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GRAPH DECOMPOSITIONS AND QUASIGROUP IDENTITIES (*)

CURT LINDNER (Auburn)

1. Introduction.

Since everybody's favorite graph decomposition is a Steiner triple system, it is certainly the starting point for a paper on the subject. (Everybody=block designers in graph theory clothing). A *Steiner triple system* (more simply, triple system) is a pair (K_n, T), where K_n is the complete undirected graph on n vertices and T is a collection of triangles which partition K_n. The *number* n is called the *order* of the triple system (K_n, T) and it has been known forever (= since 1847 [3]) that the spectrum (= set of all n such that a triple system (K_n, T) exists) of triple systems is precisely the set of all $n \equiv 1$ or 3 (mod 6). It is trivial to see that if (K_n, T) is a triple system of order n then $|T| = n(n-1)/6$.

EXAMPLE 1.1. In what follows we will denote the triangle

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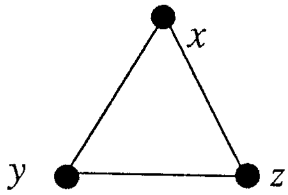


by $\{x, y, z\}$ or simply xyz.

(1) The unique triple system or order 3

1 2 3

(2) The unique (to within isomorphism) triple system of order 7

1 2 4
2 3 5
3 4 6
4 5 7
5 6 1
6 7 2
7 1 3

(3) The unique (to within isomorphism) triple system of order 9

1 2 3	1 4 7	1 5 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 4 8	3 5 7

(4) There are exactly two (to within isomorphism) triple systems of order 13 [8].

1	2	5	1	3	8
2	3	6	2	4	9
3	4	7	3	5	10
4	5	8	4	6	11
5	6	9	5	7	2
6	7	10	6	8	13
7	8	11	7	9	1
8	9	12	8	10	2
9	10	13	9	11	3
10	11	1	10	12	4
11	12	2	11	13	5
12	13	3	12	1	6
13	1	4	13	2	7

cyclic

1	2	3	3	6	13
1	4	5	3	7	11
1	6	7	3	9	10
1	8	9	4	7	9
1	10	11	4	10	13
1	12	13	4	11	12
2	4	6	5	6	10
2	5	7	5	8	11
2	8	10	5	9	13
2	9	12	6	8	12
2	11	13	6	9	11
3	4	8	7	8	13
3	5	12	7	10	12

non-cyclic

Now given a triple system (K_n, T) based on Q, we can define a binary operation «$\circ$» on Q by:

$$\begin{cases} \text{(i)} \ a \circ a = a, \text{ for all } a \in Q, \text{ and} \\ \text{(ii)} \ \text{if } a \neq b, \ a \circ b = b \circ a = c \text{ if and only if } \{a, b, c\} \in T \end{cases}$$

EXAMPLE 1.2. The following groupoids are constructed from the corresponding triple systems in Example 1.1.

(1)

∘	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

(2)

∘	1	2	3	4	5	6	7
1	1	4	7	2	6	5	3
2	4	2	5	1	3	7	6
3	7	5	3	6	2	4	1
4	2	1	6	4	7	3	5
5	6	3	2	7	5	1	4
6	5	7	4	3	1	6	2
7	3	6	1	5	4	2	7

(3)

∘	1	2	3	4	5	6	7	8	9
1	1	3	2	7	9	8	4	6	5
2	3	2	1	9	8	7	6	5	4
3	2	1	3	8	7	9	5	4	6
4	7	9	8	4	6	5	1	3	2
5	9	8	7	6	5	4	3	2	1
6	8	7	9	5	4	6	7	1	3
7	4	6	5	1	3	7	7	9	8
8	6	5	4	3	2	1	9	8	7
9	5	4	6	2	1	3	8	7	9

(4) Constructed from the *cyclic* triple system of order 13.

$\circ$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	5	8	13	2	12	9	3	7	11	10	6	4
2	5	2	6	9	1	3	13	10	4	8	12	11	7
3	8	6	3	7	10	2	4	1	11	5	9	13	12
4	13	9	7	4	8	11	3	5	2	12	6	10	1
5	2	1	10	8	5	9	12	4	6	3	13	7	11
6	12	3	2	11	9	6	10	13	5	7	4	1	8
7	9	13	4	3	12	10	7	11	1	6	8	5	2
8	3	10	1	5	4	13	11	8	12	2	7	9	6
9	7	4	11	2	6	5	1	12	9	13	3	8	10
10	11	8	5	12	3	7	6	2	13	10	1	4	9
11	10	12	9	6	13	4	8	7	3	1	11	2	5
12	6	11	13	10	7	1	5	9	8	4	2	12	3
13	4	7	12	1	11	8	2	6	10	9	5	3	13

Now, inspection of the groupoids constructed in Example 1.2 from the triple systems in Example 1.1 reveals two facts: one obvious and the other not so obvious. The obvious fact is that each of the groupoids in Example 1.2 is a *quasigroup*. The not so obvious fact is that each of these quasigroups satisfies the three identities

$$\begin{cases} x^2 = x, \\ xy = yx, \text{ and} \\ (yx)x = y. \end{cases}$$

Rather than being a coincidence, this is always the case. That is to say, if (K_n, T) is a triple system based on Q and we define a binary operation «$\circ$» on Q as above, then $(Q, \circ)$ is *always* a quasigroup satisfying the three identities $x^2 = x$, $xy = yx$, and $(yx)x = y$.

On the other hand let $(Q, \circ)$ be a quasigroup of order $|Q| = n$ satisfying the three identities $x^2 = x$, $xy = yx$, and $(yx)x = y$ and define a collection of triangles T of K_n (based on Q) by $\{a, b, c\} \in T$ if and only if $a \circ b = b \circ a = c$, $a \circ c = c \circ a = b$, and $b \circ c = c \circ b = a$. Then (K_n, T) is a triple system of order n.

EXAMPLE 1.3. The following quasigroup of order 13 satisfies the three identities $x^2 = x$, $xy = yx$, and $(yx)x = y$.

$\circ$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	3	2	5	4	7	6	9	8	11	10	13	12
2	3	2	1	6	7	4	5	10	12	8	13	9	11
3	2	1	3	8	12	13	11	4	10	9	7	5	6
4	5	6	8	4	1	2	9	3	7	13	12	11	10
5	4	7	12	1	5	10	2	11	13	6	8	3	9
6	7	4	13	2	10	6	1	12	11	5	9	8	3
7	6	5	11	9	2	1	7	13	4	12	3	10	8
8	9	10	4	3	11	12	13	8	1	2	5	6	7
9	8	12	10	7	13	11	4	1	9	3	6	2	5
10	11	8	9	13	6	5	12	2	3	10	1	7	4
11	10	13	7	12	8	9	3	5	6	1	11	4	2
12	13	9	5	11	3	8	10	6	2	7	4	12	1
13	12	11	6	10	9	3	8	7	5	4	2	1	13

If $T = \{\{a, b, c\} | a \circ b = b \circ a = c,\ a \circ c = c \circ a = b,\ b \circ c = c \circ b = a\}$, then (K_{13}, T) is the *non-cyclic* triple system of order 13 in Example 1.1 (4). Check it out!

The above comments and examples illustrate the fact that a Steiner triple system is equivalent to a quasigroup (called, not too suprisingly, a *Steiner* quasigroup) satisfying the three identities $x^2 = x$, $xy = yx$, and $(yx) = y$.

Now a triangle in K_n is also a cycle of length 3 (a 3-cycle). Couched in terms of 3-cycles, a Steiner triple system is a pair (K_n, T) where T is a collection of 3-cycles which partition K_n. Using this vernacular a Steiner triple system is called a *3-cycle system*. In general, a *k-cycle system* (kCS) is a pair (K_n, C) where C is an edge-disjoint collection of k-cycles which partition K_n. The number n is called the *order* of the k-cycle system (K_n, C) and, of course, $|C| = n(n-1)/2k$. Since there is nothing particularly sacred about the number three, the above comments on Steiner triple systems ($= 3CSs$) lead quite naturally to the following problems.

(1) For a fixed k, determine the *spectrum* of k-cycle systems; i.e., the set of all n such that a kCS of order n exists.

(2) If (K_n, C) is a kCS, based on the set Q, is it possible to define in some reasonable way a binary operation «$\circ$» from the collection of k-cycles C so that $(Q, \circ)$ is a *quasigroup*?

(3) If the answer to (2) is yes, does the quasigroup $(Q, \circ)$ satisfy a *finite* collection of 2-variable identities which allow us to recover the kCS it came from? In other words, is a kCS equivalent to a quasigroup satisfying a finite collection of 2-variable identities?

The object of this paper is a survey of what is known about the solutions of problems (1), (2) and (3) for $k = 3, 4, 5, 6$, and 7. Mind boggling details are omitted. The interested reader can refer to the original papers for details. The author's aim is to keep things as simple and understandable as possible. Hence in what follows there will be lots of examples and superficial explanations and not a lot of details. And why not? After all this *is* a survey paper.

2. $3CSs$ =triple systems.

Since the genesis of this paper is the connection between triple systems and quasigroups satisfying certain identities it is certainly the place to begin a survey on this general subject. Here goes!

We begin with the simplest construction known to man for triple systems.

The $6k+3$ *Construction.* Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$; i.e., a quasigroup satisfying the identities $x^2 = x$ (idempotent) and $xy = yx$ (commutative). Set $S = Q \times \{1, 2, 3\}$ and define a collection of triangles T of K_{6k+3} (based on S) as follows:

(1) $\{(x,1),(x,2),(x,3)\} \in T$ for every $x \in Q$, and

(2) if $x \neq y \in Q$, the three triangles $\{(x,1),(y,1),(x \circ y, 2)\}$, $\{(x,2),(y,2),(x \circ y, 3)\}$, and $\{(x,3),(y,3),(x \circ y, 1)\} \in T$.

It is straight forward to see that (K_{6k+3}, T) is a triple system. Since an idempotent commutative quasigroup $(Q, \circ)$ of order $|Q| = 2k+1$ exists for every k (just rename the Cayley table of the additive group of integers modulo $2k+1$) this construction produces a triple system of *every* order $n \equiv 3 \pmod 6$. □

The $6k+1$ *Construction.* Let $Q = \{1, 2, 3, \ldots, 2k\}$ and $H = \{\{1,2\},\{3,4\},\{5,6\},\ldots,\{2k-1, 2k\}\}$. The 2-element subsets in H are called *holes*. Let $(Q, \circ)$ be a commutative quasigroup with the property that, for each hole $h \in H$, $(h, \circ)$ is a subquasigroup. Such a quasigroup is called a *commutative quasigroup with holes* H.

$\circ$	1	2	3	4	5	6	7	8
1	1	2	8	5	4	7	6	3
2	2	1	6	7	8	3	4	5
3	8	6	3	4	7	2	5	1
4	5	7	4	3	1	8	2	6
5	4	8	7	1	5	6	3	2
6	7	3	2	8	6	5	1	4
7	6	4	5	2	3	1	7	8
8	3	5	1	6	2	4	8	7

Commutative quasigroup with holes $H=\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$.

Although not as easy to construct as idempotent commutative quasigroups (at least not when $2k \neq 2\cdot$ (odd number)), such quasigroups exist for every $2k \geq 6$ [1]. Now, let $(Q, \circ)$ be a commutative quasigroup with holes of order $2k$, set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define a collection of triangles T of K_{6k+1} (based on S) as follows:

(1) For each hole $h \in H$, construct a copy of the triple system of order 7 (Example 1.1) on $\{\infty\} \cup (h \times \{1, 2, 3\})$ and place these 7 triangles in T, and

(2) if x and y belong to *different* holes of H, the three triangles $\{(x,1),(y,1),(x \circ y,2)\}, \{(x,2),(y,2),(x \circ y,3)\}$, and $\{(x,3),(y,3),(x \circ y,1) \in T$.

This construction produces a triple system of *every* order $n \equiv 1 \pmod 6 \geq 19$. But that's alright, since we already have examples of triple systems of order 7 and 13 (Example 1.1). □

Combining the $6k+3$ and $6k+1$ Constructions (along with the triple systems of orders 7 and 13 in Example 1.1) gives a triple system of every order $n \equiv 1$ or $3 \pmod 6$.

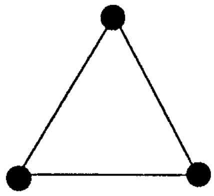
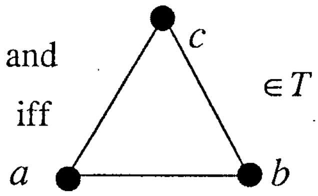
With the existence of triple systems in hand we turn our attention to suppling a few details omitted in the introduction.

Let (K_n, T) be a triple system based on Q and define a binary operation «$\circ$» on Q by $a \circ a = a$ for all $a \in Q$ *and* if $a \neq b$, $a \circ b = c$ if and only if $\{a, b, c\} \in T$. To begin with, $(Q, \circ)$ is *always* a quasigroup. This is easy to see. Let $\{a, b, c\}$ and $\{a, d, e\} \in T$. Then $a \circ a = a \neq c = a \circ b$ *and* $a \circ b = c \neq e = a \circ d$. Since $(Q, \circ)$ is *finite* (and commutative) the left cancellation law does the trick!

Trivially $(Q, \circ)$ satisfies the identities $x^2 = x$ (idempotent) and $xy = yx$ (commutative). To see that $(Q, \circ)$ also satisfies the identity $(yx)x = y$ is just as easy. Let $a \neq b \in Q$ and $\{a, b, c\} \in T$. Then $(b \circ a) \circ a = c \circ a = b$.

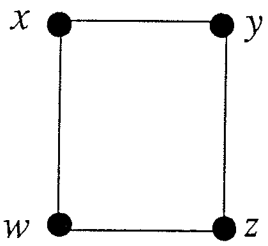
On the other hand, let $(Q, \circ)$ be a Steiner quasigroup; i.e., a quasigroup satisfying the identities $x^2 = x, xy = yx$, and $(yx)x = y$. Let K_n be based on Q and define a collection of triangles T by $\{a, b, c\} \in T$ if and only if $a \circ b = b \circ a = c$, $a \circ c = c \circ a = b$, and $b \circ c = c \circ b = a$. If $a \neq b$, then $\{a, b, a \circ b\} \in T$ (by definition) since $a \circ b = b \circ a = a \circ b$, $a \circ (a \circ b) = (a \circ b) \circ a = (b \circ a) \circ a = b$, and $b \circ (a \circ b) = (a \circ b) \circ b = a$. It follows that the edge $\{a, b\}$ belongs to exactly one triangle.

The following table is self-explanatory.

Decomposition of K_n into triangles=3 cycles 	
spectrum of 3CSs	all n ≡ 1 or 3 (mod 6) [3] Steiner triple system
quasigroup	a ∘ a=a, and a ∘ b=c, iff 
equivalent quasigroup	$\begin{cases} x^2=x, \\ xy=yx, \text{ and} \\ (yx)x=y \end{cases}$

3. $4CSs$ = **4-cycle systems.**

A *4-cycle system* ($4CS$) is a pair (K_n, C), where C is an edge disjoint collection of 4-cycles which partition K_n. It is a routine exercise to show that $n \equiv 1 \pmod 8$ is *necessary* for the existence of a $4CS$ and that $|C| = n(n-1)/8$. The *number* n is called the order of the $4CS$ (K_n, C). We will denote the 4-cycle



by any cyclic shift of (x, y, z, w) or (y, x, w, z).

EXAMPLE 3.1. ($4CS$ of order 9). Let K_9 be based on Z_9 and set $C = \{(i, 1+i, 8+i, 5+i) | i \in Z_9\}$. Then (K_9, C) is a $4CS$ of order 9.

$4CSs$ are easy to construct!

The $8n+1$ *Folk Construction.* Let $|X| = 4n$ and let $H = \{h_1, h_2, \ldots, h_n\}$ be a partition of X into subsets (called *holes*) of size 4. Let $S = \{\infty\} \cup (X \times \{1,2\})$ and define a collection of 4-cycles C of K_{8n+1} based on S as follows:

(1) For each hole $h_i \in H$, place a copy of the $4CS$ in Example 3.1 on $\{\infty\} \cup (h_i \times \{1,2\})$ and place these 9 4-cycles in C, and

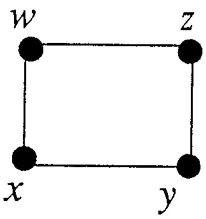
(2) if x and y belong to different holes of H, place the 4-cycle $((x,1),(y,1),(x,2),(y,2))$ in C.

It is immediate that (K_{8n+1}, c) is a $4CS$ of order $8n+1$. □

Unfortunately, there is no reasonable (or unreasonable) way to define a quasigroup from the 4-cycles of a $4CS$. There are only two ways of *attempting* to define a binary operation from the cycles of a $4CS$ (K_n, C).

One way. Let (K_n, C) be a $4CS$ and define

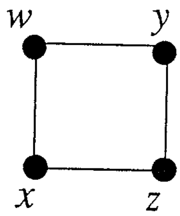
$x \circ x = x$, all $x \in Q$, and

$x \circ y = z$, iff  $\in C$.

But then $x \circ w = z$ *also* so $(Q, \circ)$ is *Always Never* a quasigroup.

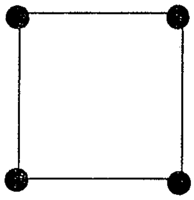
The other way. Let (K_n, C) be a $4CS$ and define

$x \circ x = x$, all $x \in Q$, and

$x \circ y =$ (toss a coin with z on one side and w on the other) iff  $\in C$.

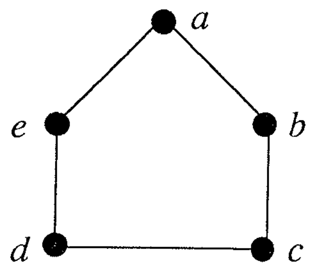
There are two things wrong with this attempt, not the least of which is that x and y may not appear in a 4-cycle as opposite vertices!

Well, so much for $4CS$!

Decomposition of K_n into 4-cycles 	
spectrum of 4CSs	all $n \equiv 1 \pmod 8$ Folk Theorem
quasigroup	
equivalent quasigroup	

4. $5CSs$ **=pentagon systems.**

A 5-*cycle system* (5CS) or *pentagon system* is a pair (K_n, P), where P is an edge disjoint collection of 5-cycles (or pentagons) which partition K_n. We will denote the pentagon



by any cyclic shift of (a, b, c, d, e) or (b, a, e, d, c).

EXAMPLE 4.1.

(1) $(K_5, P), P = \{(1,2,3,4,5), (1,3,5,2,4)\}$.

(2) (K_{11}, P_1)

$P_1=$

1	3	9	5	4
2	4	10	6	5
3	5	11	7	6
4	6	1	8	7
5	7	2	9	8
6	8	3	10	9
7	9	4	11	10
8	10	5	1	11
9	11	6	2	1
10	1	7	3	2
11	2	8	4	3

(3) (K_{11}, P_2)

$P_2=$

1	3	10	5	4
2	4	11	6	5
3	5	1	7	6
4	6	2	8	7
5	7	3	9	8
6	8	4	10	9
7	9	5	11	10
8	10	6	1	11
9	11	7	2	1
10	1	8	3	2
11	2	9	4	3

There are two *reasonable* way to define a binary operation from the cycles of a pentagon system (K_n, P) based on Q.

(1) $a \circ a = a$, for all $a \in Q$, and if $a \neq b$, $a \circ b = c$ *and* $b \circ a = e$ iff $(a,b,c,d,e) \in P$; *OR*

(2) $a \circ a = a$, for all $a \in Q$, and if $a \neq b$, $a \circ b = b \circ a = d$ iff $(a,b,c,d,e) \in P$.

We will concern ourselves with (1) *only*. The primary reason being that a *similar* definition always produces a well-defined binary operation for any k-cycle system when $k \geq 5$.

EXAMPLE 4.2. The following groupoids are constructed from the corresponding pentagon systems in Example 4.1 using definition (1).

(1)

$\circ$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

(2)

$\circ_1$	1	2	3	4	5	6	7	8	9	10	11
1	1	6	9	5	10	4	3	7	11	2	8
2	9	2	7	10	6	11	5	4	8	1	3
3	4	10	3	8	11	7	1	6	5	9	2
4	3	5	11	4	9	1	8	2	7	6	10
5	11	4	6	1	5	10	2	9	3	8	7
6	8	1	5	7	2	6	11	3	10	4	9
7	10	9	2	6	8	3	7	1	4	11	5
8	6	11	10	3	7	9	4	8	2	5	1
9	2	7	1	11	4	8	10	5	9	3	6
10	7	3	8	2	1	5	9	11	6	10	4
11	5	8	4	9	3	2	6	10	1	7	11

(3)

$\circ_2$	1	2	3	4	5	6	7	8	9	10	11
1	1	7	10	5	3	10	6	3	11	2	8
2	9	2	8	11	6	4	11	7	4	1	3
3	4	10	3	9	1	7	5	1	8	5	2
4	3	5	11	4	10	2	8	6	2	9	6
5	7	4	6	1	5	11	3	9	7	3	10
6	11	8	5	7	2	6	1	4	10	8	4
7	5	1	9	6	8	3	7	2	5	11	9
8	10	6	2	10	7	9	4	8	3	6	1
9	2	11	7	3	11	8	10	5	9	4	7
10	8	3	1	8	4	1	9	11	6	10	5
11	6	9	4	2	9	5	2	10	1	7	11

Inspection reveals that the groupoids constructed in (1) and (2) are *quasigroups*, whereas the groupoid constructed in (3) is *not*. For example $1 \circ_2 3 = 1 \circ_2 6 = 10$. So, unlike the case for Steiner triple systems (where the associated groupoid is *always* a quasigroup) *and* the case for 4-cycle systems (where it's *never* possible to define a quasigroup), sometimes the groupoid associated with a pentagon system *is* a quasigroup and sometimes *not*! The problem then is to determine, in general, under what conditions the groupoid associated with a pentagon system is a quasigroup. A bit of reflection reveals that the groupoid $(Q, \circ)$ constructed from the cycles of a pentagon system (K_n, P) is a *quasigroup* if and only if every pair of vertices are connected by a path of length 2 in *exactly* one pentagon of P.

Put another way, if any only if $(K_n, P(2))$ is a pentagon system where $P(2) = \{(a, c, e, b, d) | (a, b, c, d, e) \in P)$. Such a pentagon system is said to be 2-perfect (or Steiner).

EXAMPLE 4.3. We compute $P(2)$ for each of the pentagon systems in Example 4.1.

(1) (K_5, P) is 2-perfect, since $(K_5, P(2)) = \{(1, 3, 5, 2, 4), (1, 5, 4, 3, 2)\}$ is a pentagon system.

(2) (K_{11}, P_1) is 2-perfect since $(K_{11}, P_1(2))$ is a pentagon system.

$P_1(2) =$

1	9	4	3	5
2	10	5	4	6
3	11	6	5	7
4	1	7	6	8
5	2	8	7	9
6	3	9	8	10
7	4	10	9	11
8	5	11	10	1
9	6	1	11	2
10	7	2	1	3
11	8	3	2	4

(3) (K_{11}, P_2) is *NOT* 2-perfect since $(K_{11}, P_2(2))$ is *NOT* a pentagon system.

$P_2(2)=$

1	10	4	3	5
2	11	5	4	6
3	1	6	5	7
4	2	7	6	8
5	3	8	7	9
6	4	9	8	10
7	5	10	9	11
8	6	11	10	1
9	7	1	11	2
10	8	2	1	3
11	9	3	2	4

uncovered edges=

1	4
2	5
3	6
4	7
5	8
6	9
7	10
8	11
9	1
10	2
11	3

In 1966 Alex Rosa [10] proved that the *spectrum* for pentagon systems is precisely the set of all $n \equiv 1$ or 5 (mod 10). We give a different construction here. (A generalization of the construction given in Section 2 for triple systems).

The $10k+5$ *Construction.* Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$ and set $S = Q \times \{1,2,3,4,5\}$. Further let

$$I = \{(1,1,2,4,2), (2,2,3,5,3), (3,3,4,1,4), (4,4,5,2,5), (5,5,1,3,1)\}$$

and define a collection of pentagons P of K_{10k+5} (based on S) as follows:

(1) For each $x \in Q$, place the two pentagons $((x,1), (x,2), (x,3), (x,4), (x,5))$ and $((x,1), (x,3), (x,5), (x,2), (x,4))$ in P, and

(2) for each $x \neq y$ in Q and each $(i,i,j,k,j) \in I$ place the pentagon $((x,i), (y,i), (x,j), (x \circ y, k), (y,j))$ in P.

Then (K_{10k+5}, P) is a pentagon system. □

The $10k+1$ *Construction.* Let $Q = \{1, 2, 3, \ldots, 2k\}$, $H = \{\{1,2\}, \{3,4\}, \ldots, \{2k-1, 2k\}\}$, and $(Q, \circ)$ a commutative quasigroup with holes H. (See the $6k+1$ Construction for triple systems in Section 2). Set $S = \{\infty\} \cup (Q \times \{1, 2, 3, 4, 5\})$ and define a collection of pentagons P of K_{10k+1} (based on S) as follows:

(1) For each hole $h \in H$, construct a copy of the 2-perfect pentagon system (K_{11}, P_1) of order 11 (Example 4.1 (2)) on $\{\infty\} \cup (h \times \{1, 2, 3, 4, 5\})$ and place these pentagon in P, and

(2) if x and y belong to *different* holes of H and $(i, i, j, k, j) \in I$ (defined as above) place the pentagon $((x, i), (y, i), (x, j), (x \circ y, k), (y, j))$ in P.

Then (K_{10k+1}, P) is a pentagon system. □

These two constructions produce pentagon systems of every order except 21 and this can be taken care of by taking a block design of order 21 with block size 5 [2] and placing a copy of the pentagon system (K_5, P) in Example 4.1 (1) on each block.

THEOREM 4.4. (Alex Rosa [10]). *The spectrum for pentagon systems* (= $5CSs$) *is precisely the set of all* $n \equiv 1$ *or 5 (mod 10).* □

Remark. The proof of Theorem 4.4 is not Rosa's original proof.

Unfortunately, except for the orders $n = 5$, 13, and 21 the above constructions (as well as Rosa's original construction) *never* produce a 2-perfect (= Steiner) pentagon system. This is immediate from a cursory inspection of the $10k+5$ and $10k+1$ Constructions.

In 1983 the author and Doug Stinson [6] remedied this situation by modifying the $10k+5$ and $10k+1$ Constructions in the following manner.

The 2-perfect $10k+5$ *Construction.* In the $10k+5$ construction denote the idempotent commutative quasigroup $(Q, \circ)$ by $(Q, \circ_1)$ and let $(Q, \circ_2)$ be an idempotent self-orthogonal quasigroup which is orthogonal to $(Q, \circ_1)$. Such a pair of quasigroups exist for *every*

$2k+1 \geq 5$ [4, 13, 14, 15]. In the $10k+5$ Construction replace (2) by

(2*) for each $x \neq y$ in Q and each $(i,i,j,k,j) \in I$ place the pentagon $((x,i),(y,i),(x \circ_2 y, j),(x \circ_1 y, k),(y \circ_2 x, j))$ in P.

The proof that (K_{10k+5}, P) is 2-perfect can be found in [6]. □

The 2-perfect $10k+1$ *Construction.* In the $10k+1$ Construction denote the commutative quasigroup $(Q,\circ)$ with holes H by $(Q,\circ_1)$ and let $(Q,\circ_2)$ be a self-orthogonal quasigroup with the same holes H which is orthogonal to $(Q,\circ_1)$. (A pair of quasigroups based on Q with holes H are said to be orthogonal provided that when the partial latin squares obtained by deleting the cells $h \times h$, $h \in H$, are superimposed, the resulting collection of ordered pairs is *precisely* $(Q \times Q)\backslash\{(x,y)|x,y \in h \in H\}$.) Such a pair of quasigroup exists for every $2k \equiv 2 \pmod 4$ with a few exceptions. (See [9, 15].) In the $10k+1$ Construction replace (2) by

(2*) for each $x \neq y$ in Q and each $(i,i,j,k,j) \in I$ place the pentagon $((x,i),(y,i),(x \circ_2 y, j),(x \circ_1 y, k),(y \circ_2 x, j))$ in P.

The proof that (K_{10k+5}, P) is 2-perfect can be found in [6]. □

THEOREM 4.5. (C. C. Lindner and D. R. Stinson [6]). *The spectrum for 2-perfect pentagon systems (=Steiner pentagon systems =2 perfect 5CSs) is precisely the set of all* $n \equiv 1$ *or 5 (mod 10), except* $n = 15$ *for which no such system exists.*

Proof. Except for $10k+1$, when $2k \equiv 0 \pmod 4$, the 2-perfect $10k+5$ and $10k+1$ Constructions give 2-perfect pentagon systems of every admissable order except for a handful of exceptions. These exceptions are handled by ad hoc constructions in [6]. For $10k+1$ with $2k \equiv 0 \pmod 4$, write $10k+1 = 20m+1$ and use a block design with block size 5 (as in the $10k+1$ Construction). Finally, there does not exist a 2-perfect pentagon system of order 15, since such a system would imply the existence of a block design of order 15 with block size 5 and $\lambda = 2$. No such system exists [2]! □

Let (K_n, P) be a 2-perfect pentagon system and $(Q,\circ)$ the quasigroup defined by $a \circ a = a$ for all $a \in Q$ *and* if $a \neq b, a \circ b = c$ if and only if $(a, b, c, d, e) \in P$. It is routine to see that $(Q, \circ)$ satisfies the three identities

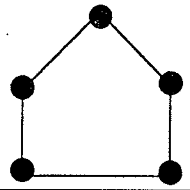
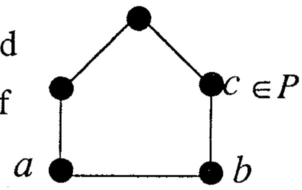
$$\begin{cases} x^2 = x, \\ (yx)x = y, \text{ and} \\ x(yx) = y(xy). \end{cases}$$

For example, if $a \neq b$ and $(a, b, c, d, e) \in P$, then $(b \circ a) \circ a = e \circ a = b$ and $a \circ (b \circ a) = a \circ e = d = b \circ c = b \circ (a \circ b)$.

On the other hand let $(Q, \circ)$ be quasigroup of order n satisfying the three identities $x^2 = x$, $(yx)x = y$, and $x(yx) = y(xy)$. It is straight forward to see that if $a \neq b$, then a, b, $a \circ b$, $b \circ a$, and $b \circ (a \circ b) = a \circ (b \circ a)$ are five *distinct* elements. This is important in the following definition. Now define a collection of pentagons P of K_n (based on Q) by $P = \{(a, b, a \circ b, b \circ (a \circ b) = a \circ (b \circ a), b \circ a) |$ all $a \neq b \in Q\}$. The proof that P is an edge disjoint collection of pentagons follows from the fact that the pentagon $(a, b, a \circ b, b \circ (a \circ b) = a \circ (b \circ a), b \circ a)$ determined by the edge $\{a, b\}$ is uniquely determined by any edge belonging to it. For example, consider the edge $\{a \circ b, b \circ (a \circ b)\}$. The pentagon defined by the edge $\{a \circ b, b \circ (a \circ b)\}$ is $(a \circ b, b \circ (a \circ b), (a \circ b) \circ (b \circ (a \circ b)) = b \circ ((a \circ b) \circ b) = b \circ a$, $(b \circ (a \circ b)) \circ (b \circ a) = (a \circ (b \circ a)) \circ (b \circ a) = a$, $(b \circ (a \circ b)) \circ (a \circ b) = b) = (a \circ b, b \circ (a \circ b)$, $b \circ a, a, b) = (a, b, a \circ b, b \circ (a \circ b), b \circ a)$. Hence (K_n, P) is a pentagon system. Now (K_n, P) will be 2-perfect if and only if $(K_n, P(2))$ is a pentagon system, and this will be true if and only if each edge of K_n belongs to a pentagon of $P(2)$. So, let $\{a, c\}$ be any edge of K_n. Since $(Q, \circ)$ is a quasigroup $a \circ b = c$ for some $b \in Q$. But then $(a, b, c, b \circ c, b \circ a) \in P$ and so $(a, c, b \circ a, b, b \circ c) \in P(2)$.

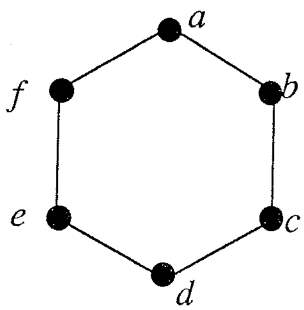
It follows that a 2-*perfect* pentagon system is *equivalent* to a quasigroup satisfying the three identities $x^2 = x$, $(yx)x = y$, and $x(yx) = y(xy)$.

We collect all of the above information in the following easy to read table.

Decomposition of K_n into pentagons=5 cycles 	
spectrum of 5CSs	all $n \equiv 1$ or 5 (mod 10) [10] Pentagon system
spectrum of 2-perfect 5CSs	all $n \equiv 1$ or 5 (mod 10) *except* n=15 [6] Steiner pentagon system
quasigroup	$a \circ a = a$, and $a \circ b = c$, iff 
equivalent quasigroup	$\begin{cases} x^2 = x, \\ (yx)x = y, \text{ and} \\ x(yx) = y(xy) \end{cases}$

5. $6CSs$ **=hexagon systems.**

A 6-cycle system ($6CS$) or *hexagon system* is a pair (K_n, H), where H is an edge disjoint collection of 6-cycles (or *hexagons*) which partition K_n. The number n is called the *order* of the hexagon system (K_n, H), and $|H| = n(n-1)/12$. We will denote the hexagon



by any cyclic shift of (a, b, c, d, e, f) or (b, a, f, e, d, c).

EXAMPLE 5.1.

(1) (K_9, H)

$H=$

1	9	2	4	5	7
2	7	3	5	6	8
3	8	1	6	4	9
1	2	6	9	8	4
2	3	4	7	9	5
3	1	5	8	7	6

(2) (K_{13}, H_1)

$H_1=$

5	9	11	8	13	12
6	10	12	9	1	13
7	11	13	10	2	1
8	12	1	11	3	2
9	13	2	12	4	3
10	1	3	13	5	4
11	2	4	1	6	5
12	3	5	2	7	6
13	4	6	3	8	7
1	5	7	4	9	8
2	6	8	5	10	9
3	7	9	6	11	10
4	8	10	7	12	11

(3) (K_{13}, H_2)

$H_2=$

1	2	13	3	12	7
2	3	1	4	13	8
3	4	2	5	1	9
4	5	3	6	2	10
5	6	4	7	3	11
6	7	5	8	4	12
7	8	6	9	5	13
8	9	7	10	6	1
9	10	8	11	7	2
10	11	9	12	8	3
11	12	10	13	9	4
12	13	11	1	10	5
13	1	12	2	11	6

It is trivial to see that a necessary condition for the existence of a hexagon system of order n is $n \equiv 1$ or 9 (mod 12). They are just as easy to construct.

The $n+12$ Folk Construction. Let (K_n, H) be a hexagon system of order n based on $\{\infty\} \cup X$ and (K_{13}, H_1) the hexagon system of order 13 in Example 5.1 (2) based on $\{\infty\} \cup Y$. In [12] Dominique Sotteau has shown (among other things) that the complete bipartite graph $K_{X,Y}$ (based on X and Y) can be decomposed into hexagons. Let B be such a decomposition and define a collection of hexagons H^* on $\{\infty\} \cup Y \cup X$ by $H^* = H \cup H_1 \cup B$.

It is immediate that (K_{n+12}, H^*) is a hexagon system. Starting with the hexagon systems of orders 9 and 13 (Example 5.1 (1) and (2)) gives a hexagon system of every order $n \equiv 1$ or 9 (mod 12).

A hexagon system (K_n, H) is said to be *2-perfect* provided $(K_n, H(2))$ is a *triple system*, where $H(2) = \{\{a,c,e\},\{b,d,f\} \mid (a,b,c,d,e,f) \in C\}$. (Put another way, if and only if the distance 2 graph covers the edges of K_n).

EXAMPLE 5.2. We compute $H(2)$ for each of the hexagon systems in Example 5.1.

(1) (K_9, H) is *NOT* 2-perfect since $(K_9, H(2))$ is *NOT* a triple system.

$H(2)=$

1	2	5	9	4	7
2	3	6	7	5	8
3	1	4	8	6	9
1	6	8	2	9	4
2	4	9	3	7	5
3	5	7	1	8	6

uncovered edges=

1 7	4 5
1 9	4 6
2 7	4 8
2 8	5 6
3 8	5 9
3 9	6 7

(2) (K_{13}, H_1) is 2-perfect since $(K_{13}, H_1(2))$ is a triple system.

$H_1(2)=$

5	11	13	9	8	12
6	12	1	10	9	13
7	13	2	11	10	1
8	1	3	12	11	2
9	2	4	13	12	3
10	3	5	1	13	4
11	4	6	2	1	5
12	5	7	3	2	6
13	6	8	4	3	7
1	7	9	5	4	8
2	8	10	6	5	9
3	9	11	7	6	10
4	10	12	8	7	11

Notice that the triple system (K_{13},H(2)) is the cyclic triple system of order 13 in Example 1.1 (3).

(3) (K_{13}, H_2) is *NOT* 2-perfect since $(K_{13}, H_2(2))$ is *NOT* a triple system.

$H_2(2)=$

1	13	12	2	3	7
2	1	13	3	4	8
3	2	1	4	5	9
4	3	2	5	6	10
5	4	3	6	7	11
6	5	4	7	8	12
7	6	5	8	9	13
8	7	6	9	10	1
9	8	7	10	11	2
10	9	8	11	12	3
11	10	9	12	13	4
12	11	10	13	1	5
13	12	11	1	2	6

uncovered edges=

1	4	1	7
2	5	2	8
3	6	3	9
4	7	4	10
5	8	5	11
6	9	6	12
7	10	7	13
8	11	8	1
9	12	9	2
10	13	10	3
11	1	11	4
12	2	12	5
13	3	13	6

As was the case with pentagon systems, if (K_n, H) is a hexagon system based on Q we will define a binary operation «$\circ$» b on Q by:

(1) $a \circ a = a$, all $a \in Q$, and

(2) if $a \neq b$, $a \circ b = c$ and $b \circ a = f$ iff $(a, b, c, d, e, f) \in H$.

Again, as was the case with pentagon systems, the groupoid $(Q, \circ)$ defined in this manner is a *quasigroup* if and only if (K_n, H) is 2-*perfect*.

EXAMPLE 5.3. The following groupoids are constructed from the corresponding hexagon systems in Example 5.1. ((1) and (2) suffice for illustrative purposes).

$\circ$	1	2	3	4	5	6	7	8	9
1	1	6	5	8	8	4	5	3	2
2	4	2	4	5	9	9	3	6	1
3	5	5	3	7	6	7	2	1	4
4	2	9	2	4	7	1	9	9	3
5	3	3	7	2	5	8	1	7	7
6	8	1	1	9	3	6	8	2	8
7	9	8	5	3	4	3	7	5	5
8	6	7	9	1	1	5	6	8	6
9	7	4	8	6	2	2	4	4	9

NOT a quasigroup since (K_9, H) is NOT 2-perfect

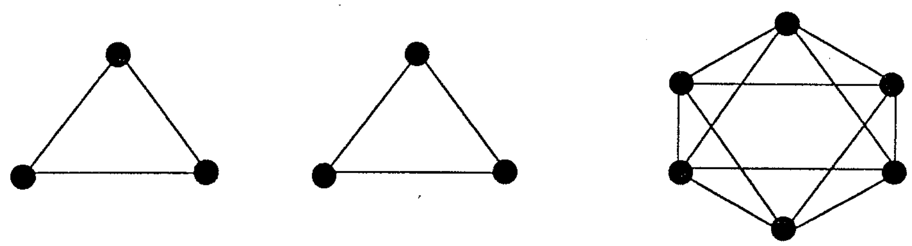
$\circ$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	10	13	2	7	5	11	9	12	4	3	8	6
2	7	2	11	1	3	8	6	12	10	13	4	4	9
3	10	8	3	12	2	4	9	7	13	11	1	6	5
4	6	11	9	4	13	3	5	10	8	1	12	2	7
5	8	7	12	10	5	1	4	6	11	9	2	13	3
6	4	9	8	13	11	6	2	5	7	12	10	3	1
7	2	5	10	9	1	12	7	3	6	8	13	11	4
8	5	3	6	11	10	2	13	8	4	7	9	1	12
9	13	6	4	7	12	11	3	1	9	5	8	10	2
10	3	1	7	5	8	13	12	4	2	10	6	9	11
11	12	4	2	8	6	9	1	13	5	3	11	7	10
12	11	13	5	3	9	7	10	2	1	6	4	12	8
13	9	12	1	6	4	10	8	11	3	2	7	5	13

IS a quasigroup since (K_{13}, H_1) is 2-perfect

The additional requirement that a hexagon system be 2-perfect allows us to construct a quasigroup from the cycles of a hexagon system. Unfortunately, the constructions of 2-perfect hexagon systems are brutally difficult, tedious, and totally out of line with the types of constructions in 'this paper. Also extremely long! For this reason we refer the interested reader to the original paper [5] for details. We will content ourselves here with a statement of what is known followed by a brief description of the method of attack.

THEOREM 5.4. (C. C. Lindner, K. T. Phelps, and C. A. Rodger [5]). *The spectrum for 2-perfect hexagon systems (=2 perfect 6CSs) is the set of all* $n \equiv 1$ *or 9 (mod 12), except* $n = 9$ *for which no such system exists, and possibly* $n = 45$ *and 57.*

Outline of Proof. It is trivial to see that there does not exist a 2-perfect hexagon system of order 9. The construction of 2-perfect hexagon systems of orders $n \geq 13$ requires the construction of a triple system (K_n, T) of order $n \equiv 1$ or 9 (mod 12) with the *additional property* that the triples in T can be partitioned into partial parallel classes of two triangles each and such that the two disjoint triangles in each partial parallel class can be superimposed so that the resulting hexagons partition K_n.



partial parallel class oriented and superimposed

This is not an easy undertaking. For one thing, a partition of the triangles in T into partial parallel classes does not necessarily guarantee that the triangles in each partial parallel class can be oriented and superimposed so that the resulting collection of hexagons forms a hexagon system. A difficult procedure whose description is somewhat out of keeping in an elementary survey paper (because of space requirements if nothing else). □

Now let (K_n, H) be a 2-perfect hexagon system based on Q and $(Q, \circ)$ the associated quasigroup; i.e., the quasigroup defined by $a \circ a = a$ for all $a \in Q$ *and* if $a \neq b, a \circ b = c$ and $b \circ a = f$ if and only if $(a, b, c, d, e, f) \in H$. Then $(Q, \circ)$ satisfies the three identities

$$\begin{cases} x^2 = x, \\ (yx)x = y, \text{ and} \\ (xy)(y(xy)) = x(yx). \end{cases}$$

For example, if $a \neq b$ and $(a, b, c, d, e, f) \in H$, then

$$(a \circ b) \circ (b \circ (a \circ b)) = c \circ (b \circ c) = c \circ d = e = a \circ f = a \circ (b \circ a).$$

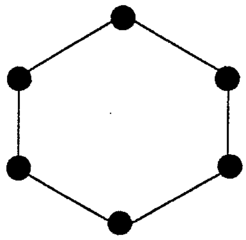
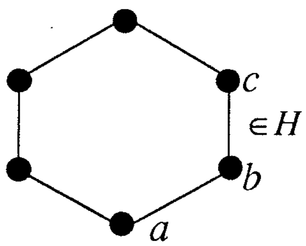
Unfortunately, a quasigroup satisfying the above three identities does not necessarily come from a 2-perfect hexagon system. This is easy to see. Let $(Q, \circ)$ be the quasigroup associated with a triple system (= $3CS$). Then $(Q, \circ)$ satisfies $x^2 = x$, $xy = yx$, and $(yx)x = y$ and therefore also satisfies the identity $(xy)(y(xy)) = x(yx)$. Since the spectrum for triple systems is different from the spectrum for 2-perfect hexagon systems, a 2-perfect hexagon system is NOT equivalent to a quasigroup satisfying the identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = x(yx)$.

However if we throw in the property of being anti-symemtric things are different. *Anti-symmetric* means $a \circ b \neq b \circ a$ for all $a \neq b \in Q$. Clearly the quasigroup $(Q, \circ)$ associated with the 2-perfect hexagon system (K_n, H) is anti-symemtric. (If $a \neq b$ and $(a, b, c, d, e, f) \in H$, then $a \circ b = c \neq f = b \circ a$). On the other hand, if $(Q, \circ)$ is a quasigroup satisfying the three identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy) = x(yx)$ *and* is also anti-symmetric then if $a \neq b \in Q$, the six elements $a, b, a \circ b$, $b \circ (a \circ b), (a \circ b) \circ (b \circ (a \circ b)) = a \circ (b \circ a)$, and $b \circ a$ are all *distinct*. Therefore (just as for 2-perfect pentagon systems) we can construct a 2-perfect hexagon system (K_n, H) by defining H to be:

$$H = \{(a, b, a \circ b, b \circ (a \circ b), a \circ (b \circ a), b \circ a | a \neq b \in Q\}.$$

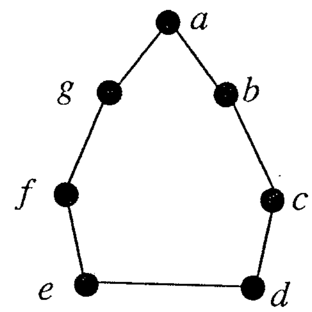
Hence a 2-perfect hexagon system is equivalent to an *anti-symmetric* quasigroup satisfying the three identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = x(yx)$. Whether or not the *property* of being anti-symmetric can be replaced with a *finite* collection of 2-variable quasigroup identities I so that $I^* = \{x^2 = x, (yx)x = y, (xy)(y(xy)) = x(yx)\} \cup I$ implies anti-symmetry but *not* $x = y$ is an open(and so it seems to the author) interesting problem.

We collect everything together in the following table.

<table>
<tr><td colspan="2">Decomposition of K_n into hexagons=6 cycles
</td></tr>
<tr><td>spectrum of 6CSs</td><td>all n ≡ 1 or 9 (mod 12) (Folk Theorem) Hexagon system</td></tr>
<tr><td>spectrum of 2-perfect 6CSs</td><td>all n ≡ 1 or 9 (mod 12) except n=9, and possible n=45 and 57 [5]</td></tr>
<tr><td>quasigroup</td><td>a o a=a, and
a o b=c, iff </td></tr>
<tr><td>equivalent quasigroup</td><td>$\left\{\begin{array}{l} x^2=x, \\ (yx)x=y, \text{ and} \\ (xy)(y(xy))=x(yx) \\ + \text{ Anti-Symmetry} \end{array}\right.$</td></tr>
</table>

6. $7CSs$ =heptagon systems.

A 7-cycle system ($7CS$) or *heptagon system* is a pair (K_n, C), where C is an edge disjoint collection of 7-cycles (or heptagons) which partition K_n. As usual, the number n is called the *order* of the heptagon system (K_n, C) and $|C| = n(n-1)/14$. In what follows we will denote the heptagon



by any cyclic shift of (a, b, c, d, e, f, g) or (b, a, g, f, e, d, c).

Some examples are in order.

EXAMPLE 6.1.

(1) (K_7, C)

C=

1	2	3	4	5	6	7
1	3	5	7	2	4	6
1	4	7	3	6	2	5

(2) (K_{15}, C) (Alex Rosa [11]).

$$C=\begin{array}{|ccccccc|}
1 & 4 & 10 & 8 & 13 & 6 & 3\\
2 & 5 & 11 & 9 & 14 & 7 & 4\\
3 & 6 & 12 & 10 & 15 & 8 & 5\\
4 & 7 & 13 & 11 & 1 & 9 & 6\\
5 & 8 & 14 & 12 & 2 & 10 & 7\\
6 & 9 & 15 & 13 & 3 & 11 & 8\\
7 & 10 & 1 & 14 & 4 & 12 & 9\\
8 & 11 & 2 & 15 & 5 & 13 & 10\\
9 & 12 & 3 & 1 & 6 & 14 & 11\\
10 & 13 & 4 & 2 & 7 & 14 & 12\\
11 & 14 & 5 & 3 & 8 & 1 & 13\\
12 & 15 & 6 & 4 & 9 & 2 & 14\\
13 & 1 & 7 & 5 & 10 & 3 & 15\\
14 & 2 & 8 & 6 & 11 & 4 & 1\\
15 & 3 & 9 & 7 & 12 & 5 & 2
\end{array}$$

It is immediate that a necessary condition for the existence of a heptagon system of order n is $n \equiv 1$ or $7 \pmod{14}$. Without any additional properties, a heptagon system is quite easy to construct. The following two constructions are just extrapolations of the constructions used for triple systems and pentagon systems.

The $14k+7$ *Construction.* Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$ and set $S = Q \times \{1,2,3,4,5,6,7\}$. Further let $J = \{(1,1,2,4,7,4,2), (2,2,3,5,1,5,3), (3,3,4,6,2,6,4), (4,4,5,7,3,7,5),$

$$(5,5,6,1,4,1,6), (6,6,7,2,5,2,7), (7,7,1,3,6,3,1)\}$$

and define a collection of heptagons C of K_{14k+7} (based on S) as follows:

(1) For each $x \in Q$ construct a copy of the heptagon system in Example 6.1 (1) on $\{x\} \times \{1,2,3,4,5,6,7\}$ and place these three heptagons in C, and

(2) for each $x \neq y$ in Q and each $(i,i,j,k,t,k,j) \in J$ place the heptagon $((x,i),(y,i),(x,j),(y,k),(x\circ y,t),(x,k),(y,j))$ in C.

Then (K_{14k+7}, C) is a heptagon system. □

The $14k+1$ *Construction*. Let $Q = \{1, 2, \ldots, 2k\}$, $H = \{\{1,2\}, \{3,4\}, \ldots, \{2k-1, 2k\}\}$, and $(Q, \circ)$ a commutative quasigroup with holes H. (See the $6k+1$ Construction for triple systems in Section 2). Let $S = \{\infty\} \cup (Q \times \{1,2,3,4,5,6,7\})$ and define a collection of heptagons C of K_{14k+1} (based on S) as follows:

(1) For each hole $h \in H$, construct a copy of the heptagon system of order 15 (Example 6.1(2)) on $\{\infty\} \cup (h \times \{1,2,3,4,5,6,7\})$ and place these heptagons in C, and

(2) if x and y belong to *different* holes of H and $(i,i,j,k,t,k,j) \in J$ (defined as above) place the heptagon

$$((x,i),(y,i),(x,j),(y,k),(x \circ y,t),(x,k),(y,j))$$

in C.

Then (K_{14k+1}, C) is a heptagon system. □

These two constructions produce heptagon systems of every order except 29 and this can be handled with a finite field construction. (See [7]).

Folk Theorem 6.2. The spectrum for heptagon systems (= $7CSs$) is *precisely* the set of all $n \equiv 1$ or 7 (mod 14). □

Just as was the case for pentagon systems and hexagon systems we will say that a heptagon system (K_n, C) is 2-*perfect* provided $(K_n, C(2))$ is a heptagon system, where

$$C(2) = \{a,c,e,g,b,d,f)|(a,b,c,d,e,f,g) \in C\}.$$

That is, if and only if, the collection of distance 2 graphs covers the edges of K_n.

EXAMPLE 6.3. Both of the heptagon systems in Example 6.1 are 2-perfect.

(1) (K_7, C) is 2-perfect since $(K_7, C(2))$ is a heptagon system.

$C(2)=$

1	3	5	7	2	4	6
1	5	2	6	3	7	4
1	7	6	5	4	3	2

(2) (K_{15}, C) is 2-*perfect* since $(K_{15}, C(2))$ is a heptagon system.

$C(2)=$

1	10	13	3	4	8	6
2	11	14	4	5	9	7
3	12	15	5	6	10	8
4	13	1	6	7	11	9
5	14	2	7	8	12	10
6	15	3	8	9	13	11
7	1	4	9	10	14	12
8	2	5	10	11	15	13
9	3	6	11	12	1	14
10	4	7	12	13	2	15
11	5	8	13	14	3	1
12	6	9	14	15	4	2
13	7	10	15	1	5	3
14	8	11	1	2	6	4
15	9	12	2	3	7	5

Now let (K_n, C) be a heptagon system based on Q and define a binary operation «o» on Q in the «usual» way by:

$$\begin{cases} (1)\ a \circ a = a, \text{ all } a \in Q, \text{ and} \\ (2) \text{ if } a \neq b, a \circ b = c \text{ and } b \circ a = g \text{ iff} \\ \qquad (a, b, c, d, e, f, g) \in C. \end{cases}$$

Then the groupoid $(Q, \circ)$ is a *quasigroup* if and only if the heptagon system (K_n, C) is 2-perfect. (How about that for a big surprise!) By now, the interested reader should be able to construct the groupoids associated with the 2-perfect heptagon systems in Example 6.1 and see that are indeed quasigroups.

Unfortunately, except for the orders $n = 7$, 15, and 29 the constructions used in Folk Theorem 6.2 *never* produce a 2-perfect heptagon system. This is obvious from looking at the $14k + 7$ and $14k + 1$ Constructions.

Quite recently (still unpublished) this situation was remedied by Elizabetta Manduchi [7]. Part of the solution is a modification of the $14k + 1$ and $14k + 7$ Constructions and (the most difficult part) the remainder consists of a large collection of ad hoc constructions. Not too surprisingly we omit the large colection of ad hoc constructions and concentrate on the modifications of the $14k + 1$ and $14k + 7$ Constructions. The interested reader can consult the original paper [7] for the ad hoc constructions.

The 2-perfect $14k + 7$ *Construction.* In the $14k + 7$ Construction replace the idempotent commutative quasigroup $(Q, \circ)$ with a pair of idempotent quasigroups $(Q, \circ_1)$ and $(Q, \circ_2)$ which are orthogonal. Further, let $(Q, \circ_1)$ be commutative. Such a pair of quasigroups exists for *every* $2k + 1 \geq 5$. (See the 2-perfect $10k + 5$ Construction in Section 4). In the $14k + 7$ Construction replace (2) by

(2*) for each $x \neq y$ in Q and each $(i, i, j, k, t, k, j) \in J$ place the heptagon

$$((x, i), (y, i), (x \circ_2 y, j), (y, k), (x \circ_1 y, t), (x, k), (y \circ_2 x, j))$$

in C.

The proof that (K_{14k+7}, C) is 2-perfect can be found in [7]. □

The 2-perfect $14k + 1$ *Construction.* The $14k + 1$ Construction replace the commutative quasigroup $(Q, \circ)$ with holes H by a pair of quasigroups $(Q, \circ_1)$ and $(Q, \circ_2)$ with holes H which are orthogonal. Further, let $(Q, \circ_1)$ be commutative. Such a pair exists for all $2k$ with 40 exceptions. (See [9, 15]). In the $14k + 1$ Construction replace (2) by:

(2*) if x and y belong to *different* holes of H and $(i, i, j, k, t, k, j) \in J$

place the heptagon

$$((x,i),(y,i),(x \circ_2 y, j),(y,k),(x \circ_1 y, t),(x,k),(y \circ_2 x, j))$$

in C.

The proof that (K_{14k+1}, C) is 2-perfect can be found in [7]. □

Theorem 6.4. (E. Manduchi [7]). *The spectrum for 2-perfect heptagon systems (=2 perfect 7CSs) is the set of all $n \equiv 1$ or 7 (mod 14), except possibly $n = 21$ and 85.*

Proof. The $14k+7$ Construction plus Example 6.1 (1) takes care of everything $\equiv 7$ (mod 14), except $n = 21$. The $14k+1$ Construction plus Example 6.1 (2) takes care of everything $\equiv 1$ (mod 14) with 40 exceptions. These 40 exceptions, with the exception of $n = 85$, are handled by ad hoc constructions in [7]. The reader is refered to [7] for the appropriate details. □

Now, let (K_n, C) be a 2-perfect heptagon system (based on Q) and define the quasigroup $(Q, \circ)$ in the (by now) usual way:

$$\begin{cases} a \circ a = a, \text{ all } a \in Q, \text{ and} \\ \text{if } a \neq b, a \circ b = c \text{ and } b \circ a = g \text{ iff} \\ (a,b,c,d,e,f,g) \in C. \end{cases}$$

A routine exercise shows that the quasigroup $(Q, \circ)$ satisfies the three identities

$$\begin{cases} x^2 = x, \\ (yx)x = y, \text{ and} \\ (xy)(y(xy)) = (yx)(x(yx)). \end{cases}$$

Now, let $(Q, \circ)$ be a quasigroup satisfying the above three identities. Then for each $a \neq b \in Q$, the seven elements

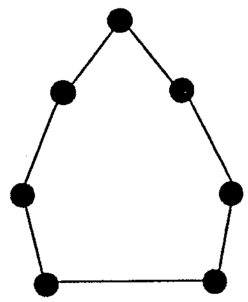
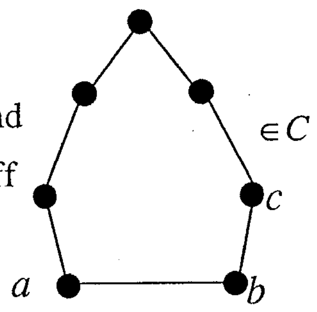
$$a, b, a \circ b, b \circ (a \circ b), (a \circ b) \circ (b \circ (a \circ b)) = (b \circ a \circ (a \circ (b \circ a)), a \circ (b \circ a),$$

and $b \circ a$ are *distinct.* If we define a collection of heptagons

$$C = \{(a, b, a \circ b, b \circ (a \circ b), (a \circ b) \circ (b \circ (a \circ b)), a \circ (b \circ a), b \circ a | \text{ all } a \neq b \in Q\},$$

then (K_n, C) is a 2-perfect heptagon system. The fact that (K_n, C) is a heptagon system is a consequence of the fact that the heptagon determined by the edge $\{a, b\}$ is uniquely determined by any edge belonging to it. The fact that (K_n, C) is 2-perfect follows from the fact that $(Q, \circ)$ is a quasigroup.

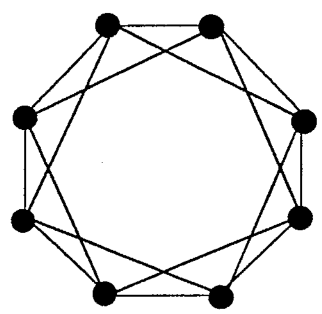
Once again we collect everything together in a table.

Decomposition of K_n into HEPTAGONS =7 cycles 	
spectrum of 7CSs	all n ≡ 1 or 7 (mod 14) (Folk Theorem) HEPTAGON system
spectrum of 2-perfect 7CSs	all n ≡ 1 or 7 (mod 14) *except* possibly n=21 and 85 Elizabetta Manduchi [7]
quasigroup	a ∘ a=a, and a ∘ b=c, iff 
equivalent quasigroup	$\begin{cases} x^2=x, \\ (yx)x=y, \text{ and} \\ (xy)(y(xy))=(yx)(x(yx)) \end{cases}$

7. Open problems.

The very first problem that comes to mind is the following. Does there exist a finite collection of 2-variable quasigroup identities such that a 2-perfect hexagon system is equivalent to a quasigroup satisfying the identities $\{x^2 = x, (yx)x = y, (xy)(y(xy)) = x(yx)\} \cup I$? Not exactly an earth-shattering problem, but never-the-less a problem of interest to people in universal algebra. (Several prominent mathematicians in universal algebra have expressed the opinion that no such collection I exists. No proofs though! Just spectulation).

It is another Folk Theorem that the spectrum for $8CSs$ (=decomposition of K_n into 8-cycles) is precisely the set of all $n \equiv 1 \pmod{16}$. An $8CS$ (K_n, C) is 2-perfect provided the collection of distance 2 graphs inside each 8-cycle cover the edges of K_n. Now the distance 2 graph inside of each octogon is a pair of disjoint 4-cycles.



Hence an $8CS$ (K_n, C) is 2-perfect if and only if $(K_n, C(2))$ is a 4-cycle system. To date not a single example of a 2-perfect $8CS$ is known. Not even for $n = 17$ (the first possible order). The author has no doubt that the spectrum for 2-perfect $8CSs$ is the same as for $8CSs$. It remains only for someone to supply a proof. Once the spectrum of 2-perfect $8CSs$ has been determined the problem of whether or not a 2-perfect $8CS$ is equivalent to a quasigroup satisfying a finite collection of 2-variable identities is immediate.

Once the problem of determing the spectrum of 2-perfect $8CSs$ has been settled (forget about 2-variable quasigroup identities for the moment) the next problem is the determination of the spectrum of 2 perfect 9-cycle systems and then 2 perfect 10-cycle systems etc. ect.

It never ends! However this paper must end. And now is as good a place as any.

Quite recently Elizabeth Billington and Peter Adame (University of Queensland (Australia) have settled the existence problem for 2-perfect $8CS_s$ by showing that the spectrum for 2-perfect $8CS_s$ is precisely all $n \equiv 1$ (mod. 16).

REFERENCES

[1] Chin-Mei Kau Fu., *The intersectios problem for pentagon systems,* Ph. D. Thesis, Auburn University (1987).

[2] Hanani H., *Balanced incomplete block designs and related designs,* Discrete Math., **11** (1975), 255-369.

[3] Kirkman T.P., *On a problem in combinations,* Cambridge and Dublin Math. J., **2** (1847), 191-204.

[4] Lindner C. C., Mullin R.C., Stinson D. R., *On the spectrum of resolvable orthogonal arrays invariant under conjugation by the Klein group K_4,* Aequationes Math., **26** (1983), 176-183.

[5] Lindner C. C., Phelps K. T. Rodger C. A., *The spectrum for 2-perfect 6-cycle systems,* J. Combinatorial Theory Ser. A, **15** (1991), 76-85.

[6] Lindner C. C., Stinson D. R., *Steiner pentagon systems,* Discrete Math., **52** (1984), 67-74.

[7] Manduchi E., *Steiner heptagon systems,* Ars Combinatoria, (to appear).

[8] Mathon R. A., Phelps K. T., Rosa A., *Small Steiner triple systems and their properties,* Ars Combinatoria, **15** (1983), 3-110.

[9] Mullin R. C., Stinson D. R., *Holey Solssoms,* Utilitas Math., **25**, (1984), 159-169.

[10] Alex Rosa., *On cyclic decompositions of the complete graph into polygons with an odd number of edges,* Casopis Pest. Mat., **91** (1966), 53-63, (in Slovak).

[11] Rosa Alex., Private communication.

[12] Sotteau D., *Decompositions of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$,* J. Combinatorial Theory Ser. B, **29** (1981), 75-81.

[13] Wang S.P., *On self-orthogonal latin squares and partial transversals of latin squares,* Ph. D. thesis, Ohio State University, (1978).

[14] Zhu L., *A few more self-orthogonal latin squares with symmetric orthogonal mates,* Congressus Numer. **42** (1984), 313-320.

[15] Zhu L., *Existence for Holey Solssoms of type 2^n,* Congressus Numer. **45**, (1984), 295-304.

Department of Algebra
Combinatorics and Analysis
Division of Mathematics
120 Mathematics Annex
Auburn University
Auburn, Alabama 36849-5307 U.S.A.