4) $|\mathscr{A}(x,\eta,\xi)|\leq K_1(x)+h_3(x)|\eta|^{p/p'}+h_4(x)|\xi|^{p/p'}$, where K_1, h_3 and h_4 are positive functions, with h_3 and $h_4\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$.
(H5) $x\mapsto g(x,\eta,\xi)$ is measurable in Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$
$(\eta,\xi)\mapsto g(x,\eta,\xi)$ is continuous in $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H6) $|g(x,\eta,\xi)|\leq K_2(x)+h_5(x)|\eta|^{p/p'}+h_6(x)|\xi|^{p/p'}$, where K_2, h_5 and h_6 are positive functions, with $h_5,h_6\in L^\infty(\Omega)$ and $K_2\in L^{p'}(\Omega,\omega)$.
(H7) $g(x,\eta,\xi)\eta\geq 0$, for all $\eta\in\mathbb{R}$.
(H8) $(g(x,\eta,\xi)-g(x,\eta',\xi'))(\eta-\eta')>0$, whenever $\eta,\eta'\in\mathbb{R}$, $\eta\neq\eta'$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x)>0$ for a.e. $x\in\mathbb{R}^n$. Every weight ω rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E)=\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4],[5] and [8]).

A class of weights, which is particulary well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [10]). These classes have found many usefull applications in harmonic analysis (see [11]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

Equations like (1) have been studied by many authors in the non-degenerate case (i.e. with $\omega(x)\equiv 1$) (see e.g. [1] and the references therein). The degenerate case with different conditions haven been studied by many authors. In [2] Drabek, Kufner and Mustonen proved that under certain condition, the Dirichlet problem associated with the equation $-\text{div}(a(x,u,\nabla u)) = h$, $h \in [W_0^{1,p}(\Omega,\omega)]^*$ has at least one solution $u \in W_0^{1,p}(\Omega,\omega)$.

The following theorem will be proved in section 3.

Theorem 1.1 Assume the conditions (H1)-(H8). If $\omega \in A_p$, with $1 < p < \infty$, and $f_j/\omega \in L^{p'}(\Omega,\omega)$ $(j = 0,1,...,n)$ then problem (P) has a solution $u \in W_0^{1,p}(\Omega,\omega)$.

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 1.2 Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$.
Proof. See Theorem 26.A in [13]. □

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \leq C_{p,\omega}$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [7],[8] or [11] for more informations about A_p-weights). The weight ω satisfies the doubling condition if $\omega(2B) \leq C\omega(B)$, for all balls $B \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. If $\omega \in A_p$, then ω is doubling (see corollary 15.7 in [8]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see corollary 4.4, chapter IX in [11]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\frac{\omega(E)}{\omega(B)}$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [8]). Therefore, if $\omega(E) = 0$ then $|E| = 0$.

Definition 2.1. Let ω be a weight, and let $\Omega\subset\mathbb{R}^n$ be open. For $0 < p < \infty$, we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p\,\omega(x)\,dx\right)^{1/p} < \infty.$$

Remark 2.2. If $\omega\in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have

$$L^p(\Omega,\omega)\subset L^1_{\text{loc}}(\Omega)$$

for every open set Ω (see Remark 1.2.4 in [12]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$. □

Definition 2.3. Let $\Omega\subset\mathbb{R}^n$ be open, $1 < p < \infty$, and let ω be an A_p-weight, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D_ju\in L^p(\Omega,\omega)$, for $j = 1,...,n$. The norm of u in $W^{1,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx + \sum_{j=1}^n \int_\Omega |D_ju(x)|^p\,\omega(x)\,dx\right)^{1/p}. \qquad (2)$$

We also define $W^{1,p}_0(\Omega,\omega)$ as the clousure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega,\omega)$, and

$$\|u\|_{W^{1,p}_0(\Omega,\omega)} = \left(\sum_{j=1}^n \int_\Omega |D_ju(x)|^p\,\omega(x)\,dx\right)^{1/p}.$$

The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$ (see [4]),

$$\begin{aligned}&W^{-1,p'}(\Omega,\omega)\\ &= \{T = f_0 - \text{div} f : f = (f_1,...,f_n),\ f_j/\omega\in L^{p'}(\Omega,\omega),\ j = 0,...,n\}.\end{aligned}$$

It is evident that the weight ω which satisfies $0 < c_1\le\omega(x)\le c_2$ for $x\in\Omega$ (c_1 and c_2 positive constants), give nothing new (the space $\mathrm{W}^{1,p}_0(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following two theorems.

Theorem 2.4 Let $\omega\in A_p$, $1<p<\infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_n\to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{n_k}\}$ and a function $\Phi\in L^p(\Omega,\omega)$ such that
(i) $u_{n_k}(x)\to u(x)$, $n_k\to\infty$, μ-a.e. on Ω;
(ii) $|u_{n_k}(x)|\le\Phi(x)$, μ-a.e. on Ω.
Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [6]. □

Theorem 2.5(The Weighted Sobolev Inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ $(n\ge 2)$ and $\omega\in A_p$ $(1<p<\infty)$. There exist constants C_Ω and δ positive such that for all $u\in C_0^\infty(\Omega)$ and all k satisfying $1\le k\le n/(n-1)+\delta$,

$$\|u\|_{L^{kp}(\Omega,\omega)}\le C_\Omega\|\nabla u\|_{L^p(\Omega,\omega)}.$$

Proof. See Theorem 1.3 in [4]. □

Definition 2.4. Let $1<p<\infty$. We say that an element $u\in W_0^{1,p}(\Omega,\omega)$ is a (weak) solution of problem (P) if

$$\begin{aligned}&\sum_{j=1}^n\int_\Omega \omega\mathscr{A}_j(x,u,\nabla u)D_j\varphi\,dx+\int_\Omega g(x,u,\nabla u)\varphi\,\omega\,dx\\&=\int_\Omega f_0\varphi\,dx+\sum_{j=1}^n\int_\Omega f_jD_j\varphi\,dx,\end{aligned}$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega)$.

3. Proof of Theorem 1.1

We define $B: W_0^{1,p}(\Omega,\omega)\times W_0^{1,p}(\Omega,\omega)\to\mathbb{R}$ and $T: W_0^{1,p}(\Omega,\omega)\to\mathbb{R}$ by

$$\begin{aligned}B(u,\varphi)&=\sum_{j=1}^n\int_\Omega \omega\mathscr{A}_j(x,u,\nabla u)D_j\varphi\,dx+\int_\Omega g(x,u,\nabla u)\varphi\omega\,dx;\\T(\varphi)&=\int_\Omega f_0\varphi\,dx+\sum_{j=1}^n\int_\Omega f_jD_j\varphi\,dx.\end{aligned}$$

Then $u\in W_0^{1,p}(\Omega,\omega)$ is a (weak) solution to problem (P) if

$$B(u,\varphi)=T(\varphi),\ \text{ for all } \varphi\in W_0^{1,p}(\Omega,\omega).$$

Step 1. We define the operators

$$F_j : W_0^{1,p}(\Omega,\omega) \to L^{p'}(\Omega,\omega) \text{ and } G : W_0^{1,p}(\Omega,\omega) \to L^{p'}(\Omega,\omega)$$

by $(F_j u)(x) = \mathscr{A}_j(x,u(x),\nabla u(x))$ and $(Gu)(x) = g(x,u(x),\nabla u)$.

We have that the operators F_j and G are bounded and continuous. In fact,
(i) Using (H4) and Theorem 2.5, we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u(x)|^{p'}\,\omega\,dx \\
&= \int_\Omega |\mathscr{A}_j(x,u,\nabla u)|^{p'}\,\omega\,dx \\
&\leq \int_\Omega \left(K_1 + h_3|u|^{p/p'} + h_4|\nabla u|^{p/p'}\right)^{p'}\omega\,dx \\
&\leq C\int_\Omega \left[(K_1^{p'} + h_3^{p'}|u|^p + h_4^{p'}|\nabla u|^p)\,\omega\right]dx \\
&= C\left[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_3^{p'}|u|^p\,\omega\,dx + \int_\Omega h_4^{p'}|\nabla u|^p\,\omega\,dx\right] \\
&\leq C\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + (C_\Omega\|h_3\|^{p'}_{L^\infty(\Omega)} + \|h_4\|^{p'}_{L^\infty(\Omega)})\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right]. \qquad (3)
\end{aligned}$$

Therefore, in (3) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega)} \leq C\left(\|K_1\|_{L^{p'}(\Omega,\omega)} + \|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\right). \qquad (4)$$

Analogously, by condition (H6), we have

$$\begin{aligned}
\|Gu\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |g(x,u,\nabla u)|^{p'}\,\omega\,dx \\
&\leq \int_\Omega \left(K_2 + h_5|u|^{p/p'} + h_6|\nabla u|^{p/p'}\right)^{p'}\omega\,dx \\
&\leq C\left[\|K_2\|^{p'}_{L^{p'}(\Omega,\omega)} + (C_\Omega\|h_5\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_6\|^{p'}_{L^\infty(\Omega)})\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right].
\end{aligned}$$

Hence,

$$\|Gu\|_{L^{p'}(\Omega,\omega)} \leq C\left(\|K_2\|_{L^{p'}(\Omega,\omega)} + \|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\right). \qquad (5)$$

(ii) Let $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$ as $n\to\infty$. We need to show that $F_j u_n \to F_j u$ and $Gu_n \to Gu$ in $L^{p'}(\Omega,\omega)$.
If $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$, then $|\nabla u_n| \to |\nabla u|$ in $L^p(\Omega,\omega)$ and $u_n \to u$ in $L^p(\Omega,\omega)$. Using Theorem 2.4, there exist a subsequence $\{u_{n_k}\}$ and functions Φ_1 and Φ_2 in $L^p(\Omega,\omega)$ such that

$$\begin{aligned}
& u_{n_k}(x) \to u(x), \ \ \mu-\text{a.e. in } \Omega,\\
& |u_{n_k}(x)| \le \Phi_1(x), \ \ \mu-\text{a.e. in } \Omega,\\
& |\nabla u_{n_k}(x)| \to |\nabla u(x)|, \ \ \mu-\text{a.e. in } \Omega,\\
& |\nabla u_{n_k}(x)| \le \Phi_2(x), \ \ \mu-\text{a.e. in } \Omega.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \|F_j u_{n_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} = \int_\Omega |F_j u_{n_k}(x) - F_j u(x)|^{p'}\,\omega\,dx \\
= & \int_\Omega |\mathscr{A}_j(x,u_{n_k},\nabla u_{n_k}) - \mathscr{A}_j(x,u,\nabla u)|^{p'}\,\omega\,dx \\
\le & \ C\int_\Omega \left(|\mathscr{A}_j(x,u_{n_k},\nabla u_{n_k})|^{p'} + |\mathscr{A}_j(x,u,\nabla u)|^{p'} \right)\omega\,dx \\
\le & \ C\Bigg[\int_\Omega \left(K_1 + h_3|u_{n_k}|^{p/p'} + h_4|\nabla u_{n_k}|^{p/p'} \right)^{p'} \omega\,dx \\
+ & \int_\Omega \left(K_1 + h_3|u|^{p/p'} + h_4|\nabla u|^{p/p'} \right)^{p'} \omega\,dx \Bigg] \\
\le & \ 2C\int_\Omega \left(K_1 + h_3\Phi_1^{p/p'} + h_4\Phi_2^{p/p'} \right)^{p'} \omega\,dx \\
= & \ \tilde{C}\left[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_3^{p'}\Phi_1^p\,\omega\,dx + \int_\Omega h_4^{p'}\Phi_2^p\,\omega\,dx \right] \\
\le & \ \tilde{C}\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_3\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^p\,\omega\,dx + \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_2^p\,\omega\,dx \right] \\
\le & \ \tilde{C}\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_3\|^{p'}_{L^\infty(\Omega)} \|\Phi_1\|^p_{L^p(\Omega,\omega)} + \|h_4\|^{p'}_{L^\infty(\Omega)} \|\Phi_2\|^p_{L^p(\Omega,\omega)} \right].
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\|Gu_{n_k} - Gu\|^{p'}_{L^{p'}(\Omega,\omega)} \ \le \ & \tilde{C}\Big[\|K_2\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_5\|^{p'}_{L^\infty(\Omega)} \|\Phi_1\|^p_{L^p(\Omega,\omega)} \\
+ \ & \|h_6\|^{p'}_{L^\infty(\Omega)} \|\Phi_2\|^p_{L^p(\Omega,\omega)} \Big].
\end{aligned}$$

By condition (H1) and (H5), we have

$$F_j u_n(x) = \mathscr{A}_j(x,u_n(x),\nabla u_n(x)) \to \mathscr{A}_j(x,u(x),\nabla u(x)) = F_j u(x), \text{ as } n\to\infty,$$
$$G u_n(x) = g(x,u_n(x),\nabla u_n(x)) \to g(x,u(x),\nabla u(x)) = Gu(x), \text{ as } n\to\infty,$$

for almost all $x\in\Omega$. Therefore, by Dominated Convergence Theorem, we obtain

$$\|F_j u_{n_k} - F_j u\|_{L^{p'}(\Omega,\omega)} \to 0 \text{ and } \|G u_{n_k} - Gu\|_{L^{p'}(\Omega,\omega)} \to 0$$

that is, $F_j u_{n_k} \to F_j u$ and $G u_{n_k} \to Gu$ in $L^{p'}(\Omega,\omega)$. By Convergence principle in Banach spaces, we have

$$F_j u_n \to F_j u \text{ in } L^{p'}(\Omega,\omega), \text{ and } G u_n \to Gu \text{ in } L^{p'}(\Omega,\omega). \tag{6}$$

Step 2. We have,

$$\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega}|\varphi|\omega\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}|D_j\varphi|\omega\,dx \\
&\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} \\
&\leq \left(C_\Omega\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
&|B(u,\varphi)| \leq \sum_{j=1}^n \int_\Omega |\mathscr{A}_j(x,u,\nabla u)||D_j\varphi|\omega\,dx + \int_\Omega |g(x,u,\nabla u|\varphi\omega\,dx \\
\leq\ & \int_\Omega \left(K_1 + h_3|u|^{p/p'} + h_4|\nabla u|^{p/p'}\right)|\nabla\varphi|\omega\,dx \\
+\ & \int_\Omega \left(K_2 + h_5|u|^{p/p'} + h_6|\nabla u|^{p/p'}\right)|\varphi|\omega\,dx \\
\leq\ & C\Bigg[\|K_1\|_{L^{p'}(\Omega,\omega)} + \|K_2\|_{L^{p'}(\Omega,\omega)} + \Big(C_\Omega(\|h_3\|_{L^\infty(\Omega)} + \|h_5\|_{L^\infty(\Omega)}) \\
+\ & \|h_4\|_{L^\infty(\Omega)} + \|h_6\|_{L^\infty(\Omega)}\Big)\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\Bigg]\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned} \tag{7}$$

Since $B(u,.)$ is linear, for each $u \in W_0^{1,p}(\Omega,\omega)$, there exists a linear and continuous operator $Au : W_0^{1,p}(\Omega,\omega) \to [W_0^{1,p}(\Omega,\omega)]^*$ such that $\langle Au,\varphi\rangle = B(u,\varphi)$, for $u,\varphi \in W_0^{1,p}(\Omega,\omega)$, and

$$\|Au\|_* \leq C\left(\|K_1\|_{L^{p'}(\Omega,\omega)} + \|K_2\|_{L^{p'}(\Omega,\omega)} + \|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\right). \tag{8}$$

Consequently, problem (P) is equivalent to the operator equation $Au = T$, with $u \in W_0^{1,p}(\Omega,\omega)$.

Step 3. Using conditions (H2) and (H8), we have

$$\begin{aligned}
& \langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1,u_1-u_2) - B(u_2,u_1-u_2) \\
= & \int_\Omega \omega\mathscr{A}(x,u_1,\nabla u_1).\nabla(u_1-u_2)\,dx - \int_\Omega \omega\mathscr{A}(x,u_2,\nabla u_2).\nabla(u_1-u_2)\,dx \\
+ & \int_\Omega \omega[g(x,u_1,\nabla u_1) - g(x,u_2,\nabla u_2)](u_1-u_2)\,dx \\
= & \int_\Omega \omega\Big(\mathscr{A}(x,u_1,\nabla u_1) - \mathscr{A}(x,u_2,\nabla u_2)\Big).\nabla(u_1-u_2)\,dx \\
+ & \int_\Omega \omega[g(x,u_1,\nabla u_1) - g(x,u_2,\nabla u_2)](u_1-u_2)\,dx \geq 0.
\end{aligned}$$

Therefore the operator A is monotone. Moreover, using (H3), (H7) and the weighted Sobolev inequality (with $k=1$), we obtain

$$\begin{aligned}
& \langle Au,u\rangle = B(u,u) \\
= & \int_\Omega \omega\mathscr{A}(x,u,\nabla u).\nabla u\,dx + \int_\Omega \omega g(x,u,\nabla u)u\,dx \\
\geq & \int_\Omega \Big(\Lambda|u|^p + \lambda|\nabla u|^p - h_1|u| - h_2\Big)\omega\,dx \\
= & \Lambda\int_\Omega |u|^p\omega\,dx + \lambda\int_\Omega|\nabla u|^p\omega\,dx - \int_\Omega h_1|u|\omega\,dx - \int_\Omega h_2\omega\,dx \\
\geq & \Lambda\int_\Omega |u|^p\omega\,dx + \lambda\int_\Omega|\nabla u|^p\omega\,dx \\
- & \left(\int_\Omega h_1^{p'}\omega\,dx\right)^{1/p'}\left(\int_\Omega |u|^p\omega\,dx\right)^{1/p} - \|h_2\|_{L^1(\Omega,\omega)} \\
\geq & C\|u\|^p_{W_0^{1,p}(\Omega,\omega)} - \|h_1\|_{L^{p'}(\Omega,\omega)}\|u\|_{L^p(\Omega,\omega)} - \|h_2\|_{L^1(\Omega,\omega)} \\
\geq & C\|u\|^p_{W_0^{1,p}(\Omega,\omega)} - C_\Omega\|h_1\|_{L^{p'}(\Omega,\omega)}\|\nabla u\|_{L^p(\Omega,\omega)} - \|h_2\|_{L^1(\Omega,\omega)} \\
\geq & C\|u\|^p_{W_0^{1,p}(\Omega,\omega)} - C_\Omega\|h_1\|_{L^{p'}(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega)} - \|h_2\|_{L^1(\Omega,\omega)}.
\end{aligned}$$

Hence, since $1 < p < \infty$, we have

$$\frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega)}} = C\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega)} - C_\Omega\|h_1\|_{L^{p'}(\Omega,\omega)} - \frac{\|h_2\|_{L^1(\Omega,\omega)}}{\|u\|_{W_0^{1,p}(\Omega,\omega)}},$$

and

$$\frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega)}} \to +\infty,$$

as $\|u\|_{W_0^{1,p}(\Omega,\omega)} \to +\infty$, that is, A is coercive.

Step 4. We need to show that the operator A is continuous. Let $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$ as $n \to \infty$. We have,

$$\begin{aligned}
& |B(u_n,\varphi) - B(u,\varphi)| \\
\leq\ & \sum_{j=1}^{n} \int_\Omega |\mathscr{A}_j(x,u_n,\nabla u_n) - \mathscr{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx \\
+\ & \int_\Omega |g(x,u_n,\nabla u_n) - g(x,u,\nabla u)||\varphi|\,\omega\,dx \\
=\ & \sum_{j=1}^{n} \int_\Omega |F_ju_n - F_ju||D_j\varphi|\,\omega\,dx + \int_\Omega |Gu_n - Gu||\varphi|\,\omega\,dx \\
\leq\ & \sum_{j=1}^{n} \|F_ju_n - F_ju\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} + \|Gu_n - Gu\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^{p'}(\Omega,\omega)} \\
\leq\ & C\left(\sum_{j=1}^{n} \|F_ju_n - F_ju\|_{L^{p'}(\Omega,\omega)} + \|Gu_n - Gu\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)},
\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$. Hence,

$$\|Au_n - Au\|_* \leq C\left(\sum_{j=1}^{n} \|F_ju_n - F_ju\|_{L^{p'}(\Omega,\omega)}\right) + \|Gu_n - Gu\|_{L^{p'}(\Omega,\omega)}.$$

Therefore, using (6), we have $\|Au_n - Au\|_* \to 0$ as $n \to +\infty$.

Therefore, by Theorem 1.2, the operator equation $Au = T$ has a solution u in $W_0^{1,p}(\Omega,\omega)$ and u is the solution for problem (P).

4. Example Let $\Omega = \{(x,y) \in \mathbb{R}^n \ : x^2 + y^2 < 1\}$, and consider the weight function $\omega(x,y) = (x^2+y^2)^{-1/2}$ ($\omega \in A_2$), the functions $\mathscr{A} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathscr{A}((x,y),\eta,\xi) = h_4(x,y)\xi$$
$$g((x,y),\eta,\xi) = \eta\cos^2(xy).$$

where $h_4(x,y) = 2e^{x^2+y^2}$. Let us consider the partial differential operator

$$\begin{aligned} Lu(x,y) &= -\mathrm{div}\left[\omega(x,y)\mathscr{A}((x,y),u,\nabla u)\right] + \omega(x,y)g((x,y),u,\nabla u) \\ &= -\frac{\partial}{\partial x}\left[\omega(x,y)h_4(x,y)\frac{\partial u}{\partial x}\right] - \frac{\partial}{\partial y}\left[\omega(x,y)h_4(x,y)\frac{\partial u}{\partial y}\right] \\ &+ \omega(x,y)u(x,y)\cos^2(xy). \end{aligned}$$

Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x,y) &= \frac{\cos(xy)}{(x^2+y^2)^{1/5}} - \frac{\partial}{\partial x}\left(\frac{\sin(xy)}{(x^2+y^2)^{1/9}}\right) - \frac{\partial}{\partial y}\left(\frac{\sin(xy)}{(x^2+y^2)^{1/9}}\right), \text{ on } \Omega \\ u(x,y) &= 0, \text{ on } \partial\Omega \end{cases}$$

has a solution $u \in W_0^{1,2}(\Omega,\omega)$.

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ALBO CARLOS CAVALHEIRO
Department of Mathematics
State University of Londrina
86051-990, Londrina - PR, Brazil
e-mail: `accava@gmail.com`