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LINEAR SPACES: HISTORY AND THEORY

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Linear spaces belong to the most fundamental geometric and combinatorical structures. In this paper I would like to give an overview about one of the most important aspects of the theory of linear spaces, namely the problem of embedding finite linear spaces in finite projective planes. I shall not only present some of the known results but shall also try to indicate some of the (in my opinion) interesting open questions.

1. The Past.

A *linear spaces* S consists of a set P of *points*, a set L of *lines* and a symmetric *incidence relation* $I \subseteq P \times L$ satisfying the following two simple axioms:

(L1) *Any two distinct points are incident with precisely one common line.*

(L2) *Any line is incident with at least two points.*

There is an abundance of examples of linear spaces: projective planes, projective spaces, affine planes, affine spaces, the euclidean plane, the hyperbolic plane, complete graphs, Steiner triple systems, $2-(v, k, 1)$ esigns – in short: every geometer and every combinatorialist works every day with linear spaces.

But probably people thought for a long time that these structures are too weak to be considered in their own right. So it was only in 1964 when Libois [13] coined the name «espace linéaire». In the meantime, this name has been smuggled into different languages: *linear space, spazio lineare, linearer Raum.*

Here, a little philological remark is in order. All these names want to suggest that the structures under consideration are spaces endowed with *lines*. Thus, from a philological point of view expression like *line space, spazio di rette, Geradenraum* (or *Inzidenzraum*) would have been better choices. One might complain about this, but the established name in now *linear space* – if one likes it or not.

But linear spaces have been studied long before they were baptized. I would like to describe here in particulat two results.

1.1. M. Hall's free extension process.

One of the papers which is fundamental for the modern theory of projective planes is M. Hall's famous 1943 paper [11].

A *projective plane* is a linear space satisfying the following two additional axioms:

(P1) *Any two distinct lines are incident with a (necessarily unique) common point.*

(P2) *There exist four points, no three of which are incident with a common line.*

M. Hall proved – among many other things – that any linear space **S** can be embedded into a projective plane. Let us recall this process. Let **S** be a linear space containing four points, no three of which are collinear. (Such a linear space is called *non-degenerate*).

If **S** is already a projective plane, then there is nothing to show.

If $S = \mathbf{S}_0$ is not a projective plane, then there are pairs of lines which have no point in common. To any such pair we adjoin a *new point* being incident with precisely those two lines.

So we get a structure $\mathbf{S}_0'$ which has the property that any two lines meet uniquely. But there are now pairs of points which have no line in common. (Otherwise S_0' would be a projective plane, but any new point is on just two lines, which is impossible in a projective plane).

Now we join any two points P, Q which are not on a common line by the *new line* $\{P, Q\}$. So we obtain a new linear space $\mathbf{S}_1$ which has pairs of non-intersecting lines.

Repeating this process we get a series

$$\mathbf{S} = \mathbf{S}_0 \subseteq \mathbf{S}_1 \subseteq \mathbf{S}_2 \subseteq \dots$$

of linear spaces, none of which is a projective plane. Let now **P** be the union of all these (infinitely many) linear spaces. *Then* **P** *is a projective plane*. (For any two points P, Q of **P** are contained in some $\mathbf{S}_i$, so they are joined in $\mathbf{S}_{i+1}$; dually, any two lines are contained in some $\mathbf{S}_j$, so they intersect in any $\mathbf{S}_k$ with $k \geq j + 1$. Hence in **P** any two points are joined and any two lines meet; thus **P** is a projective plane).

To sum *up, any linear spaces* **S** *can be embedded in a projective plane*. However, if **S** is not itself a projective plane, then the above constructed projective plane **P** is always *infinite*, even if **S** is *finite*.

It is tempting to ask whether a finite linear space can be embedded in a *finite* projective plane. In fact, this is a well-known conjecture which probably goes back to Hall's paper (although it is not stated there):

CONJECTURE. *Any finite linear space is embeddable in a finite projective plane.*

One would even like to ask a more precise *question: Given a finite linear space* **S**. *What is the least order of a projective plane* **P** *such that* **S** *is embeddable into* **P**?

The above conjecture is the leitmotiv in the theory of finite linear spaces. If the conjecture should be true, then today we are still far apart from answering it completely, although we shall see that there have been obtained many interesting and deep results. In the following I would like to point out some key embedding results; finally I shall try to indicate why this conjecture is a very difficult one.

The result which is most quoted in the theory of linear spaces is

1.2. The Hanani-de Bruijn-Erdös Theorem.

From now on, we shall exclusively consider *finite* linear spaces. These are linear spaces with a finite number v of points and hence also a finite number b of lines. The Hanani-de Bruijn-Erdös theorem says that a non-trivial finite linear space has at least as many lines as points.

THEOREM [6]. *Let* **S** *be a finite linear space with* v *points and* $b > 1$ *lines. Then* $b \geq v$ *with equality if and only if* **S** *is a projective plane or a near-pencil.*

[A *near-pencil* is a linear space on v points which has a line with $v - 1$ points (all other lines having just two points)].

Therefore, in particular, linear spaces with the minimal number of lines are projective planes, so they are trivially embeddable in a finite projective plane. For the history of the Hanani-de Bruijn-Erdös theorem and in particular for the history of its proofs we refer to [22].

2. The Present.

I would like to present here two types of embedding results, global results and local results. By a *global* result I mean a theorem which has as its hypotheses only conditions on v and b. For instance, the result discussed in the preceding section. *If $b = v$, then the lienar space is embeddable* is a result of global type. A *local* result has as its hypotheses also conditions on the degrees of the points and the lines. (The *degree* of a point P is the number r_P of lines through it; dually, the *degree* of the line l is the number k_l of points on l). A typical result in this direction is the *Kuiper-Dembowski theorem* on semiaffine planes [7] which might our context be formulated as follows:

Kuiper-Dembowski theorem [7]. If a finite linear space **S** satisfies

$$r_p - k_l \leq 1 \text{ for any non-incident point-line pair } (P, l),$$

then **S** is embeddable in a finite projective plane.

Before going a little bit in details, I would like to mention that I want to deal here only with embeddings of proper linear spaces, that is only considering the structure of points and lines – not of planes. Embeddings of so-called *planar spaces* has also attracted much attention, the reader is referred to [3], [10], [17].

2.1. *Global Results.*

A deep generalization of the Hanani-de Bruijn-Erdös theorem was obtained by Erdös, Mullin, Sós, Stinson [9] and Metsch [18]. In order to formulate the result we need a definition.

Let **S** be a finite linear space with v points. Denote by n the uniquely defined positive integer satisfying

$$n^2 - n + 1 = (n-1)^2 + (n-1) + 1 < v \leq n^2 + n + 1.$$

We define the number $B(v)$ as follows:

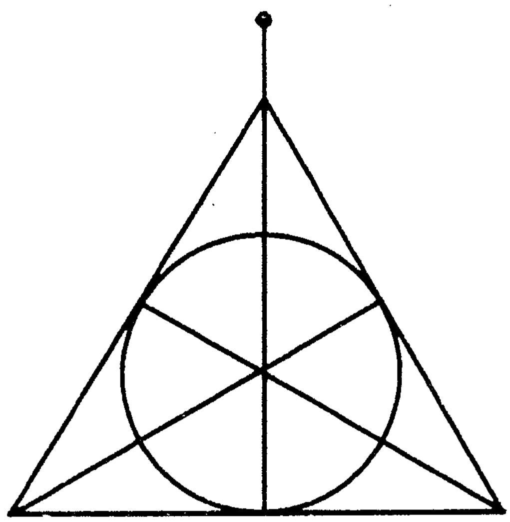
$$B(v) = \begin{cases} n^2 + n - 1, & \text{if } v = n^2 - n + 2 \neq 4 \\ n^2 + n, & \text{if } n^2 - n + 3 \leq v \leq n^2 + 1 \text{ or } v = 4 \\ n^2 + n + 1, & \text{if } n^2 + 2 \leq v. \end{cases}$$

Now the above mentioned result reads as follows.

THEOREM 2.1.1. [9], [18]. *Let* **S** *be a finite linear space with* v *points and* b *lines and let* $B(v)$ *be the above defined number. Then*

$$b \geq B(v).$$

Moreover, equality implies that either **S** *is embeddable in a projective plane of order* n*, or that* **S** *is the following exceptional linear space which satisfies* $n = 3$ *and is embeddable in the projective plane of order 4.*



An exceptional linear space with $v = 8$.
(Lines of size 2 are not drawn.)

Theorem 2.1.1. describes linear spaces with few lines. Now we turn to linear spaces with many points. Let **S** be a finite linear space and denote by $n+1$ the maximum point degree of **S**; the so-defined number n is called the *order* of **S**.

A relatively old but very useful result is

THEOREM 2.1.1. [21]. *Let* **S** *be a linear space of order* n*. If* $v \geq n^2$*, then* **S** *is embeddable in a projective plane of order* n*.*

The proof of 2.1.2. is relatively simple and consists of two steps.

First one has to establich the existence of a line of degree n. Then, any such line lies in a unique parallel class. One of the problems in proving the following results is that in general one cannot assume that there exists an n-line.

This theorem has attracted much attention and was generalized several times (see for instance [8]). The latest version reads as follwos.

THEOREM 2.1.3. [18]. *Let* **S** *be a linear space of order* n. *If* $v \geq n^2 - \frac{1}{2}n + 1$ *and* $n > 222$, *then* **S** *is embeddable in a projective plane of order* n.

Now we turn to linear spaces in which the maximum point degree is bigger than $n+1$. Here, the closed complements of a Baer subplanes are the most prominent examples. Let **P** be a projective plane of order n having a *Baer subplane* (that is a subplane **B** of order $\sqrt{n}$). Then any point of **P** outside **B** is on precisely one line of **B**. So, the lines of **B** form a parallel class of $P - B$. If $P - B$ is completed by one infinite point which is incident with exactly the lines of **B**, we obtain the so-called *closed complement of a Baer subplane*. These structures have been characterized in a very satisfactory way. The result should be compared with 2.1.3.

THEOREM 2.1.4. [16] *Let* **S** *be a linear space with* $b \leq n^2 + n + 1$ *and* $v > n^2 - \frac{1}{2}n + 1$. *If some point is on more than* $n+1$ *lines, then* **S** *can be embedded in the closed complement of a Baer subplane in a projective plane of order* n. *In particular,* n *is a perfect square.*

Finally we turn to the so-called *restricted* linear spaces; these are linear spaces satisfying $(b-v)^2 \leq v$. In the classification, inflated affine planes play an important role. A *completely projectively inflated affine plane* is an affine plane together with a (possibly degenerate) projective plane defined on all its points at infinity. The following theorem is a corollary of Totten's classification theorem.

THEOREM 2.1.5. [19]. *Let* **S** *be a restricted linear space. Then either* **S** *is embeddable (in a very natural way) in a finite projective plane or* **S** *is a completely inflated affine plane.*

As corollaries one gets the classification of all linear spaces satisfying $b = v + 1$ [5], $b = v + 2$ [23] and $b = v + 3$ [20].

We mention that recently, Metsch [18] has also determined all linear spaces satisfying $(b - v)^2 \leq b$.

2.2. Local Results.

We call a linear space a-*semiaffine*, if a is the maximum number of lines through a point P outside a line l which are disjoint to l. The semiaffine planes considered by Dembowski are just the 1-semiaffine linear spaces. Also 2-semiaffine linear spaces have been classified (see [12], [15]).

Notice that every finite linear spaces **S** is a-semiaffine for some positive integer a, namely the maximum value of $r_k - k_l$, where (P, l) is a non-incidnet point-line pair of **S**. A general asymtotic result is the following.

THEOREM 2.2.1. [4] *Let* **S** *be a finite a-semiaffine linear space of order n. If*

$$4n > 6a^4 + 9a^3 + 19a^2 + 8a,$$

then **S** *is embeddable in a projective plane of order n.*

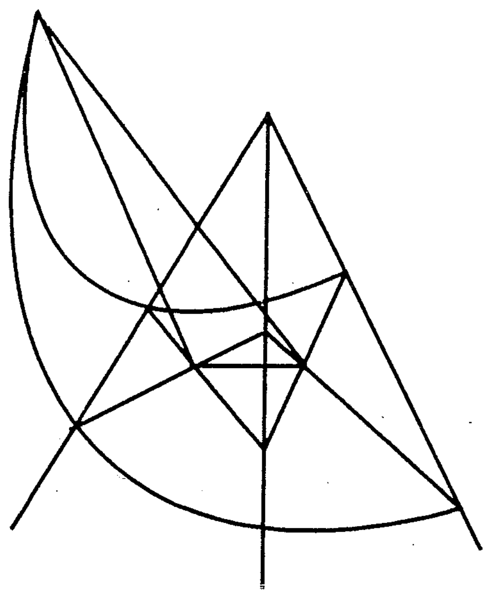
As corollaries one gets the classification of linear spaces with bounded line degree and not too many points ([1], [2]).

3. The Future.

3.1. What can we (not) hope for?

3.1.1. Desarguesian planes.

The nicest class of projective planes are the Desarguesian planes. Can one hope to embed any finite linear space in a Desarguesian plane? Clearly, the answer is «no», since for instance the linear spaces which is the «anti-Desargues-configuration» cannot be embedded in a Desarguesian plane.



The Anti-Desargues Configuration.

Of course, this configuration can be embedded in any non-Desarguesian plane. Hence the least order of a projective plane, the non-Desargues configuration can be embedded in, is 9.

Clearly, this argument generalizes. If a class C of projective planes is characterized by the validity of one of certain configurational theorems, then there is a finite linear space which cannot be embedded in any plane belonging to C.

3.1.2. Projective spaces.

Some people believe that, if a linear space **S** is not embeddable in a finite projective plane, it is perhaps embeddable in a finite projective space, since a space «has more freedom». This is, of course, not true.

Suppose that a linear space **S** is embeddable in a finite projective space $\Pi = PG(d, q)$. Denote by **P** any plane of Π. Then there exists a projective space $\Pi^* = PG(d, q^*)$ of order q^* with the following properties:

- Π is emdedded in Π^* (that is, q^* is a power of q),
- there exists a point X of Π^* such that
  - no line of **S** passes through X,
  - no plane which has at least two lines of **S** passes through X.

(The existence of X follows if the order q^* is big enough with respect to q, see for instance [14]).

Then the projective of **S** from X on **P** gives an embedding of **S** in the Desarguesian projective plane **P**.

To sum up, *if a linear space is embeddable in a finite projective space of dimension* $d \geq 3$, *then it is even embeddable in a Desarguesian projective plane. So, by 3.1.1, not every finite linear space is embeddable in a projective space of dimension* $d \geq 3$.

3.2. Can one do it by induction?

It is natural, but perhaps naive, to try to prove the conjecture by induction. The induction step would be as follows: A part $\mathbf{S}_1$ of **S** is embedded in a finite projective plane $\mathbf{P}_1$. *Problem*: Extend $\mathbf{P}_1$ to a bigger, but still finite, projective plane $\mathbf{P}_2$, in which a bigger part $\mathbf{S}_2$ of **S** is embedded.

However, it is not known in general whether a finite projective plane can be embedded in a bigger but still finite projective plane. This is only known for classes of relatively well understood projective planes. So, the question is: Can one hope to embed all finite linear spaces in finite projective planes belonging to a «well-behaved» class of projective planes?

3.2.1. The naive approach: Induction on v.

Of course, linear spaces with a small number v of points are embeddable.

Suppose now that **S** is a linear space with v points and assume that every linear space with $v-1$ points is embeddable.

Consider a point X of **S** and the lines $l_1, \ldots, l_r$ of **S** throught X.

By induction, $\mathbf{S}' = \mathbf{S} - X$ is embeddable in some finite projective plane **P**. It is clear that the lines $l_1, \ldots, l_r$ form a parallel class in the embedded $\mathbf{S}'$. But as lines of **P** they mutually intersect, and, in general, in *distinct points*. If all these lines would intersect each other in the same point, we would have an emebdding of **S**.

So, the *problem* is the following: Modify **P** in such a way that

- the lines $l_1, \ldots, l_r$ intersect in one common, point,
- $\mathbf{S}'$ remains unchanged,
- the modification is still a finite projective plane.

This seems to be an extremely difficult task.

3.2.2. A more sophisticated approach: Induction on $-b$.

Let me show you another argument which reduces the whole embedding problem to a silly assertion to be proved.

Any linear space **S** on v points has at most $v(v-1)/2$ lines with equality if and only if **S** is a complete graph.

Let's try to prove the conjecture by induction on the number $c := v(v-1)/2 - b$. If $c = 0$, then **S** is a complete graph. Take any Desarguesian projective plane **P** of order $\geq v-1$. Then any conic of **P** has at least v points, no three of which are collinear. This gives an embedding of **S**.

Let now **S** be a linear space with $c := v(v-1) - b > 0$ and suppose that the assertion is true for any linear space having $c' < c$.

Since $c > 0$, **S** is not a complete graph. So, **S** has a line l with $k \geq 3$ points. Now replace in **S** the line l by a near-pencil. We obtain a linear space $\mathbf{S}'$ on the same set of points having $b' = b + k - 1 > b$ lines. So, by induction, $\mathbf{S}'$ is embeddable in a projective plane.

It remains to show that also **S** is embeddable in a finite projective plane. This is left as homework for the reader.

3.3. Some of the linear spaces I would most like to see embedded.

Notice that most (all) embedding theorems stated in section 2 embed a linear space **S** of order n in a projective plane *of the same order* n!

I am convinced that the reason why we cannot solve the embedding conjecture is that we do not understand the embedding of linear spaces of order n in projective planes of an order m «unrelated» (or, at least, unequal) to n. Here are some challenges.

3.3.1. Inflated affine planes.

Let **A** be an affine plane with line of infinity l_∞. Let **S** be a linear space which is defined on some of the points of l_∞. Then the linear space whose points and lines are the points and lines of **A** and **S** is called an *affine plane inflated by* $\mathbf{S}_\infty$. If $\mathbf{S}_\infty$ is a near-pencil, it is called a *simply inflated affine plane*, and if $\mathbf{S}_\infty$ is a projective plane, it is called a *projectively inflated affine plane* [19].

Problem Embed an inflated affine plane in a finite projective plane.

The only example I know of is the (simply) inflated affine plane of order 2, alias a quadrangle with non-collinear diagonal points. This structure is embeddable in $PG(2,3)$, in fact, by the theorem of Gleason in any finite projective plane which is not $PG(2,2^a)$ for some positive integer a.

3.3.2. Closed Complements. Problem: Embed a closed complement of a Baer subplane of a finite projective plane in a finite projective plane.

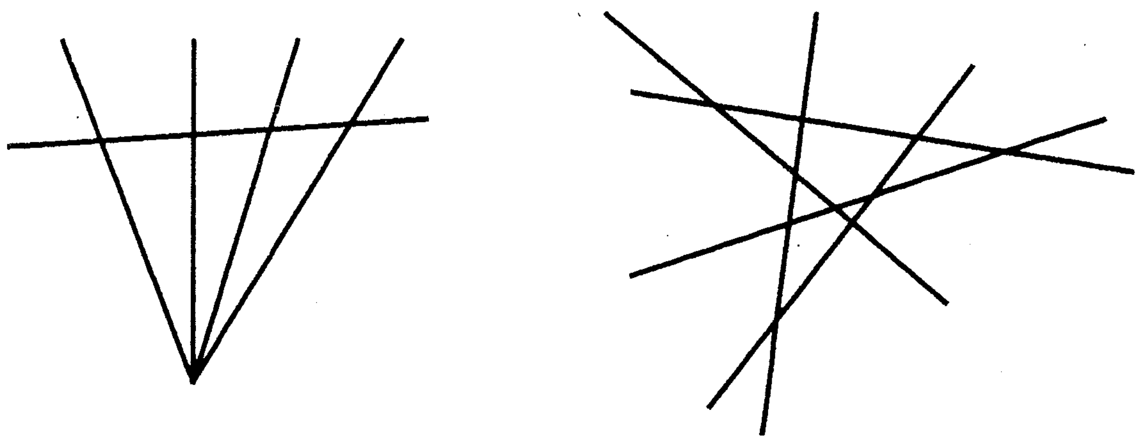
This is unsolved even for Desarguesian planes and even for the smallest case $n = 4$.

3.3.3. Affine planes with a strange point at infinity.

Let **A** be an affine plane, and denote by Π a parallel class of **A**. In the natural embedding, the lines of Π meet in a common point. But this is not necessarily the case in every embedding. The lines of Π might form an configuration which is the dual of a (possibly degenerate) linear space (since, clearly, any two lines of Π meet).

Problem: Embed an affine plane in which the lines of one parallel class form a certain configuration in a finite projective plane.

Examples of such configurations are:



An example of such an embedding is the embedding of the affine plane of order 3 (alias the unital of order 2) in the projective plane of order 4.

REFERENCES

[1] Batten L.M., *Linear spaces with line range* $\{n-1, n, n+1\}$ *and at most* n^2 *points,* J. Austral., Math. Soc. A **30** (1980), 215-228.

[2] Batten L., Totten J., *On a class of linear spaces with two consecutive line degrees,* Ars combinatoria **10** (1980), 107-114.

[3] Beutelspacher A., *Embedding planar spaces in projective spaces,* Finite Geometries, Marcel Dekker, New York and Basel, (1985), 9-17.

[4] Beutelspacher A., Metsch K., *Embedding finite linear spaces in projective planes II,* Discrete Math. **66** (1987), 219-230.

[5] Bridges W.G., *Near 1-designs,* J. Combinat. Theory A **13** (1972), 116-126.

[6] de Bruijn N.G., Erdös P., *On a combinatorial problem,* Indag. Math. **10** (1948), 1277-1279.

[7] Dembowski P., *Semiaffine Ebenen,* Arch. Math. **13** (1962), 120-131.

[8] Dow S., *An improved bound for extending partial projective planes,* Discrete Math. **45** (1983), 199-207.

[9] Erdös P., Mullin R.C., Sós V.T., Stinson D.R., *Finite linear spaces and projective planes,* Discrete Math. **47** (1983), 49-62.

[10] Frank R.D., *Projective Embedding of Certain Linear Spaces,* J. Combinat. Theory A **48** (1988), 266-269.

[11] Hall M., *Projectiv planes,* Trans. Amer. Math. Soc. **54** (1943), 229-277.

[12] Hauptmann W., *Endliche [0,2]-Ebenen,* Geom. Dedicata **9** (1980), 77-86.

[13] Libois P., *Quelques espaces linéaires,* Bull. Soc. Math. Belg. **16** (1964), 13-22.

[14] Limbos M., *Projective Embeddings of Small "Steiner Triple Systems",* Annals Discrete Math. **7** (1980), 151-173.

[15] Lo Re P.M., Olanda D., *On [0,2]-semiaffine planes,* Simon Stevin **60** (1986), 157-182.

[16] Metsch K., *An Optimal bound for Embedding Linear Spaces Into Projective planes,* Discrete Math. **70** (1988), 53-70.

[17] Metsch K., *Embedding Finite Plaanr Spaces into 3-Dimensional Projective spaces,* J. Combinat. Th. A **51** (1989), 161-168.

[18] Metsch K., *Linear spaces with few lines,* To appear.

[19] Totten J., *Classification of restricted linear spaces,* Canad. J. Math. **28** (1976), 321-333.

[20] Totten J., *Finite linear spaces with three more lines than points,* Simon Stevon **47** (1976), 35-47.

[21] Vanstone S.A., *The Extendability of* $(r-1)$*-designs,* Proc. Third Manitoba Conf. on Numerical Math., Winnipeg **1973**, 409-418.

[22] de Witte P., *Combinatorial properties of Finite Linear Spaces II,* Bull. Soc; Math. Belg. **27** (1975), 115-155.

[23] de Witte P., *Finite linear spaces with two more lines than points,* J. reine angew. Math. **288** (1976), 66-73.

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