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ON TWO CONJECTURES TO GENERALIZE VIZING'S THEOREM

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Vizing's Theorem states that for a simple graph G, the chromatic index $q(G)$ is equal to the maximum degree $\Delta(G)$ or to $\Delta(G) + 1$. To extend this theorem to some classes of hypergraphs, we suggested two conjectures, non-comparable, but, in some sense, dual, which are discussed in the present paper [1].

1. Basic definitions.

A *hypergraph* on a finite set X is a family $H = (E_1, E_2, \ldots, E_m)$ of non-empty subsets («the edges») whose union is X («the vertices»). An *intersecting family* of H is a family of pairwise intersecting edges, and we denote by $\Delta_0(H)$ the maximum cardinality of an intersecting family.

A *star* is a family of edges having a common vertes x (called a *center* of the star). For instance, if H is a graph or a multigraph, an intersecting family is either a star or a triangle (with or without multiple edges). We denote by $d_H(x)$ the *degree* of x, i.e. the number of edges containing x; we put $\Delta(H) = \max d_H(x)$.

The *chromatic index* $q(H)$ (or «edge-chromatic number») of a hypergraph H is the least number of colors needed to color the edges so that no two intersecting edges have the same color. Clearly,

$$q(H) \geq \Delta_0(H) \geq \Delta(H).$$

A hypergraph H with $q(H) = \Delta(H)$ is said to have the *edge coloring property.*

The bipartite graphs have the edge-coloring property by König's Theorem, but also the interval hypergraphs (whose vertices are points on a line, and whose edges are intervals of the line), by Gallai's Theorem. The complete graph K_{2p} with an even number of vertices has the edge coloring property (by a theorem of Lucas), but also the complete 3-uniform hypergraph K^3_{3p} (R. Peltesohn [1933]). More generally, Baranyai [1] has shown that K^r_n has the edge coloring property if and only if n is divisible by r. A similar theorem deals with the *complete* r*-partite hypergraph* $Kr_{n_1,n_2,\dots,n_r}$ (which generalizes the complete bipartite graph K_{n_1,n_2}): its vertex-set $X = X^1 \cup X^2 \cup \dots \cup X^r$ is the union of disjoint sets X^i with $|X^i| = n_i$, and its edge-set is the family of all subsets $E \subset X$ with $|E \cap X^i| = 1$ for $i = 1, 2, \dots, r$.

We have shown in [2] that the complete r-partie hypergraphs have the edge coloring property.

Let $H = (E_1, E_2, \dots, E_m)$ be a hypergraph on X. And let k be a positive integer. The *«strict» k-section* is a hypergraph $H_{(k)}$ on X defined by all the subsets F of X such that

(i) $1 \leq |F| \leq k$,

(ii) $F \subset E_i$ for some i.

So, $H_{(2)}$ is a graph (with a loop attached to each vertex). The *hereditary closure* is the hypergraph $\hat{H}$ defined by all the non-empty sets F satiosfying (ii).

A hypergraph H such that $H = \hat{H}$ is *hereditary.*

An important conjecture is:

Chvátal's conjecture: *Every hypergraph* H *satisfies*

$$\Delta_0(\hat{H}) = \Delta(\hat{H}).$$

Schönheim [12] proved that if H is a star, then $\Delta_0(\hat{H}) = \Delta(\hat{H})$. We can prove more:

THEOREM 1. *Let H be a star. Then $q(\hat{H}) = \Delta(\hat{H})$.*

Proof. Let $x_0 \in X$ be a center of the star H on X. We shall show, by induction on $|\hat{H}|$, that one can color the edges of $\hat{H}$ with a number of colors equal to the degree of x_0 in $\hat{H}$.

Let A be a maximal subset of X such that $A = E \cup F$, $E, F \in \hat{H}$. Clearly, $x_0 \in A$. Put

$$\mathcal{B} = (E | E \in \hat{H},\ E \cup F = A \text{ for some } F \in \hat{H}).$$

Case 1: $\hat{H} = \mathcal{B}$

It is possible to color the edges of $\mathcal{B}$ with $\frac{1}{2}|\mathcal{B}|$ colors, using the same color twice (for E and for $A - E$). Hence one can color the edges of $\hat{H}$ with a number of colors equal to the degree of x_0, and the proof is achieved.

Case 2. $\hat{H} \neq \mathcal{B}$

We first show that $\hat{H} - \mathcal{B}$ *is a hereditary hypergraph.* Let $E \in \hat{H} - \mathcal{B}$, and let $E' \subset E$. Since $E' \in \hat{H}$, we have only to show that $E' \notin \mathcal{B}$. Otherwise, there exists some $F' \in \hat{H}$ such that $E' \cup F' = A$; by the maximality of A, we have $E \cup F' = A$, and therefore $E \in \mathcal{B}$: contradiction. Now, we show that the maximal edges of $\hat{H} - \mathcal{B}$ *constitute a star with x_0 as a center.* Otherwise there exists a maximal edge E of $\hat{H} - \mathcal{B}$ with $E \not\ni x_0$. Since $E \cup \{x_0\} = E_0 \in \hat{H}$, and since $E_0 \notin \hat{H} - \mathcal{B}$ (because of the maximality of E), we have $E_0 \in \mathcal{B}$, and there exists some $F_0 \in \hat{H}$ with $E_0 \cup F_0 = A$. Hence

$$E \cup (F_0 \cup \{x_0\}) = A$$

$$(F_0 \cup \{x_0\}) \in \hat{H}.$$

So, $E \in \mathcal{B}$: contradiction.

By the induction hypothesis, we can color the edges of $\hat{H} - \mathcal{B}$ with $d_{\hat{H}-\mathcal{B}}(x_0)$ colors; and we have seen (case 1) that one can color the edges of $\mathcal{B}$ with $d_{\mathcal{B}}(x_0)$ colors. Hence

$$\Delta(\hat{H}) \leq q(\hat{H}) \leq d_{\hat{H}-\mathcal{B}}(x_0) + d_{\mathcal{B}}(x_0) = d_{\hat{H}}(x_0) \leq \Delta(\hat{H})$$

This shows that $q(\hat{H}) = \Delta(\hat{H})$. ∎

In particular, the family of all the subsets of X has the edge-coloring property.

For other cases where the Chvátal conjecture holds true, see [1], [2], [4], [5], [6], [9], [12], [14], [15], [16], [17].

2. The two conjectures.

Vizing has shown that if G is a simple graph (no loops, no multiple edges), then the chromatic index is either $\Delta(G)$ or $\Delta(G)+1$. Since the hereditary closure $\hat{G}$ has one loop at each vertex, Vizing's Theorem is equivalent to the edge-coloring property for $\hat{G}$, i.e.:

$$q(\hat{G}) = \Delta(G) + 1 = \Delta(\hat{G})$$

An interesting extension of the concept of a «simple graph» is the concept of «linear hypergraph». A hypergraph $H = (E_1, E_2, \dots, E_m)$ is *linear* if no two edges intersect in more that one point, i.e;

$$|E_i \cap E_j| \leq 1 \qquad (i \neq j)$$

A linear hypergraph H can have *loops* (i.e. edges with cardinality 1), and a loop $\{x\}$ can be repeated several times. Clearly, every sub-hypergraph of H is linear.

All finite projective planes, all Steiner triple systems, and more generally all Steiner systems $S(2,k,n)$ are linear hypergraphs.

Since the properties of linear hypergraphs are in many respect similar to those of simple graphs, we suggested several years ago the following conjecture:

Conjecture A: *Every linear hypergraph H satisfies*

$$q(\hat{H}) = \Delta(\hat{H}).$$

A weaker statements is: *every linear hypergraph H satisfies* $\Delta_0(\hat{H}) = \Delta(\hat{H})$.

This has been proved by Sterboul [16] for H uniform and linear, and by Stein [15] for every linear H. The conjecture A can be proved only in a few particular cases.

EXAMPLE: Let H be the lines of a finite projective plane with seven points, denoted $a, b, c, \ldots, g$. We have $\Delta(\hat{H}) = 3 + 6 + 1 = 10$. The edges of $\hat{H}$ can be colored with 10 colors denoted (1), (2),...,(10), as follows:

$(1):$	abc, de, f, g	$(6):$	cde, ab, fg
$(2):$	adg, ef, bc	$(7):$	cfg, ad, be
$(3):$	aef, cd, bg	$(8):$	ag, cf, bd, e
$(4):$	bdf, cg, ae	$(9):$	af, ce, dg, b
$(5):$	beg, ac, df	$(10):$	eg, bf, a, c, d

Now, let us consider a similar conjecture, which was stated independently by Meyniel (unpublished), Berge [4], Füredi [10], and was anticipated by Colbourn and Colbourn [7] for the Steiner triple systems:

Conjecture B: *Every linear hypergraph H with no repeated loop satisfied* $q(H) \leq \Delta(H_{(2)})$

Clearly, if H is a simple graph, we have $\Delta(H_{(2)}) = \Delta(H) + 1$, and consequently the conjecture B reduces to Vizing's Theorem.

A weaker form of the conjecture B which was proved by Füredi [10] is: Every linear hypergraph H satisfies $\Delta_0(H) \leq \Delta(H_{(2)})$.

The conjecture B itself seems to be difficult, and can be proved in very few cases, as the following:

THEOREM 2. *Let H be an r-uniform linear hypergraph with maximum degree $\Delta(H) \leq r$. Then*

$$q(H) \leq \Delta(H_{(2)}) \tag{1}$$

Furthermore, a connected H satisfies (1) with equality if and only if H is either a finite projective plane or a graph isomorphic to an odd cycle.

Proof. If every edge of H is of cardinality $\leq r$, i.e. if H is of «rank» r, the degree of any vertex e in the line-graph $L(H)$ is at most $r(\Delta(H)-1)$, because of the linearity. Put $\Delta(H)=\Delta$, and apply Brooks' Theorem to the line-graph. Then

$$q(H)=\gamma[L(H)] \leq \Delta(L(H)) \leq r(\Delta-1)=r\Delta-r \leq$$

$$\leq \Delta(r-1)=\Delta(H_{(2)}).$$

The inequality (1) follows.

If (1) holds with equality, we get a system of equalities which imply $r=\Delta$ and $\gamma(L(H))=\Delta[L(H)]$. Since $L(H)$ is a connected graph (if H is connected), Brooks' theorem asserts that the graph $L(H)$ is either a clique K_n, or an odd cycle; so, H is either a finite projective plane with n points, or an odd cycle C_n.

The conjecture B can also be considered for the «dual» of H.

The *dual* of a hypergraph $H=(E_1,E_2,\dots,E_m)$ on $X=\{x_1,x_2,\dots,x_n\}$ is a hypergraph $H^*=(X_1,X_2,\dots,X_n)$ whose vertices $e_1,e_2,\dots,e_m$ correspond respectively to the edges $E_1,E_2,\dots,E_m$ of H, and whose edges are

$$X_i=\{e_j \mid E_j \ni x_i \text{ in } H\}$$

Clearly, H^* is a hypergraph on $\{e_1,e_2,\dots,e_m\}$, and the incidence matrix of H^* is the transpose of the incidence matrix of H. We have $(H^*)^* \simeq H$.

The rank $r(H^*)=\max\limits_{E\in H^*}|E|$ is equal to $\Delta(H)$, and a *loop* in H^* corresponds to *a vertex of degree* 1 in H. The dual H^* is *regular* («every vertex has the same degree») if and only if H is *uniform* («every edge has the same cardinality»).

Also:

PROPOSITION. *The dual H^* of a linear hypergraph H is linear.*

Let H be a linear hypergraph on $X=\{x_1,x_2,\dots,x_n\}$. Two edges X_i and X_j of H^* cannot have two distinct vertices e_1 and e_2 in their intersection, because, in H, the edges E_1 and E_2 would both contain $\{x_1,x_2\}$, and this contradicts the linearity of H.

Note also that the line graph $L(H)$ of a hypergraph H is the 2-section of H^* (with all loops removed). Thus, $\Delta(H^*_{(2)}) = \Delta(L(H)) + 1$, and the conjecture B is equivalent to:

Conjecture B': *Let H be a linear hypergraph with no edge containing more than one vertex of degree 1, and let $\gamma(H)$ denote its «strong chromatic number» (i.e. the least number of colors needed to color the vertices so that no two vertices with the same color are in the same edge). Then $\gamma(H) \leq \Delta(L(H)) + 1$.*

A weaker conjecture was proposed by Erdös, Farber and Lovász during the Hypergraph Seminar in Columbus, Ohio (see Erdös [8]).

Conjecture of Erdös - Farber - Lovász: *If H is a linear hypergraph consisting of m edges with cardinality m, then it is possible to color the vertices with m colors so that no two vertices with the same color are contained in the same edge.*

This is an easy consequence of conjecture B'; since $\Delta(L(H)) \leq m-1$, the validity of the conjecture B' implies $\gamma(H) \leq m$;

The conjecture of Erdos - Farber - Lovász was proved up to $m = 10$ by computer search (Hindman [11]), and also for the dual of cyclic Steiner System (Colbourn and Colbourn [7]).

Remark 1 We do not even know if the conjecture B holds true when a few edges have cardinality 3, all the other edges having cardinality 2. For such a hypergraph H, only two cases can occur.

Case 1. $\Delta(\hat{H}) = \Delta(H_{(2)})$. Then the conjecture A is stronger than the conjecture B, because the validity of the conjecture A for H implies:

$$q(H) \leq q(\hat{H}) = \Delta(\hat{H}) = \Delta(H_{(2)})$$

Case 2. $\Delta(\hat{H}) = \Delta(H_{(2)})+1$. Then the conjecture B is stronger than the conjecture A, because the validity of the conjecture B implies:

$$q(\hat{H}) \leq q(H_{(2)}) + 1 = \Delta(H_{(2)}) + 1 = \Delta(\hat{H}) \leq q(\hat{H})$$

Remark 2. For any linear hypergraph H, the conjecture B holds true either for H or for its dual H^*.

In fact, only two cases can occur.

Case 1. $q(H) \leq \gamma(H)$. Then, from Brooks Theorem, $q(H) \leq \gamma(H) \leq \Delta(H_{(2)})$, and the conjecture B holds true for H.

Case 2. $\gamma(H) \leq q(H)$. Then $q(H^*) \leq \gamma(H^*)$, and, as before, we see that conjecture B holds true for H^*

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