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# SUPPORTS OF $(v, 4, 2)$ DESIGNS

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The spectrum of possible numbers of repeated blocks in $(v, 4, 2)$ designs is determined for all $v \geq 121$.

## 1. Background.

A $(v, k, \lambda)$ *design* is a pair $(V, \mathbf{B})$, where $V$ is a set of $v$ *elements*, and $\mathbf{B}$ is a collection of $k$-subsets of $V$ called *blocks*; each 2-subset of $V$ appears in precisely $\lambda$ of the blocks. This definition permits *repeated* blocks.

The *support* of a design $(V, \mathbf{B})$ is the set $\mathbf{B}^*$ of distinct blocks in the design. While $b = |\mathbf{B}|$ is always $\lambda v(v-1)/k(k-1)$, the *support size* $b^* = |\mathbf{B}^*|$ is not determined uniquely by the parameters in general. In part due to a number of statistical applications, there has been much recent activity in determining the spectrum of support sizes for $(v, k, \lambda)$ designs. For triple systems ($k = 3$), the problem is essentially settled for all $\lambda \leq 8$ [5, 6, 7, 16].

We study the spectrum of support sizes for $(v, 4, 2)$ designs in this paper. In fact, we consider an equivalent problem: the determination of the possible numbers of repeated blocks. If a $(v, 4, 2)$ design has $r$

repeated blocks, its support size is $v(v-1)/6 - r$.

$(v, 4, 2)$ designs exist if and only if $v \equiv 1 \pmod 3$ [9]. Let $M_v$ be $v(v-1)/12$ if $v \equiv 1, 4 \pmod{12}$, or $(v(v-1)-42)/12$ if $v \equiv 7, 10 \pmod{12}$. Let $R_4(v)$ denote the set of numbers of repeated blocks in $(v, 4, 2)$ designs. We prove the following:

THEOREM *For* $v \equiv 1, 4$ *(mod 12),* $v \geq 88$,

$$R_4(v) = \{0, 1, \ldots, M_v\} \backslash \{M_v - 5, M_v - 3, M_v - 2, M_v - 1\},$$

*and for* $v \equiv 7, 10$ *(mod 12),* $v \geq 127$,

$$R_4(v) = \{0, 1, \ldots, M_v\} \backslash \{M_v - 2, M_v - 1\}.$$

Our proof of the Main Theorem relies on a number of known results; we recall these here. A *transversal design* $TD(k, n)$ is a set of $k$ disjoint sets of $n$ elements each (called *groups*), and a set of blocks of size $k$, so that each block intersects each group in precisely one element, and every pair of elements from different groups appears in a block. An *incomplete* transversal design $ITD(k, n, m)$ also has $k$ groups of size $n$; in each group, $m$ elements are distinguished as belonging to a *hole*. A pair of elements from different groups then appears in one block if they are not both in the hole, and zero blocks if both elements are in the hole. Such incomplete designs could be obtained, for example, by removing a sub-$TD(k, m)$ from a $TD(k, n)$; however, there are other examples when the $TD(k, m)$ does not itself exist.

The following is well known:

LEMMA A [1,18]: *There exists a* $TD(5, n)$ *for all* $n \notin \{2, 3, 6\}$ *except possibly* $n = 10$. □

We also use incomplete transversal designs:

LEMMA B [10]: *For* $n \geq 3m$ *and* $m \geq 1$, *there exists an* $ITD(4, n, m)$ *except for* $(n, m) = (6, 1)$. □

We also employ a powerful theorem of Rees and Stinson:

LEMMA C [14]: *Let* $v, w \equiv 1$ *(mod 3),* $v(v-1) \equiv w(w-1)$ *(mod 12), and* $v \geq 3w$. *Then there exists a pairwise balanced design on* $v$ *elements having one block of size* $w$ *and all other blocks of size four.*□

Omitting the $w$-block leaves a *hole* of size $w$, forming an incomplete $PBD$ which we denote as a $(v, w; 4) - IPBD$.

We use Lemmas A, B and C as building blocks in recursive constructions for determining $R_4(v)$. In section 2, we establish the necessity of the conditions in the Main Theorem, and dispense with the (easier) case when $v \equiv 1, 4$ (mod 12). In section 3, we then use Lemmas A and B in examining the intersection problem for transversal designs $TD(4, n)$. In section 4, we prove sufficiency in the Main Theorem using the results obtained on intersection of transversal designs, and using Lemma C to take care of the values missed in this way.

We assume familiarity with constructions using pairwise balanced designs and group divisible designs; see [1] for further background.

## 2. Necessary Conditions.

Consider the graph $G$ whose edges are the pairs which do *not* appear in repeated blocks. The graph $2G$, obtained from $G$ by taking each edge twice, must have a partition into $K_4$'s. Hence $G$ must have all vertex degrees $\equiv 0$ (mod 3). Moreover, the partition of $2G$ must have no repeated $K_4'$s, and hence no vertex degree is equal to 3. $G$ is then a simple graph on $v$ points with degrees 0, 6, 9 and so on.

When $v \equiv 1, 4$ (mod 12), $G$ has 0 (mod 6) edges, and when $v \equiv 7, 10$ (mod 12), $G$ has 3 (mod 6) edges. Hence we have

LEMMA 2.1. $\{M_v - 5, M_v - 3, M_v - 2, M_v - 1\} \cap R_4(v) = \emptyset$ *for* $v \equiv 1, 4$ *(mod 12).*

*Proof.* If $M_v - s \in R_4(v)$, the graph $G$ defined has $6s$ edges. Now $G$ must contain a vertex of degree at least six, and hence must have at least seven vertices of degree at least six. Then if $s \neq 0$, $s \geq 4$. We must rule out $s = 5$. In this case, $G$ has 30 edges, and must have 10 vertices each of degree 6.

Suppose that $G$ has degree sequences $6^m$. Choose a vertex $x$ and let the six neighbours of $x$ be $y_1, \dots, y_6$. Since $x$ must appear in four distinct quadruples, there are at least ten edges of the form $\{y_i, y_j\}$. Now $g$ has $l = m - 7$ further vertices, whose total degree is $6l$. Since edges from the $\{y_i\}$ account for at most 10 of this total degree, we obtain that $l(l-1) \geq 6l - 10$. But then $l$ cannot equal 3 or 4, and hence $G$ cannot have degree sequence $6^{10}$ or $6^{11}$. □.

LEMMA 2.2. $\{M_v - 2, M_v - 1, M_v + 1, M_v + 2, M_v + 3\} \cap R_4(v) = \emptyset$ *for* $v \equiv 7, 10$ *(mod 12).*

*Proof.* If $M_v - s \in R_4(v)$, the graph $G$ has $6s + 21$ edges. $G$ has at least one vertex of nonzero degree, since $G$ has at least three edges. Hence $G$ has at seven vertices of nonzero degree, and at least 21 edges. Hence $s \geq 0$. If $s = 1$, $G$ has 27 edges and has 9 vertices of degree 6. Consider the 2-regular complement of $G$ on these nine vertices. If the complement contains a triangle or a 4-cycle, it is routine to verify that $2G$ has no partition into $K_4$'s. If the complement forms a 9-cycle, one can verify that $G$ only contains nine distinct $K_4$'s in it. These nine do not partition $2G$, and hence no such partition exists.

For $s = 2$, $G$ has 33 edges. It could have two vertices of degree 9 and eight of degree 6, or it could have eleven of degree 6. The latter case is eliminated in the proof of Lemma 2.1. In the first case, by considering a $K_4$ involving neither vertex of degree 9, one concludes that $G$ must contain an edge which appears in no $K_4$'s at all. Hence the first case is ruled out. □

Now we dispense with sufficiency in the easier case, when $v \equiv 1, 4$ (mod 12).

LEMMA 2.3. *For* $v \equiv 1, 4$ *(mod 12) and* $v \geq 40$,

$$R_4(v) = \{0, 1, \dots, M_v\} \setminus \{M_v - 5, M_v - 3, M_v - 2, M_v - 1\},$$

*except possibly for* $M_{73} - 4 \in R_4(73)$, $M_{85} - 4 \in R_4(85)$ *and* $M_v - 7 \in R_4(v)$ *for* $v \in \{40, 49, 52, 61, 64, 73\}$.

*Proof.* Colbourn, Hoffman and Lindner [4] proved that for $v \equiv 1, 4$

(mod 12) and $v \geq 40$, there are two $(v,4,1)$ designs having $r$ blocks in common for all $0 \leq r \leq M_v$ *except* $r = M_v - s$ for $s \in \{1,2,3,4,5,7\}$. Taking the union of these $(v,4,1)$ designs establishes $r \in R_4(v)$. Hence we need only consider $M_v - 4$ and $M_v - 7$. For the first, consider the unique graph $G$ on eight vertices which is 6-regular (the «cocktail party graph» on eight vertices). The multigraph $2G$ has a partition into eight distinct $K_4$'s. We use this to show that $M_{40} - 4 \in R_4(40)$. Form an $ITD(4,9,2)$; add four points at infinity. On each group together with the four points, place a $(13,4,2)$ design with a hole on the four added points, having every block repeated. Place a repeated block on the four added points. The pairs left to partition form $2G$, and hence yield 8 nonrepeated blocks. Using Lemma C, we obtain $M_v - 4 \in R_4(v)$ for all $v \equiv 1,4$ (mod 12) and $v \geq 121$. A similar strategy settles $v \in \{52,88,100\}$. Instead placing one point at infinity settles the cases $v \in \{49,61,97,109\}$. Placing no points at infinity handles $v \in \{64,112\}$. For $v = 76$, we use a packing by quadruples whose leave is the cocktail party graph, given in [8].

For $M_v - 7$, we use a group divisible design with blocks of size 4 and group-type $6^2 3^4$ [15]. Add a point at infinity to extend each group. The result is a pairwise balanced design on 25 elements with two blocks of size 7 and all others of size four. Double each 4-block and replace each 7-block by the blocks of a (7,4,2) design. This shows $M_{25} - 7 \in R_4(25)$. By lemma C, we have $M_v - 7 \in R_4(v)$ for all $v \equiv 1,4$ (mod 12), $v \geq 76$. □

## 3. Intersection of transversal designs.

Our main strategy in dealing with the remaining congruence classes is to use a simple quadrupling construction using incomplete transversal designs $ITD(4,n,m)$. Our particular concern is with the case $m \in \{1,2\}$. However, we state a number of the results more generally.

For uniformity, we view a $TD(4,n)$ as an $ITD(4,n,0)$. Now two $ITD(4,n,m)$'s are said to *intersect* in $s$ blocks if the *ITDs* have groups on the same sets of points, their holes on the same sets of points, and $s$ blocks in common. We denote by $TI(n,m)$ the set of intersection sizes of $ITD(4,n,m)$'s. The maximum value in $TI(n,m)$ is $n^2-m^2$, provided of course that an $ITD(4,n,m)$ exists at all.

Now we develop some straightforward constructions:

LEMMA 3.1. *If an $ITD(4,n,m)$ exists, then for $0 \leq i \leq n-m$, $i \neq n-m-1$, and $0 \leq j \leq m$, $j \neq m-1$,*

$$in + j(n-m) \in TI(n,m).$$

*Proof.* Let one group of the $ITD(4,n,m)$ be $\{1,2,\ldots,n\}$ so that the hole lies on $\{1,2,\ldots,m\}$. Form a second $ITD$ by applying the permutation fixing $1,\ldots,j$ and $m+1,\ldots,m+i$, and moving the remaining elements so that elements in the hole remain in the hole. This is applied to a single group. Since each element in the hole meets $n-m$ blocks, and each outside the hole meets $n$ blocks, we obtain the intersection $in+j(n-m)$ as required. □

We can also fill the holes in an $ITD$:

LEMMA 3.2. *If $r \in TI(n,m)$ and $s \in TI(m,u)$ then $r+s \in TI(n,u)$.*

*Proof.* Fill the hole of an $ITD(4,n,m)$ with an $ITD(4,m,u)$. □

Next we describe some applications of Wilson's fundamental construction [19]:

LEMMA 3.3. *For $1 \leq i \leq n^2-m^2$, let $r_i \in TI(k,0)$. Then if an $ITD(4,n,m)$ exists,*

$$\sum_{i=1}^{n^2-m^2} r_i \in TI(kn, km).$$

*Proof.* Give every point in the $ITD(4,n,m)$ weight $k$; each block then becomes a $TD(4,k)$. For each of the $n^2-m^2$ blocks, choose two $TD(4,k)$'s which intersect in $r_i$ blocks. □

This can be extended in a useful way; In an $ITD(4,n,m)$, define a *holey parallel class* to be a set of $n-m$ pairwise disjoint blocks involving no elements from the hole.

LEMMA 3.4. *Let* $t = n^2 - m^2 - l(n-m)$. *For* $1 \leq i \leq t$ *let* $r_i \in TI(k, 0)$, *and for* $1 \leq j \leq l(n-m)$ *let* $s_j \in TI(k+1, 1)$. *If* $k > 1$ *and an* $ITD(4, n, m)$ *exists having* $l$ *disjoint holey parallel classes, then*

$$\sum_{i=1}^{t} r_i + \sum_{j=1}^{l(n-m)} s_j \in TI(kn + l, km + l).$$

*Proof.* Give every point of the $ITD(4, n, m)$ weight $k$; blocks not in a holey parallel class become $TD(4, k)$'s. Add $l$ points at infinity to each group; then blocks in the $l$-th holey parallel class become $ITD(4, k+1, 1)$'s placing the hole on the $l$-th additional point in each group. □

We use Lemma 3.4 primarily in the case when $m = 0$; in this case, a $TD(5, n)$ can be used to form a resolvable $TD(4, n)$. Naturally, an $ITD(5, n, m)$ could also be used here.

Now we turn to results for small values to be used applying these lemmas. It is evident that if $s \geq 1$ and $s \in TI(n, 0)$ then $s - 1 \in TI(n, 1)$. $TI(1, 0) = \{1\}$ and $TI(2, 0)$ is empty. Colbourn, Hoffman and Lindner [4] established that $TI(3, 0) = \{0, 1, 3, 9\}$, and that $TI(4, 0) = \{0, 1, 2, 4, 8, 16\}$. Using these results, we obtain a strong general result:

LEMMA 3.5. *Let* $x$ *be a positive integer,* $x \notin \{2, 3, 6, 10\}$. *Let* $0 \leq y \leq x$. *Let* $M = (3x + y)^2 - y^2$ *and let* $0 \leq s \leq M$, $s \neq M - t$ *for* $t \in \{1, 2, 3, 4, 5, 7, 10, 11, 13, 19\}$. *Then* $s \in TI(3x + y, y)$.

*Proof.* Apply Lemma 3.4 with weight 3 to a resolvable $TD(4, x)$ (from Lemma A). Use $TI(3,\ 0) = \{0,\ 1,\ 3,\ 9\}$ and $TI(4,\ 1) = \{0,\ 1,\ 3,\ 7,\ 15\}$. □

The same construction with $y = x$ leaves further possible exceptions, namely $M - T$ for $t \in \{6, 9, 17, 18, 21, 25, 33\}$.

LEMMA 3.6. *Let* $n = 9$ *or* $n \geq 12$, $n \neq 14$. *Then if* $0 \leq s \leq n^2$ *and* $s \neq n^2 - t$ *for* $t \in \{1, 2, 3, 4, 5, 7, 10, 11, 13, 19\}$, *then* $s \in TI(n, 0)$.

*Proof.* When $n \notin \{9, 17, 20, 23\}$, this follows directly from Lemma 3.5. When $n = 9$, this follows from Lemma 3.3. When $n = 23$, apply weight 3 to an $ITD(4,7,1)$ having two holey parallel classes (use a $TD(6,7)$ to obtain this). Fill the remaining hole with a $TD(4,5)$. For $n = 20$, remove four points from a block and one further point from a (25,5,1) to from a group divisible design of type $3^4 1^8$ having block sizes four and five. Use this to form a $TD(4,20)$ with four disjoint $TD(4,3)$'s.

For $n = 17$, start with a group-divisible design block size 4 and group type $3^5$. We add two extra points, and apply weight four, using 3 $ITD(4,6,2)$'s, a $TD(4,5)$, 4 $TD(4,3)$'s 5 $TD(4,1)$'s and 12 $TD(4,4)$'s minus a parallel class. □

Remark that for $n = 14$, one can obtain an almost complete determination of $TI(14,0)$ using the $HSOLS(2^5 3^1)$ of Stinson and Zhu [17].

Now we turn to the determination of $TI(n,2)$. In general, we can apply Lemma 3.5 with $y = 2$, or with $y \geq 6$; in the latter case, an $ITD(4,y,2)$ is then used to fill the hole and leave a (smaller) hole. This enables us to prove

LEMMA 3.7. *Let* $n \in \{14,17,18,21\}$ *or* $n \geq 23$. *Let* $0 \leq s \leq n^2 - 4$ *and* $s \neq n^2 - 4 - t$ *for* $t \in \{1,2,3,4,5,7,10,11,13,19\}$. *Then* $s \in TI(n,2)$.

*Proof.* For $n \notin \{18,21,24,25,28,32,36,37\}$, this follows directly from Lemma 3.5 using $y = 2$ or $y \geq 6$. For $n \in \{18,21,24,36\}$, apply weight 3 in Lemma 3.3 to an $ITD(4,n/3,2)$, and fill the hole with an $ITD(4,6,2)$. For $n = 25$, apply Lemma 3.4 with weight 3 to an $ITD(4,7,1)$ with four holey parallel classes, using an $ITD$ (4,7,2) to fill the hole. For $n = 28$, do the same using an $ITD(4,8,1)$ with four holey parallel classes, and for $n = 32$ use an $ITD(4,9,1)$ having five holey parallel classes. The basic ingredients here exist using $TD(8,7)$, $TD(8,8)$ and $TD(9,9)$ designs. Finally for $n = 37$, apply weight 9 to a $TD(4,4)$ having a parallel class, filling with three $ITD(4,10,1)$'s and one $ITD(4,10,2)$. □

## 4. Small Orders.

In [4], numerous results are settled for the small cases $v = 25$, 28 and 37 not covered by Lemma 2.3. In this section, we examine the small cases for $v \equiv 7, 10 \pmod{12}$.

For $v = 7$ and $v = 10$, it is easy to see that the $(v, 4, 2)$ designs are unique up to isomorphism, and have no repeated blocks. The first nontrivial case is therefore $v = 19$. At the present time, very little is known about this case. From Lemma 2.2, we have $R_4(19) \subseteq \{0, \dots, 22\} \cup \{25\}$. Moreover, since no (19,4,2) can contain a (7,4,2) subdesign, one cannot obtain a (19,4,2) with 25 repeated blocks. This leaves many possibilities in principle. However, by local optimization techniques, we have thus far been able to settle only four cases affirmatively:

LEMMA 4.1. $\{0, 1, 2, 3\} \subset R_4(19)$.

*Proof.* There is a (19,4,2) design with no repeated blocks [9]; for example, the starter blocks $\{\{0, 1, 4, 15\}, \{0, 6, 11, 13\}, \{0, 7, 9, 10\}\}$ in $Z_{19}$ give a block-transitive (19,4,2) design. For the remaining values, we found designs by computer using local optimization techniques. We present the list of blocks using elements $\{a, b, \dots, s\}$.

For $1 \in R_4(19)$, take the design with blocks: adgm adgm frps epra ifrm odbn naji hken sigb aslk cpob fiqe cgfk haks jcmq idrs jlrh gbjf brch hqso ofaq kbqi fbnl dipj debp gjse lcie ngkr oigh fhdl moen sejo ghcp pmqh ebkm fhed rgqe qdkc mhni rlom kjfo mpkj orkd jldq glop qnps sndc jcnr cfms coai abrq blms alce bhaj gqln fpna plik.

For $2 \in R_4(19)$, take the design with blocks: adgm adgm bcef bcef asbk incl hgkb srcm qsca ehlq sgnf lhes reja dqjh pfml mqer ihpr peod icdn qolg qsid nalb pdes brld prkl oegi nmej osrk frqa djob ohbm cjpg hnpa rhgc fndr rjno mocl nqpb aofh ipmk hijf qckj egkn smnh aopc jgls sfoi khcd sjpb pfgq mfkj kqno aeik lkdf qmbi lija grib.

For $3 \in R_4(19)$, take the design with blocks: adgm adgm bcef bcef hijk hijk lbgj nmje disc mekp rdnq qgfj shlq ianf qofm obsr mlbk aqjc aofk engh fjrs pcha nogk spkq elao drkc ksfg riom idlf ndjb pigc jpol cnkl dhlf seja phfm shmb qcml pdes slan paib ergh qahr qioe dohb qgbi hcno mcjr odjp nmsi lrgp qbnp qdke ogsc karb eril frpn. □

Turning to $v = 22$, the situation is much more satisfactory:

LEMMA 4.2. $\{0,1,\ldots,26\} \cup \{35\} \subseteq R_4(22)$.

Our general strategy is to select two Kirkman triple systems of order 15, having parallel classes $P_1,\ldots,P_7$ and $Q_1,\ldots,Q_7$, respectively. Seven new points $x_1,\ldots,x_7$ are added; next we assign each of the seven points to a different parallel class from the $\{P_i\}$,a nd assign each to a different parallel class from the $\{Q_i\}$. A (22,4,2) is then formed by adding $x_i$ to each block in the parallel classes to which $x_i$ is assigned, and finally placing a (7,4,2) design on the $\{x_i\}$. The number classes of repeated blocks is then the sum for $i = 1,\ldots,7$ of the number of blocks in common between the two parallel classes to which $x_i$ is assigned. In order to obtain different numbers of repeated blocks, there is much freedom in this approach – one can choose different Kirkman triple systems, and different assignments of extra points to parallel classes.

By taking the two Kirkman triple systems identically, but varying the assignment of extra points to parallel classes, we obtain (22,4,2) designs with 0, 5, 10, 15, 20, 25 and 35 repeated blocks. Taking any Kirkman triple system, and a copy of it in which two elements are transposed, and varying the assignment of extra points to parallel classes, we obtain designs with 3, 6, 8, 9, 11, 12, 14, 17 and 23 repeated blocks. Taking any Kirkman triple system, and a copy of it in which the permutation $x \to y \to z \to x$ has been applied to the elements of a block $\{x,y,z\}$ and again varying the assignment, we obtain designs with 2, 4, 6, 7, 8, 9, 11, 13 and 17 repeated blocks.

Now take two Kirkman systems as follows. One has parallel classes {bcd, eim, afk,gln,hjo}, {bef, cik, dno, agj, hlm}, {bgh, cjl,a dm, eko, fin}, {bij, cmo, dfg, ael, hkn}, {bkl, acn, deh, fjm, gio}, {bmn, ceg, djk, flo, ahi}, {abo, cfh, dil, ejn, gjm}. The second has the first three parallel classes the same, while the remaining four are {bij, acn, deh, flo, gkm}, {bkl, cmo, dfg, ejn, ahi}, {bmn, cfh, djk, ael, gio}, {abo, ceg, dil, fjm, hkn}. Now assigning extra points to the parallel classes of these two Kirkman systems in various ways gives designs with 1, 16, 18, 19 and 21 repeated blocks (and some numbers among those already achieved).

Next take two Kirkman systems as follows. The first is {bcd, eim, afk,gln, hjo}, {bef, cik, dno, agj, hlm}, {bgh, cmo, djk, ael, fin}, {bmn, cjl, dfg, eko, ahi}, {bij, ceg, adm, flo, hkn}, {bkl, acn, deh, fjm, gio}, {abo, cfh, dil, ejn, gkn}. The second system has the same first four parallel classes, and three further parallel classes as follows: {bij, acn, deh, flo, gkm}, {bkl, cfh, adm, ejn, gio}, {abo, ceg, dil, fjm, hkn}. Assigning extra points as before gives designs with 22, 24 and 26 repeated blocks. □

For $v = 31$, we use a similar strategy. Mathon, Phelps and Rosa [12] have enumerated thirty of the Kirkman triple systems of order 21; we make extensive use of their results.

LEMMA 4.3. $\{0,\ldots,53\} \cup \{55,56,58,60,62,64,70,74\} \subseteq R_4(31)$.

*Proof.* To obtain the bulk of the values, we proceed as follows. Take two Kirkman triple systems of order 21, having parallel classes $P_1,\ldots,P_{10}$ and $Q_1,\ldots,Q_{10}$. Add ten points at infinity; each of these ten points is added to the blocks of one of the $\{P_i\}$ and one of the $\{Q_j\}$. Finally, a (10,4,2) subdesign is placed on the ten additional points. Hence we must consider intersections among the parallel classes of two Kirkman triple systems. To do this, form a $10 \times 10$ matrix $X$ in which the $(i,j)$-entry is the number of blocks in common between $P_i$ and $Q_j$. The sum of the entries in any transversal of $X$ is then a support size for a $(v,4,2)$ design.

Consider, for example, the Kirkman designs labelled A.3a and A.3b in [12]. The $10 \times 10$ matrix of intersections for these two designs is

$$\begin{matrix}
7 & & & & & & & & & \\
& 7 & & & & & & & & \\
& & 7 & & & & & & & \\
& & & 4 & 1 & & & 1 & 1 & \\
& & & 1 & 1 & 1 & 1 & 2 & 1 & \\
& & & & 2 & 2 & 2 & & 1 & \\
& & & 1 & & 3 & 3 & & & \\
& & & & 2 & 1 & & 3 & 1 & \\
& & & 1 & 1 & & 1 & 1 & 3 & \\
& & & & & & & & & 7
\end{matrix}$$

The transversal (1,1), (2,2), (3,3), (4,4), (6,5), (7,6), (5,7), (8,8), (9,9), (10,10) has sum 44, and hence we obtain $44 \in R_4(31)$. It is an easy verification that using this matrix we have $\{0,\ldots,44\} \subseteq R_4(31)$.

Similarly, using designs A.3a and A.3c from [12], we obtain $\{46,48,52,56,58\} \subseteq R_4(31)$. Designs A.3l and A.3m give $\{47,51,55\}$.

For 49 repeated blocks, we permute three points in the hole of a $(31,10;4)-IPBD$ to obtain a second (isomorphic) $IPBD$; taking their union and filling the hole with a (10,4,2) design gives the desired result. For 50 repeated blocks we permute three points of the hole in a $(31,7;4)-IPBD$. For 60 repeated blocks, we use a group-divisible design of type $6^5$ with blocks of size four [3], taking each block twice. We add a new point, and on each group together with the new point, we place a (7,4,2) design.

For 62 and 64 repeated blocks, we observe that in design A.3a of [12], the union of parallel classes #1 and #8 forms a disconnected graph with components of size 9 and 12. Hence we can produce a distinct Kirkman triple system by exchanging the two parallel classes in either component while leaving the other unchanged. From the design with 64 repeated blocks, exchanging two of the resulting parallel classes gives a design with 53 repeated blocks, and from the one with 62 repeated blocks we obtain one with 45.

Finally, taking each block of a $(31,10;4)-IPBD$ twice and filling the hole with a (10,4,2) design gives 70 repeated blocks; using a $(31,7;4)-IPBD$ similarly gives 74 repeated blocks. □

Unlike the cases $v = 22$ and 31, the case $v = 34$ is not constructed using a Kirkman triple system. Nevertheless, we can use similar techniques to handle many of the cases here.

LEMMA 4.4. *If* $s \in \{0,\ldots,87\} \cup \{90\}$, *then* $s \in R_4(34)$ *except possibly for* $s =$ 43, 49, 59, 65, 67, 73, 74, 75, 77, 79, 80, 81, 82, 83, 85 *or* 87.

*Proof.* Rees and Stinson [13] construct a resolvable pairwise balanced deisign on 24 points having ten parallel classes of triples and one parallel class of quadruples as follows. On $(Z_{10} \cup \{a,b\} \times Z_2$, place 5 quadruples taken by developing $\{(0,0),(0,1),(5,0),(5,1)\}$

modulo $(10,-)$, and place a quadruple on $\{a,b\} \times Z_2$. Now let $P_0$ be the set of blocks $\{(a,0),(0,0),(2,1)\}$, $\{(b,0),(5,0),(9,1)\}$, $\{(1,0),(7,0),(8,1)\}$ and $\{(3,0),(4,0),(6,0)\}$ developed modulo $(-,2)$. Let $P_i$ be obtaining from $P_0$ by adding $(i,0)$ modulo $(10,-)$ to each point (where $a+i=a$ and $b+i=b$). This forms the required resolvable pairwise balanced design. As with a Kirkman system, we can extend the parallel classes of triples by adding ten additional points.

We examine the effect of applying a permutation to form a second (but isomorphic) resolvable design, and then extending parallel classes of each as before. In this case, unlike that of Kirkman systems, we must also consider the effect on the quadruples which do not involve one of the additional points. For example, choose the permutation mapping $(5,1) \to (6,1) \to (7,1) \to (8,1) \to (9,1) \to (5,1)$, and fixing the remaining points of the resolvable design. This mapping fixed one quadruple, namely $\{(a,0),(b,0),(a,1),(b,1)\}$; it maps each parallel class $P_i$ to a parallel class $Q_i$. As before, we consider the intersections of the two sets of parallel classes; by varying the assignment of parallel classes to additional points, we obtain designs with $r$ repeated blocks for $r \in \{1,2,39\}$ and $r \in \{4,\dots,33\}$.

Applying instead the permutation $(6,1) \to (7,1) \to (8,1) \to (9,1) \to (6,1)$, we fix two quadruples from the parallel class, and varying assignments of the parallel classes of triples yield designs with $r$ repeated blocks for $r \in \{34,35,36,37,38,47\}$. Now applying $(7,1) \to (8,1) \to (9,1) \to (7,1)$, fied three quadruples from the parallel class and gives designs with $r$ repeated blocks for $r \in \{3,40,41,44,45,46,56\}$. Applying $(8,1) \to (9,1) \to (8,1)$ gives $r \in \{48,53,54,66\}$. Applying the identity yields designs with 62, 70 and 86 repeated blocks.

Now observe that in the Rees-Stinson design, the elements $\{a,b\} \times Z_2$ appear with 1-factors of edges on $Z_{10} \times Z_2$; in fact, each such edge is either on the set $\{0,2,4,6,8\} \times Z_2$ or on $\{1,3,5,7,9\} \times Z_2$; we call these edges *even* and *odd*, respectively.

To obtain a (possibly nonisomorphic) resolvable pairwise balanced design, one can apply a permutation to the elements of $\{a,b\} \times Z_2$ only on the blocks containing even edges in the 1-factors. For example, interchanging $(a,0)$ and $(a,1)$ just in those blocks containing even 1-factor edges, one obtains a $(34,4,2)$ with 76 repeated blocks. Permuting three points instead gives a $(34,4,2)$ with 71 repeated blocks. In this design, three points of $\{a,b\} \times Z_2$ each appear in 6 repeated blocks and the last appears in 11; the points of $\{0,2,4,6,8\} \times Z_2$ each appear in 8. The points of $\{1,3,5,7,9\} \times Z_2$ each appear in 11. The point associated with $P_i$ appears in 5 if $i$ is even, 8 if $i$ is odd. Hence permuting the parallel classes, we obtain designs with 61 and 55 repeated blocks.

Next we use a $(34,7;4)-IPBD$, constructed by Brouwer [2]. Removing the 7 points of the hole gives a pairwise belanced design having seven parallel classes of triples, and the remaining blocks (27 of them) are quadruples. We can from a second $IPBD$ by permuting the points of the hole and the points outside the hole independently. Taking the union of the two $IPBDs$ and then placing a (7,4,2) design on the hole yields a (34,4,2) design. Permuting points in the hole gives designs with 63, 72 and 90 repeated blocks. Permuting points of a block not intersecting the hole gives designs with 50 and 60 repeated blocks.

Now in Brouwer's design, two of the parallel classes have the property that their union consists of three connected components on nine vertices each. Hence we can interchange portions of these two parallel classes to form the second design, and of course we can also permute classes as well. This gives designs with 42, 51, 57, 69, 78 and 84 repeated blocks. Instead interchanging a portion and then permuting points of a block neither in the hole or the changed portion gives 58 and 64 repeated blocks; doing so in the changed portion gives 52 and 68 repeated blocks. □

While we have by no means settled all of the values for small cases, we have the following useful general result.

LEMMA 4.5. *Let $t$ and $v$ be integers satisfying*

*(i)* $t = 4$ *and* $v \geq 31$, *or*

*(ii)* $t \in \{22,23,24,25\}$ *and* $v \geq 58$, *or*

*(iii)* $9 \leq t \leq 35$ *and* $v \geq 67$.

Then $M_v - t \in R_4(v)$.

*Proof.* Choose $w = 10$, 19 or 22 depending on which case (i), (ii) or (iii) is required. Form a $(v,w;4)-IPBD$ using Lemma C and take each block twice. Fill the hole with a $(w,4,2)$ design having $M_w - t$ repeated blocks, provided by the $(10,4,2)$ design, or by Lemma 4.1 or Lemma 4.2. □

Naturally, any improvement in the state of knowledge about $R_4(19)$ or $R_4(22)$ would lead to a stronger result here.

## 5. Supports: $v \equiv 7, 10$ (mod 12).

We use the results on $TI(n,m)$ for $m \in \{0,1,2\}$ to determine $R_4(v)$ for $v \equiv 7, 10$ (mod 12). We also first prove an easy result:

LEMMA 5.1. *Let $v \equiv 7, 10$ (mod 12). Then if $v \geq 22$, there is a $(v,4,2)$ design containing a sub-(7,4,2) design in which exactly one block is repeated, and one in which every block outside the subsystem is repeated. And if $v \geq 31$, there is a $(v,4,2)$ design containing a sub-(10,4,2) in which exactly one block is repeated, and one in which every block outside the subsystem is repeated.*

*Proof.* To form the designs with repeated blocks, form a pairwise balanced design on $v$ elements with blocks of size four and one block of size 7 or 10 [14]. Replace each block of size $b$ by a $(b,4,2)$ design. To obtain exactly one block repeated, take the same pairwise balanced design. It follows from results of Lindner and Street [11] that a second pairwise balanced design can be formed which has the large block, but only one block of size four, in common with the original. The union of the two pairwise balanced designs, and the replacement of the large block of size $b$ by a $(b,4,2)$ design yields the required $(v,4,2)$ design. □

LEMMA 5.2. *Let $s_i \geq 0$, $s_i + 1 \in R_4(x+3)$, $i = 1,2,3,4$ and let $t \in TI(x,0) \cup TI(x,2)$. Then if $t < x^2$, $t + s_1 + s_2 + s_3 + s_4 \in R_4(4x+3)$, and if $t > 0$, $t - 1 + s_1 + s_2 + s_3 + s_4 \in R_4(4x+3)$.*

*Proof.* Form the union of the $TD(4,x)$'s or $ITD(4,x,2)$'s; for the latter, fill the remaining hole with the decomposition of the cocktail party graph. Place the resulting design on points $\{v_{ij} : 1 \leq 4 \leq, 1 \leq j \leq x\}$, so that a block appears on $\{v_{11}, v_{21}, v_{31}, v_{41}\}$. For the first case of the lemma, choose this block to be nonrepeated; for the second, choose the block to be repeated. In either event, omit one copy of the block. Now add three points $a, b, c$, and for $i = 1, 2, 3, 4$, take a $(x+3, 4, 2)$ deisgn and omit a repeated block. Place the result on $\{v_{ij} : 1 \leq j \leq x\} \cup \{a, b, c\}$, leaving the hole of size 4 on $\{a, b, c, v_{i1}\}$. Finally, on $\{a, b, c, v_{11}, v_{21}, v_{31}, v_{41}\}$ place a (7,4,2) design omitting a block on $\{v_{11}, v_{21}, v_{31}, v_{41}\}$. □

From this lemma, we obtain a nearly complete solution for the case $v \equiv 7 \pmod{12}$. However, a further construction along the same lines also proves to be useful:

LEMMA 5.3. *Let* $t \in TI(x,0) \cup TI(x,2)$, $0 \leq s \leq 4$ *and* $x \equiv 0, 3$ *(mod 12),* $x \geq 7$. *Then* $t + sM_{x+7} \in R_4(4x+7)$.

*Proof.* We form two $TD(4,x)$'s or $ITD(4,x,2)$'s sharing $t$ blocks on points $\{v_{ij} : i = 1, 2, 3, 4; 1 \leq j \leq x\}$. If $ITD$'s are used, fill the hole with the decomposition of the cocktail party graph. Now seven further points are added. For each $i = 1, 2, 3, 4$, on the points $\{v_{ij}\}$, $1 \leq j \leq x$ and the seven additional points, a $(x+7, 4, 2)$ is placed, omitting a (7,4,2) subdesign on the seven extra points. By Lemma 5.1, this can be done so as to repeat all or none of the blocks for each $i$. □

Now we can settle the case $v \equiv 7 \pmod{12}$.

LEMMA 5.4. *For* $v \equiv 7$ *(mod 12),* $v \geq 43$,

$$R_4(v) = \{0, 1, \ldots, M_v\} \backslash \{M_v - 2, M_v - 1\}$$

*with the possible exception of* $M_v - t$ *for* $t \in \{3, 5, 7\}$ *and* $v = 67$, *and* $t \in \{3, 5\}$ *and* $v = 79$.

*Proof.* When $x = (v-3)/4 \equiv 1, 10 \pmod{12}$, we apply Lemma 5.2, using Lemma 2.3 to provide $R_4(x+3)$, and Lemmas 3.6 and 3.7 to provide $TI(x,1)$ and $TI(x,2)$. When $x \in \{10, 13, 22, 25, 34\}$, the lemmas cited do not provide solutions; however, it is a routine exercise to verify that with the results of [4] for small cases not handled by Lemma 2.3, and the existence of $TD(4, x)$, $ITD(4, x, 2)$ and $ITD(4, x, 3)$ (Lemma B), each required number of repeated blocks can be realized.

When $x \equiv 4, 7 \pmod{12}$, $x \neq 16$, proceeding in a similar way, using Lemma 5.1 to construct two extreme values in $R_4(x + 3)$, yields all values except $M_v - t$ for $t < 14$ and $t \in \{15, 16\}$. Since we have a complete solution for $v = 43$, applying Lemma C for $v \geq 139$ yields

$$\{M_v - 147, \ldots, M_v - 3\} \cup \{M_v\} \subseteq R_4(v).$$

Hence the lemma holds for $v \geq 139$.

For $v = 127$, we modify Lemma 5.2 to employ a $TD(4, 31)$ or $ITD(4, 31, 2)$ with a sub-$TD(4, 7)$ (both are easily constructed). We align the sub-$TD(4, 7)$ with a $(7, 4, 2)$ in each $(34, 4, 2)$ added, and replace the (7,4,2)'s and the $TD(4, 7)$ by a (28,4,2) design. The case $v = 115$ is handled similarly using $TD(4, 28)$ and $ITD(4, 28, 2)$ with a sub-$TD(4, 7)$.

For $v = 79$, using an $ITD(4, 18, 6)$ and a pairwise balanced design on 25 points with two 7-blocks meeting in a point and 4-blocks elsewhere, one can form a pairwise balanced design on 79 points with one 25-block, one 7-block and 4-blocks otherwise; Using the (partial) determination of $R_4(25)$ resulting from [4] and Lemma 2.1, we obtain $M_{79} - t$ for $t \in \{6, 7, 8, 12, 15\}$. Hence for $v = 79$, the only values in doubt are $M_{79} - t$ for $t \in \{3, 4, 5, 9, 10, 11, 13, 16\}$. Lemma 4.5 then provides $M_{79} - t$ for $t \in \{4, 9, 10, 11, 13, 16\}$.

For $v = 67$, we use Lemma 5.4; this leaves as possible exceptions $M_{67} - t$ for $t \in \{3, 5, 7, 10, 11, 13, 19\}$. Lemma 4.5 provides designs for $t \in \{10, 11, 13, 19\}$. □

Now we turn to the case $v \equiv 10 \pmod{12}$. We adopt a similar strategy here.

LEMMA 5.5. *Let $s_i \geq 0$, $s_i \in R_4(x+2)$, $i = 1, 2, 3, 4$. Let $t \in TI(x, 2)$. Then $t + s_1 + s_2 + s_3 + s_4 \in R_4(4x + 2)$.*

*Proof.* Form two $ITD(4, x, 2)$'s on the same groups (and with the same hole) having $t$ blocks in common, and form their union. Add two extra points, and on each group together with the two extra points take the blocks of an $(x+2,4,2)$ omitting a repeated block on the two extra points and the two points of the hole. Fill the final hole with the blocks of a (10,4,2) design. □

This lemma is sufficient to establish the following:

LEMMA 5.6. *For* $v \equiv 10$ *(mod 12),* $v \geq 130$,

$$R_4(v) = \{0, 1, \ldots, M_v\} \backslash \{M_v - 2, M_v - 1\}.$$

*Proof.* Using Lemma 3.7 to provide the $ITD$s, and Lemmas 2.3, 5.1 and 5.4 to provide the smaller orders, we realize all values up to $M_v - 38$ (and some larger than this as well). Then using Lemma 5.4 to provide $R_4(43)$ and embedding using Lemma C, we obtain all values larger than $M_v - 148$. Hence all values are covered. □

When $v < 130$, the values near the largest pose a serious problem. For $v = 118$, we can use $ITD(4, 29, 5)$'s adding two extra points. The groups are filled with a (22,4,2) missing a (7,4,2), and the final hole is filled with a (22,4,2). This handles all but $M_{118} - t$ for $t \in \{3, 5, 7, 11, 13\}$. Lemma 4.5 handles the cases when $t \in \{11, 13\}$.

For $v = 106$, Lemma 5.5 handles all but $M_{106} - t$ for $t \in \{3, 5, 6, 7, 8, 9, 11, 14, 15, 17, 23\}$. Using instead $ITD(4, 25, 4)$ and adding six points at infinity, filling groups with a (61, 4, 2) misisng a (10, 4, 2) and filling the final hole with a (22,4,2) yields $M_v - t$ for $t \in \{6, 8, 9, 14, 15, 17, 23\}$, leaving only $t \in \{3, 5, 7, 11\}$. Lemma 4.5 handles $t = 11$.

When $v = 94$, again Lemma 5.5 handles all but $M_{94} - t$ for $t \in \{3, 5, 7, 8, 9, 11, 14, 15, 17, 23\}$. Lemma 4.5 handles all remaining cases for $t \geq 9$.

When $v = 82$ use $ITD(4, 20, 5)$'s with two extra points; the ingredients are a (22,4,2) missing a (7,4,2), and a (22,4,2). Using Lemma 3.5, $TI(20, 5)$ contains $375 - t$ except possibly for $t \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 18, 19, 21, 25, 33\}$. Lemma 4.5 handles the remaining cases when $t \geq 9$ or $t = 4$.

For $v = 70$, use $ITD(4, 16, 1)$'s adding six points to obtain

$M_{70} - t$ except possibly for $t \in \{1,2,3,5,6,7,8,9,11,15,17\}$. Lemma 4.5 handles $t \in \{9,11,15,17\}$.

For $v = 58$, Lemma 5.5, together with $R_4(16)$ [4], leaves the cases $M_{58} - t$ for $t \in \{1,2,3,5,6,7,8,9,11,13,14,17\}$.

For $v = 46$, Lemma 5.5 togetehr with $R_4(13)$ [4], leaves the cases $M_{46} - t$ for $t \in \{1,2,3,5,6,7,8,9,11\}$. In his case, Lemma 3.7 does ot provide $TI(11,2)$; however, it is an easy exercise using Lemma 3.1 to obtain the values stated.

The determination for $v \equiv 10 \pmod{12}$ is far from complete; however, all of the remaining values are near the maximum, and we expect that most would have to be settled by computer search. Nevertheless, a more complete determination of $R_4(19)$ and $R_4(22)$ may provide useful building blocks in the construction of these designs.

## 6. Concluding Remarks.

We have given a solution to the support size problem for $(v,4,2)$ designs which is complete for all $v \geq 121$. At the present time, the only method which we see to obtain a complete determination for the small values is by exhaustive computation; this seems to be out of reach. We expect that the Main Theorem holds for all $v \geq 40$. A proof of this, however, will likely await the development of more efficient methods for the construction of designs with block size four.

In closing, we remark on an interesting property of this problem. In considering triple systems for $v \equiv 1,3 \pmod 6$, the solutions for the intersection problem and the support problem are the same. In our problem, however, the solution to the support problem admits additional solutions, which are necessarily indecomposable. This indicates a difficulty of support problems with block size four that was not encountered with triple systems.

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**Note Added in Proof** (May 1991): We have now improved Lemma 4.1 to establish that

$$\{0,1,2,3,4,5,6,7,9\} \subseteq R_4(19).$$

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