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ON CERTAIN TRANSFORMATIONS OF TRIPIE q-SERIES

R.K. SAXENA - R.K. GUPTA (Jodhpur)(*)

Certain transformations involving the Lauricella functions of three variables are given by Saran [4,5] and Exton [2]. The object of this paper is to derive the basic analogue of some of these transformations.

1. Introduction and preliminaries.

In a recent paper the authors [6] have established a number of transformations for the basic hypergeometric function of three variables. In the present paper we derive three new transformation of triple q-series which provide the basic analogue of certain known transformations of the hypergeometric function of three variables due to Saran [4,5] and Exton [2].

The results established are of general character and give rise to known as well as unknown transformations for the ordinary triple hypergeometric functions. Some of the interesting special cases are pointed out.

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For α real or complex and $|q| < 1$, let us take

$$[\alpha; q]_n = [q^\alpha; q]_n = (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}), [\alpha; q]_0 = 1.$$

We define a basic hypergeometric function in the following form:

$$\phi\begin{bmatrix} a_1, & a_2, \dots, a_r; & q; & x \\ b_1, & b_2, \dots, b_s; & q; & q^\lambda \end{bmatrix} \tag{1}$$
$$= \sum_{n\geq 0} \frac{[a_1; q]_n [a_2; q]_n \dots [a_r; q]_n x^n q^{\lambda n(n+1)/2}}{[b_1; q]_n [b_2; q]_n \dots [b_s; q]_n [q; q]_n}$$

valid for either $\lambda > 0$ or $|x| < 1$ when $\lambda = 0$.

The following results will be required in the sequel.

$$ {}_2\phi_1\begin{bmatrix} \lambda, & \alpha; & & \\ & & q; & x \\ \beta; & & & \end{bmatrix} = \frac{[\lambda x; q]_\infty}{[x; q]_\infty}\, {}_2\phi_2\begin{bmatrix} \lambda, & \beta/\alpha; & q; & (-\alpha x) \\ & \lambda x, & \beta; & q \end{bmatrix} \tag{2}$$

(Jackson [3]) where the symbol $[\alpha; q]_\infty$ stands for $\prod_{n=0}^{\infty} (1-\alpha q^n)$.

Also, we need the following q-analogue of binomial theorem

$$ {}_1\phi_0[a; -; q; z] = \frac{[az; q]_\infty}{[z; q]_\infty} \tag{3}$$

(Slater [7; IV II], p. 248)

In what follows the other notations and symbols employed in this paper have their usual meanings.

2. Transformations.

The following transformations for the triple basic series will be established here.

First Transformation.

$$\sum_{m,n,p\geq 0} \frac{[\alpha; q]_m \left[\frac{\nu}{\alpha}; q\right]_{n+p} [\beta; q]_{m+p} [\beta'; q]_n x^m y^n z^p}{[\nu; q]_{m+n+p} [q; q]_m [q; q]_n [q; q]_p} \tag{1}$$

$$= \frac{[\beta' y;q]_\infty [\beta z;q]_\infty}{[y;q]_\infty [z;q]_\infty} \sum_{m,n,p\geq 0} \frac{[\alpha;q]_{m+n+p}[\beta;q]_{m+p} x^m}{[\nu;q]_{m+n+p}[q;q]_m}$$

$$\cdot \frac{[\beta';q]_n (-cy/a_1)^n (-czq^n/a_1)^p q^{n(n+1)/2} q^{p(p+1)/2}}{[\beta z;q]_{m+p}[\beta' y;q]_n [q;q]_n [q;q]_p}$$

Second Transformation.

(2)
$$\sum_{m,n,p\geq 0} \frac{[\alpha;q]_{m+n+p}[\beta;q]_{m+p}[\beta';q]_n x^m y^n z^p}{[\alpha;q]_m[\alpha;q]_n[\alpha;q]_p[q;q]_m[q;q]_n[q;q]_p}$$

$$= \frac{[\alpha y;q]_\infty}{[y;q]_\infty} \sum_{m,n,p\geq 0} \frac{[\alpha;q]_{m+n+p}[\beta;q]_{m+p}}{[\alpha;q]_m[\alpha;q]_n[\alpha;q]_p}$$

$$\cdot \frac{[\alpha/\beta';q]_n q^{n(n+1)/2}(-\alpha q^{m+p})^n x^m z^p}{[\alpha y;q]_{m+n+p}[q;q]_m[q;q]_n[q;q]_p}$$

Third Transformation.

(3)
$$\sum_{m,n,p\geq 0} \frac{[\alpha;q]_{m+n+p}[\beta;q]_m[\beta';q]_{n+p} x^m y^n z^p}{[\nu;q]_m[\beta';q]_n[\beta';q]_p[q;q]_m[q;q]_n[q;q]_p}$$

$$= \frac{[\alpha z;q]_\infty}{[z;q]_\infty} \sum_{m,n,p\geq 0} \frac{[\alpha;q]_{m+n+2p}}{[\nu;q]_m}$$

$$\cdot \frac{[\beta;q]_m[\beta';q]_{n+p}(\beta' zy)^p q^{p^2} x^m y^n}{[\alpha z;q]_{m+n+2p}[\beta';q]_p[\beta';q]_{n+p}[q;q]_m[q;q]_n[q;q]_p}$$

Proof. of (2.1):

To prove (2.1) we see that the L.H.S. of (2.1) can be written as

$$\sum_{m,n\geq 0} \frac{[\alpha;q]_m \left[\frac{\nu}{\alpha};q\right]_n [\beta;q]_m[\beta';q]_n}{[\nu;q]_{m+n}[q;q]_m[q;q]_n} {}_2\phi_1 \left[\begin{matrix} \nu q^n/\alpha, & \beta q^m; & \\ & & q;z \\ & \nu q^{m+n}; & \end{matrix}\right]$$

Now, if we apply the transformation (1.2) and replace c, a, b

and x by $\beta q^m, \nu q^n/\alpha$, νq^{m+n} and z respectively, then we obtain

$$\frac{[\beta z;q]_\infty}{[z;q]_\beta}\sum_{m,n\geq 0}\frac{[\alpha;q]_{m+p}\left[\frac{\nu}{\alpha};q\right]_n[\beta;q]_{m+p}}{[\nu;q]_{m+n+p}[q;q]_p}$$
$$\cdot\frac{[\beta';q]_n(-\nu z/\alpha)^p q^{np}q^{p(n+1)/2}}{[q;q]_m[q;q]_n[\beta z;q]_{m+p}}$$

The result (2.1) now follows by again using the transformation (1.2) with proper choice of a, b, c and x. Proof of (2.2) can be developed in the same way.

Proof. (2.3) To prove (2.3), we write its L.H.S. as

$$\sum_{m,n\geq 0}\frac{[\alpha;q]_{m+n}[\beta;q]_m[\beta';q]_n x^m y^n}{[\nu;q]_m[\beta';q]_n[q;q]_m[q;q]_n}\,{}_2\phi_1\left[\begin{matrix}\alpha q^{m+n}, & \beta' q^n; & \\ & & q;\quad z\\ \beta'; & & \end{matrix}\right]$$

Now, transforming the ${}_2\phi_1$ with the help of (1.2) with proper choice of a, b, c and x replacing n by $n+p$, the R.H.S. is obtained.

3. Special cases.

It is interesting to observe that the transformations (2.1) to (2.3) provide q-analogues of important transformations of ordinary hypergeometric functions.

As $q\to 1$ in (2.1) to (2.3), we obtain the following known transformations given by Saran [4] and Exton [2].

(1)
$$F_T[\alpha,\nu-\alpha,\nu-\alpha,\beta,\beta',\beta;\nu,\nu,\nu;x,y,z]$$
$$=(1-y)^{-\beta'}(1-z)^{-\beta}F_1[\alpha,\beta,\beta';\nu;(x-z)/(1-z)/(1-z),y/(y-1)]$$

(Saran [4; eq. 6.5]; p. 90)

(2)
$$F_F[\alpha,\alpha,\alpha,\beta,\beta',\beta;\alpha,\alpha,\alpha;x,y,z]$$
$$=(1-x-z)^{-\beta}(1-y)^{-\beta'}H_3[\beta,\beta';\alpha;xz/(1-x-z)^2,xy/(1-y)]$$

(Exton [2; 4.1.17]; p. 116).

$$
\begin{aligned}
&F_E[\alpha,\alpha,\alpha,\beta,\beta',\beta;\nu,\beta',\beta';x,y,z] \\
&\quad = (1-y-z)^{-\alpha} H_4[\alpha,\beta;\beta';\nu;yz/(1-y-z)^2, x/(1-y-z)]
\end{aligned} \tag{3}
$$

(Exton [2; 4.1.16]; p. 116).

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Department of Mathematics and Statistics
University of Jodhpur
Jodhpur-342001 (Raj.)
INDIA