LE MATEMATICHE
Vol. XLVI (1991) - Fasc. II, pp. 655-673

# DISCRETE FINITE ELEMENTS METHOD IN SPACE-TIME DOMAIN FOR PARABOLIC LINEAR PROBLEMS

MARIA MORANDI CECCHI (Padova)
ROSARIA NOCIFORO (Catania)(*)

Theory, error-bound and applications of Discrete Finite Element Method is given to solve a class of linear one and two-dimentional parabolic problems on Sobolev space-time domains, with non homogenous discontinuous initial data and general boundary conditions.

## 1. Introduction.

A Discrete Finite Element Method in a space-time domain is presented here to solve linear differential parabolic problems with non homogeneous discontinuous initial conditions and general boundary conditions.

The method presented is an evolution of a method that was formulated in [12] to reach a better way of solution considering the possibility of applying finite elements both in space and in time

---

Key Words - Finite Element Method, Space-time, Heat equations.
(*) Entrato in Redazione il 4 ottobre 1991.

allowing the definition of variable slabs in time and discontinuous initial data.

To be more specific the discrete method solves the problem by a unique space-time finite element solver not using a Semidiscrete Galerkin method, but chosing the time depending test functions defined in a space different from the solution space.

After giving a theoretical justification and an error estimates, the method is applied to the solution of the heat conduction problem.

An iterative algorithm is generated to solve the problem into different slabs of time, with the solution of the linear system characterized by a symmetric bounded matrix.

Numerical results are given of one and two-dimensional test problems and they are compared with the exact solution.

## 2. Weak Formulation of the linear parabolic problems.

Let $\Omega$ be an open bounded domain in $R^N$ and let $\Gamma$ be the boundary of $\Omega$, regular of class $C^\infty$, with $\Gamma$ partitioned in $\Gamma_1$ and $\Gamma_2$, where $\Gamma_2 = \Gamma - \Gamma_1$ (measure $\Gamma_1 > 0$).

Let $H^1(\Omega)$ and $H_0^1(\Omega)$ be two real Sobolev spaces

$$H^1(\Omega) = \{u : u, D_x u \in L^2(\Omega)\}$$

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u(\Gamma) = O\}$$

Setting $V = \{u \in H^1(\Omega) : u(\Gamma_1) = 0\}$, $V$ results a real closed subspace such that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.

The inner product and the norm in $H^1(\Omega)$ are respectively

$$(u, v)_{H^1(\Omega)} = \int_\Omega [Du(x)Dv(x) + uv]dx$$

$$||u||_{H^1(\Omega)} = (u, u)^{1/2}_{H^1(\Omega)} = \left[\int_\Omega [|Du(x)|^2 + |u(x)|^2]dx\right]^{1/2}$$

The inner product and the related norm in $V$ are the following

$$(u, v) = \int_\Omega Du(x)Dv(x)dx, \quad ||u||_v = (u, u)_v^{1/2}$$

Let be $Q = \Omega\times]0,T[$ and let be $A$ a differential operator of order two, of the type

$$(1)\quad A\left(x,t,\frac{\partial}{\partial x}\right) = -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} a_i(x,t)\frac{\partial}{\partial x_i} + a_0(x,t)$$

where

$$(1.1)\qquad a_{ij}, a_i, a_0 \in L^{\infty}(Q) \quad Q = \Omega\times]0,T[$$

$$(1.2)\quad \sum a_{ij}(x,t)\xi_j\xi_i \geq \alpha(|\xi_1|^2 + |\xi_2|^2 + \ldots + |\xi_N|^2)\ \alpha > 0\ \xi_\iota \in C \text{ a.e. in } Q$$

Let $T_0(t)$ be a family of operators such that

$$T_0(t) \in \mathcal{L}(H^{1/2}(\Gamma_2); H^{-1/2}(\Gamma_2))$$

and the function $(T_0(t)u, v)$ whit $u, v \in H^{1/2}(\Gamma_2)$, is measurable on $(0,T)$ and

$$|(T_0(t)u, v)| \leq c||u||_{H^{1/2}(\Gamma_2)}||v||_{H^{1/2}(\Gamma_2)}$$

The initial-boundary value parabolic problem that will be analyzed consists in finding a function $u(x,t) \in H^1(Q)$ solution of the problem

$$(2)\qquad A\left(x,t,\frac{\partial}{\partial x}\right)u(x,t) + \frac{\partial}{\partial t}u(x,t) = f(x,t) \qquad u(x,t) \in Q$$

$$(2.1)\qquad u(x,0) = u_0(x) \qquad x \in \Omega$$

$$(2.2)\qquad u(x,t) = 0 \qquad x \in \Gamma_1\ t \in (0,T)$$

$$(2.3)\qquad \frac{\partial}{\partial v_A}u(x,t) + T_0(t)u = 0 \qquad x \in \Gamma_2\ t \in (0,T)$$

where $\frac{\partial}{\partial v_A} = \sum a_{ij}(x,t)\cos(nx_i)\,\frac{\partial}{\partial x_j}$, $f(x,t)$ is a given function of $L^2(Q)$ and $u_0(x)$ is a function of $L^2(\Omega)$.

One defines now a weak formulation for the problem (2)-(2.3).

Let us set $W = L^2(0,T;V)$; the inner product and the norm in $W$ are respectively

$$(u,v)_W = \int_Q D_x u(x,t) D_x v(x,t) dx dt$$

$$||u||_W = \left[\int_Q [D_x u(x)]^2 dx dt\right]^{1/2} = \left[\int_0^T ||u(t)||_V^2 dt\right]^{1/2}$$

One indicates by $U \subset W$ the subspace of functions of $W$ that satisfy the initial condition of (2.1). Let $a(t;u,v)$ be a of sesquilinear form depending on the parameter $t > 0$, while $u,v \in W$ satisfying the following continuity condition:

$$|a\langle t;u,v\rangle| \leq M||u||_V||v||V \quad \forall u,v \in W$$

where $M$ is a constant independent on $t,u$ and $W$.

Let $\Phi$ be the space of test functions defined as the subspace of functions of $W$ that are square integrable on $(0,T)$ with their first derivatives respect to $x$ and respect to $t$, and that satisfy to $\phi(x,T) = 0$.

The norm in $\Phi$ is so defined

$$||\phi||_\Phi^2 = \int_0^T ||\phi(t)||_V^2 dt + |\phi_0|_H^2$$

where

$$|\phi_0|_H^2 = \int_\Omega |\phi(0)|^2 dx, \; H = L^2(\Omega)$$

Performing an inner product on $H$ of both sides of the equation (2) by a function $\phi \in \Phi$, and integrating on $t$ between 0 and $T$, it results

$$\int_0^T \left[\left(A\left(x,t\frac{\partial}{\partial x}\right)u(t),\phi(t)\right)_H + (u'(t),\phi(t))_H\right] dt = \int_0^T (f(t),\phi(t))_H dt$$

The first term of this equation can be transformed into a sesquilinear form in $U \times \Phi$ so that the equation assumes the following equivalent form

$$(3) \qquad \int_0^T [a(t;u,\phi) - (u(t),\phi'(t))_H]dt = \int_0^T (f(t),\phi(t))_H dt + (u_0,\phi(0))_H$$

where

$$a(t;u,\phi) = \sum_{ij=1}^N \int_\Omega a_{ij}(x,t) D^j u(x) D^i \phi(x) dx + \sum_{i=1}^N \int_\Omega a_i(x,t)(D^i u(x))\phi(x)dx +$$
$$+ \int_\Omega a_0(x,t)u(x)\phi(x)dx + (T_0(t)\gamma_0 u, \gamma_0 \phi)_{\Gamma_2}$$

Now let us assume that $T_0(t)$ is a family of differential tangential operators of $\Gamma_2$ with continously differentiable coefficients, so that, for every $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ for which

$$|(T_0(t)\gamma_0 u, \gamma_0\phi)| \le \varepsilon ||u||_V^2 + c(\varepsilon)|u|_H^2 \quad \forall \varepsilon > 0$$

and
$$(T_0 u, \gamma_0 u) \le \int_{\Gamma_2} |\gamma_0 u|^2 dt \ (c_1 \text{ constant})$$

The first member of the equation (3) can be written as the following sesquilinear form

$$E(\phi,\phi) = \int_0^T [a(t;u,\phi) - (u(t),\phi'(t))_H]dt$$

where

$$E(\phi,\phi) = \int_0^T a(t;\phi(t),\phi(t))dt - \frac{1}{2}\int_0^T \frac{d}{dt}|\phi_0|^2 dt =$$
$$= \int_0^T a(t;\phi(t),\phi(t))dt + \frac{1}{2}|\phi_0|_H^2$$

The second member of (3) can be written as the following linear form

$$L(\phi) = \int_0^T (f(t),\phi(t))_H dt + (u_0,\phi(0))_H$$

that is defined and continuous on $\phi$ whit respect to the norm $||\cdot||_\Phi$

Therefore the equation (3) can be written as

$$(4) \qquad E(u,\phi) = L(\phi) \quad u \in U \quad \forall \phi \in \Phi$$

This equation is the weak formulation of the parabolic problem (2)-(2.3) and now the problem is to find the function $u \in U$ that satisfies equation (4).

The existence and the uniqueness of solution of (4) is proved by Lions [10] (Theoreme 1.1 p. 46)

Since the form $a(t;\phi,\phi)$ results weakly coercive for every $o < t < T$, that it, there exists two constants $\lambda \in R$ and $k > 0$ such that

$$a(t;u,v) + \lambda|u|_H^2 \geq k||u||_V^2,$$

we have for $\phi \in \Phi$

$$E(\phi,\phi) \geq k\int_0^T ||\phi||_V^2 dt - \lambda\int_0^T |u|_H^2 dt + \frac{1}{2}|\phi_0|_H^2 \geq K||\phi||_\phi^2 - \lambda\int_0^T |u|_H^2 dt$$

where $K = \inf(k, 1/2)$.

As it well know, we may assume $\lambda = 0$ without loss of generality [10] (p. 46).

## 3. A stability condition for the weak formulation.

By an application of Theorem 1.1 [10] for the weak formulation the linear parabolic problem, it follows the interesting result [10]:

If $U, W$ and $\Phi$ are the space above defined and:

- let $E_n(u,\phi)$ and $E(u,\phi)$ are the sesquilinear form defined in $W \times \Phi$, continuous in $W$ for every $\phi \in \Phi$ so that

$$\begin{aligned} |E_n(\phi,\phi)| &\geq \alpha||\phi||_\Phi^2 \\ |E(\phi,\phi| &\geq \alpha||\phi||_\Phi^2 \end{aligned} \qquad \alpha > 0 \quad \phi \in \Phi$$

- the function

$$R_n(\phi) = E_n(u,\phi) - E(u,\phi) \quad \text{is continuous on } \Phi \text{ and}$$
$$||R_n(\phi||_\Phi \to 0 \qquad n \to \infty$$

where

$$||R_n(\phi)||_\Phi = \frac{\sup|R_n(\phi)|}{||\phi||_\Phi} \; \phi \in \Phi, \; \phi \neq 0$$

– the semilinear form $L_n$ and $L$ are continuous on $\Phi$ such that

$$||L_n - L||_\Phi \to 0 \quad n \to \infty$$

– let $u_n$ be the solution of the problem with

$$E_n(u_n, \phi) = L_n(\phi) \quad \forall \phi \in \Phi$$

– let $u$ be the solution of the problem

$$E(u, \phi) \quad \forall \phi \in \Phi$$

then

$$u_n \to u \quad \text{on } W \quad \text{if } n \to \infty$$

## 4. Discrete finite element method.

The finite element method here is applied to computing an approximate solution of a weak formulation of parabolic problems.

In this method the piecewise polynomial functions are defined on a finite number of subdomains to approximate the exact solution of the problem.

This method is applied to obtain a discrete formulation of the weak problem.

Let $P_h$ be a projection operator that maps the subset $U$ of $W$, into a finite dimensional subspace $U_h$ of $U$, and maps the subspace $\phi$ of $W$, into $\Phi_h$.

The approximating problem becomes to find the function $u_h$ of $U_h$ such that

$$E(u_h, \phi_h) = L(\phi_h) \quad \forall \phi_h \in \Phi_h \tag{5}$$

In the Finite Element Method the approximating spaces $U_h$ and $\Phi_h$ are spaces of piecewise polynomial functions defined on a given partition of $\Omega \times [0,T]$. The functions $u_h$ and $\phi_h$ are constructed by local interpolation across the elements from the nodal values.

Notice that here the two subspaces $U_h$ and $\Phi_h$ are not coincident, because the test functions $\phi_h$ are chosen so that $\phi_h\ (T) = 0$. We set:

(6) $$u_h = \sum_{j=1}^{n_h} \alpha_j^h e_j^h \quad \phi_h = \sum_{j=1}^{n_h} \beta_j^h \eta_j^h$$

where $n_h$ is the dimension of the subspaces $U_h$ and $\Phi_h$. Here $\{e_j^h\}$ and $\{\eta_j^h\}$ are given piecewise polynomial functions on $U_h$ and $\Phi_h$, and $(\alpha_j^h)$ e $(\beta_j^h)$ are unknown parameters.

Moreover we set:

$$\phi_h' = \sum_{j=1}^{n_h} \beta_j^h \psi_j^h$$

where the $\{\psi_j^h\}$ is determined by performing an appropriate transformation of the $\{\eta_j^h\}$ so that may be mantained the same coefficients $(\beta_j^h)$:

$$\phi(0) = \sum_{j=1}^{n_h} \beta_j^h \xi_j^h$$

Substituting in the discrete problem (5) one obtains:

$$\alpha^T \mathbf{M} \beta^T = \mathbf{P}^T \beta \quad \forall \beta \in R^n$$

where $\alpha = \{\alpha_j^h\}$ e $\beta = \{\beta_j^h\}$

$$\mathbf{M} = \{M_{ij}\}, \quad M_{ij} = \int_0^T [a(t; e_i^h, \eta_j^h) - (e_j^h, \psi_j^h)]dt$$

$$\mathbf{P} = \{P_j\}, \quad P_j = \int_0^T (f, \eta_j^h)dt + (u(0), \eta_j^h)$$

and, for any $\beta, \alpha$ is the solutions of the linear system

(7) $$\mathbf{M}^T \alpha = \mathbf{P}$$

Substituting $\alpha$ into (6) one computes the approximate solution $u_h$.

Notice that, owing the coerciveness of $E(\phi, \phi)$ and the special finite element discretization, the matrix $\mathbf{M}$ is a square, positive definite and symmetric matrix.

## 5. Space of lagrange approximation and error estimates.

In the application of Discrete Finite Element Method in space-time domain for linear parabolic problems, the Lagrange approximation in several variables leads to interesting results on the solutions.

From the coerciveness in $\Phi$ of sesquilinear form $E(u,v)$ and from the definition of the norm $|| \ ||_\Phi$, it follows that $E(u,v)$ is coercive even in $W$

$$k||\phi||_W^2 \leq E(\phi,\phi) \quad \forall\phi \in W \tag{8}$$

Let $U_n$ and $\Phi_h$ be a finite dimentional subspace respectively of $U$ and $\Phi$, such that:

$$E(u_n,\phi) = L(\phi) \quad u_n \in U_n \ \forall\phi \in \Phi_h$$

and let $u$ is the exact solution of the weak problem:

$$E(u,\phi) = L(\phi) \quad u \in U \quad \forall\phi \in \Phi$$

if $W_n$ is the linear span of finite elements and suppose that $w_n \in W_n$ assumes the same values of $u_n$ at nodes and if $\theta_j$ is a generic finite element of the domain $Q = \Omega \times (O,T)$, where $E$ is the total number of finite elements of the mesh, since the finite element method takes the local interpolation on each $\theta_j$ we can write [4]

$$||u_n - w_n||_w = \left( \sum_{j=1}^{E} ||u_n - w_{nj}||^2_{H^1(\theta_j)} \right)^{1/2} \tag{9}$$

where $H^1(\theta_j)$ is the projection over the finite element $\theta_j$ of the functions of $w_n \in W_n$ and $w_{nj}$ is the interpolant of $u_n$ in $\theta_j$, therefore the problem of finding the estimate for the error $||u - u_n||w$ is reduced to a local interpolation problem.

For solving a local interpolation problem the following theorem holds [8]:

THEOREM *"Given an integer $k \geq 1$ and let $u_n \in H^{k+1}(\theta_j)$ the restriction of the approximate solution $u_n \in U_n$ where $\theta_j$ is a close* *convex hull of a k-unisolvent set* $\sum = \{a_i\}_i \; 1 \leq i \leq n$ *of* $R^N$*, verteces of* $\theta_j$.

*If* $w_{nj}$ *is the unique Lagrange interpolating polynomial of degree* $\leq k$ *of* $u_n$*, in sense that* $w_{nj}$ *is the unique element of* $H^{k+1}(\theta_j)$ *such that*

$$w_{nj}(a_i) = u_n(a_i) \quad 1 \leq i \leq n$$

*for* $h_j$ *=diameter of* $\theta_j$ *and* $\rho_j = |sup\{$*diameter of the spheres in* $\theta_j\}$ *then, for* $h_j$ *small, exists a constant* $C$ *independent on* $\theta_j$ *such that*

$$(10) \qquad ||u_n - w_{nj}||_{H^1(\theta_j)} \leq C||u_n||_{H^{k+1}(\theta_j)} \frac{h_j^{k+1}}{\rho_j}$$

The error bound for the FEDM is derived by a straightforward combination of the applications (9) and (10).

It is clear that the estimate (10) is better when the ratio $h_j/\rho_j$ is small.

The intuitive significance of this is that, for a good approximation, one should not consider $k$-unisolvent sets $\sum$ whose finite element $\theta_j$ are too flat.

In practice in the finite element method one takes the family $\{\sum_j\}_{j\in I}$ of $k$-unisolvent sets, with associated parameters $h_j$ and $p_j$, the element of a particular set $\sum_j$ are verteces of a particular finite element $\theta_j$. A family $\{\sum_j\}_{j\in I}$ is defined a *regular family* if

$$h_j \leq \alpha\rho_j \quad \text{for all } j \in I \quad \text{for some constant } \alpha > 0.$$

For such regular families the error bounds of (11) can be written at once into the following form

$$(11) \qquad ||u_n - w_{nj}||_{H^1(\theta_j)} = O(h_j^k)$$

and this is the form of the interpolation error in this method.

Then the FEDM approximation error gives

$$||u - u_n||_W \leq B||u_n - w_n||_W = O(h^k) \quad \text{with } h = \max_{1\leq j\leq E} h_j.$$

## 6. The application of the Discrete Finite Element Method to the heat conduction problems.

Here the Discrete Finite Element Method is applied to the following heat conduction problem

(12) $u_t(x,t) - k\nabla u(x,t) + zu(x,t) = f(x,t) \qquad u(x,t) \in Q$

(12.1) $u(x,0) = u_0 \qquad x \in \Omega \quad t = 0$

(12.2) $u(x,t) = 0 \qquad x \in \Gamma_1 \quad t \in (0,T)$

(12.3) $k\dfrac{\partial}{\partial x}u(x,t) + Hu = 0 \qquad x \in \Gamma_2 \quad t \in (0,T)$

This problem is a linear parabolic problem of the type (2)-(2.3), where

– the coefficient of the differential operator (1)

$$A\left(x,t,\frac{\partial}{\partial x}\right) = -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} a_i(x,t)\frac{\partial}{\partial x_i} + a_0(x,t)$$

are respectively $a_{ij}(x,t) = k$ (const.> 0), $a_i(x,t) = 0$ and $a_0(x,t) = z$ (const.).

– the linear operator $T_0(t)$ are the constant value $H$.

The corresponding form is

$$a(t;u,\phi) = k\sum_{i,j=1}^{N}\int_\Omega D^j u(x) D^i\phi(x)dx + z\int_\Omega u(x)\phi(x)dx + (Hu(x),\phi(x))_{\Gamma_2}$$

Assuming the $\phi(x) = 0$ for $x \in \Gamma_1$ the equation (3) becomes

$$\int_0^T\int_\Omega \left[k\sum_{j=1}^{N}\sum_{i=1}^{N}\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_i} + zu\phi - u\phi'\right] dxdt = \tag{13}$$
$$= -\int_0^T \langle Hu,\phi\rangle_{\Gamma_2} dt + \int_0^T\int_\Omega f\phi dxdt - \int_0^T \frac{1}{T}\int_\Omega u(0)\phi(0)dxdt$$

this is the weak formulation of the heat conduction problem (12)-(12.3).

The domain $Q = \Omega \times (0,T)$ is partitioned into subdomains $\theta_j = \Omega_e \times \Delta t_m$, where the domain $\Omega$ is partitionated into element $\Omega_e$, with $1 \leq e \leq E$, where $E$ is the number of the finite elements of the mesh and the time domain $(0,T)$ is partitioned as $\Delta t_m = (t_m, t_{m+1})$, with $1 \leq m \leq L$, where $L$ is the number of time-slabs.

Applying the Lagrange approximation to (13) and using of piecewise polynomial functions defined in each element as

$$u_h = \mathbf{w}^T\mathbf{a} \quad \phi_h = \mathbf{w}^T\mathbf{b}$$

where $\mathbf{w}^T$ is a row vector containing the terms of the approximating polynomial and $\mathbf{a}$ and $\mathbf{b}$ are vectors of coefficients.

Let suppose that:

$$\mathbf{w}\mathbf{a}_i^T = \frac{\partial u_h}{\partial x} \quad \mathbf{w}\mathbf{b}_i^T = \frac{\partial \phi_h}{\partial x} \quad i = 1 \ldots N$$

$$\mathbf{w}\mathbf{b}_t^T = \frac{\partial \phi_h}{\partial t} \quad \mathbf{w}\mathbf{b}_0^T = \phi_0$$

substituting the previous positions in the (13) and assembling for all $\theta_j$ of the domain $Q$, one gets

(14)

$$\sum_e \int_0^T \int_{\Omega_e} \left[ k \sum_{i,j=1}^N \mathbf{a}_i \mathbf{w}^T \mathbf{w}\mathbf{b}_j^T - a\mathbf{w}^T\mathbf{w}\mathbf{b}_t^T + z\mathbf{a}\mathbf{w}^T\mathbf{w}\mathbf{b}^T \right] dxdt -$$

$$\sum_e H \int_0^T \mathbf{a}_{N+1}\mathbf{w}^T\mathbf{w}\mathbf{b}^{N+1} dt = \sum_e \int_0^T \int_{\Omega_e} \left[ f\mathbf{w}\mathbf{b}^T - \frac{u_0}{T}\mathbf{w}_0\mathbf{b}_0^T \right] dxdt$$

Let be

$$K_e = \int_0^T \int_{\Omega_e} \mathbf{w}^T\mathbf{w} dxdt \quad f_e = \int_0^T \int_{\Omega_e} f(x,t)\mathbf{w} dxdt \quad u_0 = \int_{\Omega_e} u_0(x)\mathbf{w}_0 dxdt$$

where $\mathbf{w}_0$ is the restriction of $\mathbf{w}$ to the terms not containg the variable $t$, with $\mathbf{w} = \{\mathbf{w}_0, \mathbf{w}_1\}$, $\mathbf{b} = \{\mathbf{b}_0, \mathbf{b}_1\}$ e $\mathbf{u}_1 = \{\mathbf{u}_0, \mathbf{0}\}$; obviously $\mathbf{u}_0'\mathbf{b}_0 = \mathbf{u}_0\mathbf{b}$.

Posing

$$\mathbf{a}_i^T = G_i\mathbf{a}^T \quad i = 1, \ldots N$$

$$\mathbf{b}_i^T = G_j\mathbf{b}^T \quad j = 1, \ldots N \quad \mathbf{b}_t^T = G_T\mathbf{b}^T$$

and substituing into (14)

$$\sum_e (\mathbf{a}M\mathbf{b}^T - \mathbf{P}^T\mathbf{b}^T) = 0$$

where

$$M = \sum_{ij=1}^{N} kG_i^T K_e G_j - \sum_{j=1}^{N} K_e G_j + zK_e$$

$$\mathbf{P} = \mathbf{f}_e - \mathbf{u}_0$$

Now the boundary conditions are imposed taking into account the terms comming from the partial derivatives

$$\left[\frac{\partial u_h}{\partial v_A}\right]_{\Gamma_2} \quad \text{and} \quad [\phi_h]_{\Gamma_2}$$

that are added into the elements of the matrix $M$ considering the rows corresponding to the nodal values of the boundary.

If the unknown coefficients $\mathbf{a}$ and $\mathbf{b}$ are expressed by

(15) $$\mathbf{a} = \mathbf{N}\mathbf{u}_e \quad \mathbf{b} = \mathbf{N}\boldsymbol{\phi}_e$$

where $N$ is a row vector containing the terms of the test functions of the polynomial approximation and $\mathbf{u}_e$ and $\boldsymbol{\phi}_e$ are vectors coefficients, containing the nodal values of $\mathbf{u}$ and $\boldsymbol{\phi}$, and its partial derivatives, one get

$$\sum_e (\mathbf{u}_e^T\mathbf{N}^T M\mathbf{N} - \mathbf{P}^T\mathbf{N})\boldsymbol{\phi}_e = 0$$

By the standard procedure of assembling the equations of the single elements, this equation becomes

(16) $$(\mathsf{U}^T M - \mathbf{P}^T)\boldsymbol{\phi} = 0$$

The (16) must be verified for every $\phi \in \Phi$, therefore it is:

(17) $$M^T\mathsf{U} = \mathsf{P}$$

The solution of this equation in $\mathbf{U}$, provides the parameters needed in (16) for the computation of the approximate solution.

## 7. Iterative numerical method.

In this application the Discrete Finite Element Method solve numerically the heat conduction problems. For such reason we have implemented an iterative algorithm, that computes the unknown temperature on the first step starting from know initial data, and it proceeds iteratively.

The algorithm generated for one-dimentional linear heat conduction problems is presented here in an analogous way the algorithm for two-dimentional problems is generated . The one-dimentional heat conduction problem is given by

(18) $u_t(x,t) - ku_{xx}(x,t) + zu(x,t) = f(x,t) \qquad u(x,t) \in Q$

(18.1) $u(x,O) = u_0 \qquad x \in \Omega \quad t = 0$

(18.2) $u(x,t) = 0 \qquad x \in \Gamma_1 \quad t \in (0,T)$

(18.3) $k\frac{\partial}{\partial x}u(x,t) + Hu = 0 \qquad x \in \Gamma_2 \quad t \in (0,T)$

where $\Omega$ is an open bounded domain of $R^1$.

The weak formulation of this problem is

$$\int_0^T \int_\Omega k\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x}dxdt - \int_0^T \int_\Omega u\frac{\partial \phi}{\partial t}dxdt + \int_0^T \int_\Omega zu\phi dxdt = -\int_0^T \int_\Omega f\phi dxdt + k\int_0^T \left[\frac{\partial u}{\partial x}\phi\right]_{\Gamma_2} dt + \int_\Omega u_0\phi_0 dx \tag{19}$$

where the last two terms of this equation contain respectively the boundary condition (18.3) and the initial condition (18.1). where

$$\int_0^T k\left[\frac{\partial u}{\partial x}\phi\right]_{\Gamma_2} dt = -\int_0^T H[u\phi]_{\Gamma_2} dt \tag{20}$$

substituing (20) in (19)

$$\int_0^T \int_\Omega k \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} dx dt - \int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} dx dt + \int_0^T \int_\Omega z u \phi dx dt = \int_0^T \int_\Omega f \phi dx dt - H \int_0^T [u\phi]_{\Gamma_2} dt + \int_\Omega u_0 \phi_0 dx \tag{21}$$

Expressing the unknown functions $u$ and $\phi$ by

$$u = \mathbf{N}\mathbf{u} \quad \text{e} \quad \phi = \mathbf{N}\mathbf{\Phi}$$

where $\mathbf{N}$ is the vector of the shape functions and $\mathbf{u}$ and $\phi$ are the vectors of the nodal values of $u$ and $\phi$, taking

$$\mathbf{K} = \int_0^T \int_\Omega k \frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial \mathbf{N}}{\partial x} dx dt - \int_0^T \int_\Omega \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial t} dx dt + z \int_0^T \int_\Omega \mathbf{N}^T \mathbf{N} dx dt = \mathsf{K}_x - \mathsf{K}_t - \mathsf{K}_z$$

$$\mathbf{V} = -H \int_0^T [\mathbf{N}^T\mathbf{N}]_{\Gamma_2} dt = -H \int_0^T \mathsf{N}_R^T \mathsf{N}_R dt + H \int_0^T \mathsf{N}_L^T \mathsf{N}_L dt = \mathsf{V}_L - \mathsf{V}_R$$

$$\mathsf{Z} = \int_0^T \int_\Omega \mathsf{N} dx dt \qquad \mathsf{E} = \int_\Omega \mathsf{N}_0^T \mathsf{N}_0 dx$$

where the index $R$ and $L$ of the matrix indicate respectively the values on the right and on the left boundary of one-dimentional domain. Therefore the equation (22) gives

$$\mathbf{u}^T\mathbf{K}\boldsymbol{\phi} = f^T\mathbf{Z}\boldsymbol{\phi} - \mathbf{u}_R^T\mathbf{V}_R\boldsymbol{\phi}_R + \mathbf{u}_L^T\mathbf{V}_L\boldsymbol{\phi}_L + \mathbf{u}_O^T\mathbf{E}\boldsymbol{\phi}_O$$

Partitioning this equation in two parts, one indipendent on time, with index 1, and the other time depending, with index 2, and taking

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix} \quad \mathbf{u}_R = \begin{Bmatrix} \mathbf{u}_{R1} \\ \mathbf{u}_{R2} \end{Bmatrix} \quad \mathbf{u}_L = \begin{Bmatrix} \mathbf{u}_{L1} \\ \mathbf{u}_{L2} \end{Bmatrix}$$

$$\boldsymbol{\phi} = \begin{Bmatrix} \boldsymbol{\phi}_1 \\ \mathbf{0} \end{Bmatrix} \quad \boldsymbol{\phi}_R = \begin{Bmatrix} \boldsymbol{\phi}_{R1} \\ 0 \end{Bmatrix} \quad \boldsymbol{\phi}_L = \begin{Bmatrix} \boldsymbol{\phi}_{L1} \\ 0 \end{Bmatrix}$$

$$\mathbf{K} = \begin{Bmatrix} \mathbf{K}_{11} \\ \mathbf{K}_{21} \end{Bmatrix} \quad \mathbf{V}_R = \begin{Bmatrix} \mathbf{V}_{R1} \\ \mathbf{V}_{R2} \end{Bmatrix} \quad \mathbf{V}_L = \begin{Bmatrix} \mathbf{V}_{L1} \\ \mathbf{V}_{L2} \end{Bmatrix}$$

$$\mathbf{Z} = \begin{Bmatrix} \mathbf{Z}_{11} \\ \mathbf{Z}_{21} \end{Bmatrix}$$

the (22) gives

$$\mathbf{u}_1^T\mathbf{K}_{11}\boldsymbol{\phi}_1 + \mathbf{u}_2^T\mathbf{K}_{21}\boldsymbol{\phi}_1 = f^T\mathbf{Z}_{11}\boldsymbol{\phi}_1 + f^T\mathbf{Z}_{21}\boldsymbol{\phi}_1 +$$

$$(23) \qquad - \mathbf{u}_{R1}^T\mathbf{V}_{R1}\boldsymbol{\phi}_{R1} - \mathbf{u}_{R2}^T\mathbf{V}_{R2}\boldsymbol{\phi}_{R1} +$$

$$+ \mathbf{u}_{L1}^T\mathbf{V}_{L1}\boldsymbol{\phi}_{L1} + \mathbf{u}_{L2}^T\mathbf{V}_{L2}\boldsymbol{\phi}_{L1} + \mathbf{u}_0^T\mathbf{E}\boldsymbol{\phi}_0$$

Considering that $\phi$ is any test function and taking the time-independent terms at the right hand side, we obtain

$$\mathbf{K}_{21}^T\mathbf{u}_2 + \mathbf{V}_{L2}^T\mathbf{u}_{R2} - \mathbf{V}_{L2}^T\mathbf{u}_{L2} = -\mathbf{K}_{11}^T\mathbf{u}_1 + \mathbf{Z}_{11}^T f + \mathbf{Z}_{21}^T f -$$

$$(24) \qquad - \mathbf{V}_{R1}^T\mathbf{u}_{R1} + \mathbf{V}_{L1}^T\mathbf{u}_{L1} + \mathbf{E}^T\mathbf{u}_1$$

This is the equation of the iterative process.

It is a system of $E$ equations in $E$ unknowns, where $E$ is the number of the finite elements; the unknown parameter of this system is the vector $\mathbf{u}_2$ and the temperature values contained in the right hand side are that computed at preceding time slab.

If $L$ time slabs are given, to determine the temperature at the time $T$, $L$-systems as (23) are evaluated by the program.

## 8. Example.

The Discrete Finite Element Method for linear heat conduction problems is implemented on VAX 8650 in Fortran 77.

In the program for solving one-dimentional linear heat conduction problems, bilinear rectangular finite elements in the space-time with Lagrange interpolation polinomial are used.

In the program for solving two-dimentional linear heat conduction problems bilinear prismatic finite elements are used.

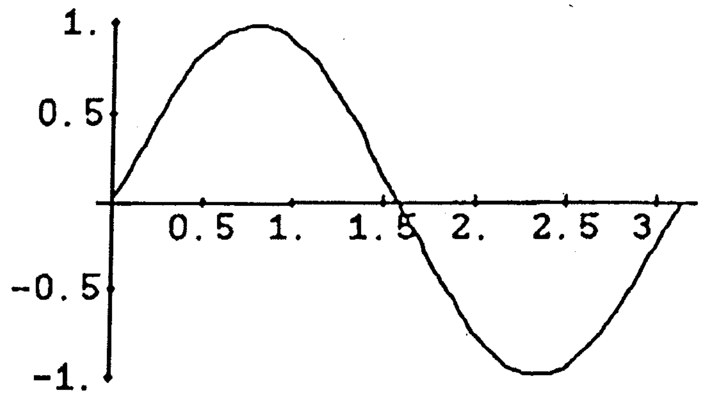
The results presented for same one-dimentional and two-dimentional test problems are plotted.

EXAMPLE 1.

$$2u_{xx}(x,t) - u_t(x,t) = \sum n^{-2} e^{-nt} \sin(nx) \qquad u \in [0,1] \times [0,1]$$

$$u(0,t) = u(1,t) = 0 \qquad 0 < t < 1$$

$$u(x,0) = \sin(2x) \qquad 0 < x < 1$$




EXAMPLE 2.

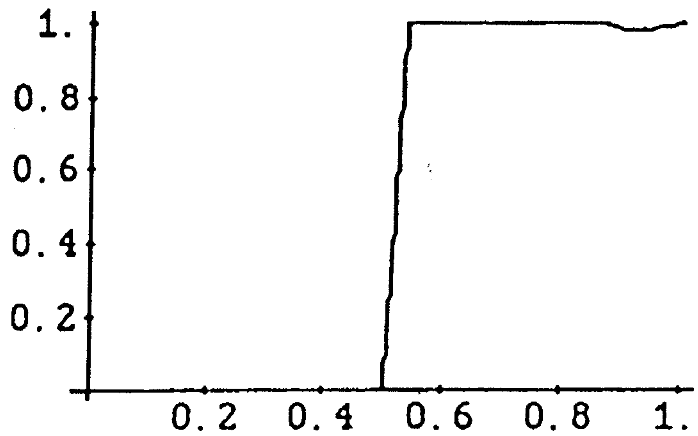
$$10^{-8} u_{xx}(x,t) = u_t(x,t) \qquad x \in [0,1]\ t \in [0,1]$$

$$u(0,t) = 0. \qquad 0 < t < 1$$

$$u(1,t) = 1. \qquad 0 < t < 1$$

$$u(x,t) = 0. \qquad 0 < x < .5$$

$$u(x,1) = 1. \qquad .5 < x < 1$$

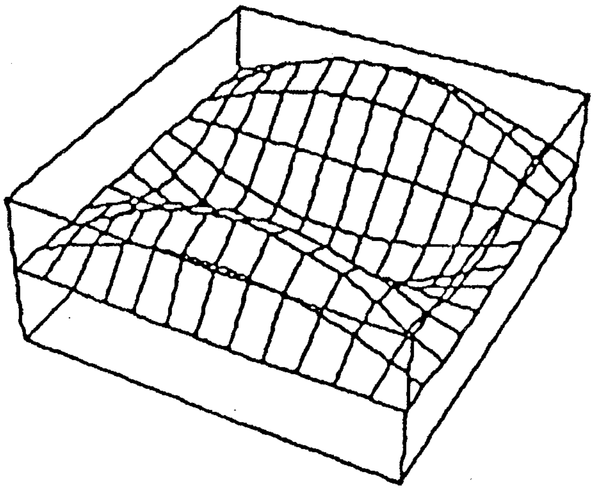


EXAMPLE 3.

$$u_{xx}(x,y,t) + u_{yy}(x,y,t) = u_t(x,y,t) \quad (x,y) \in \Omega \; t \in [0,1]$$

$$u(x,y,t) = 0 \quad (x,y) \in \partial\Omega \; 0 < t < 1$$

$$u(x,y,0) = 5 \sin x \sin 3y \quad (x,y) \in \Omega$$



## 9. Acknowledgments.

This work has been done with the support of CNR undo the contract CN92.00530.01 and 40% MURST on the Roject "Analisi Numerica e matematica Conputazionale".

REFERENCES

[1] Babuska I., *Error-Bounds for Finite Element Method* in Numer. Math., Vol **16**, (1971) 322-333.

[2] Bramble J., Hilbert S., *Bounds for a class of linear functionals with applications to Herrnite interpolation* in Numer. Math. Vol. **16**, (1971) 362-339.

[3] Brezis H., *Analisi Funzionale* - Liguori Editore - Napoli,1986.

[4] Cea J., *Approximation variationelle des problemes aux limites* in Ann. Inst. Fourier, Vol **14**, (1964) 345-444.

[5] Cella A., Lucchesi M., Pasquinelli G., *Space-Time elements for the shock wave propagation problem* in Int. Jour. for Numerical Methods in Engineering, Vol **15**, (1980) 1475-1488.

[6] Cella L., Lucchesi M., *Space-time finite elements for wave propagation problem* Meccanica, Vol 10, (1975) pp. 168-17.

[7] Ciarlet P.G., *The finite element method for elliptic problem* North-Holland, Amsterdam, 1978.

[8] Ciarlet P.G., Raviart P.A., *General Lagrange and Hermite Interpolation in $R^n$ with Applications to Finite Element Methods* in Arch. Rational Mech. Anal., Vol. **46**, (1972) 177-199.

[9] Johnson L.W., Riess R.D., *Numerical Analysis* Addison-Wesley, Sydney, 1982.

[10] Lions J.L., *Equations differentielles operationelles et Problemes aux Limites* Springer, Berlin, 1961.

[11] Morandi Cecchi M., Cella A., *An extended theory for the finite element method* in Variational Methods in Engineering, Vol. 1, eds. A. Brebbia and H. Tottenham, Southampton University Press, 1973.

[12] Morandi Cecchi M., Cella A., *A Ritz-Galerkin approach to heat conduction method and results* Proceedings 4th CAMCAM Conf., Ecole Polytechnique, Montreal, 1973, Session H, pp. 767-768.

[13] Morandi Cecchi M., *Error estimates for Finite Element solutions of heat trasfer problems in space-time domain* in The Math of finite Elements and Applications, MAFELAP, Ed. J. R. Whiteman, 1975, pp. 201-208.

[14] Morandi Cecchi M., Nociforo R., *Alcuni problemi parabolici mediante il metodo degli elementi finiti* in Atti del Convegno Nazionale di Analisi Numerica, Roma, 26-28 settembre 1988, pp. 331-341.

[15] Vemuri V, Karplus W., *Digital Computer Treatment of Partial Differential Equations* Prentice-Hall, New Jersey, 1981.

[16] Zienkiewicz O.C., *The Finite Element Method* MacGraw-Hill, London, 1988.

*Maria Morandi Cecchi*
*Dipartimento di Matematica Pura ed Applicata*
*Universita' di Padova*
*Via Belzoni, 7*
*35100 PADOVA ITALY*

*Rosaria Nociforo*
*Dipartimento di Matematica*
*Citta' Universitaria*
*Universita' di Catania*
*Viale Andrea Doria, 6*
*95125 CATANIA ITALY*