LE MATEMATICHE
Vol. XLV (1990) - Fasc. I, pp. 119-140

GRAPH DECOMPOSITIONS

CHRIS A. RODGER (Auburn)

This paper surveys some recent results and new techniques concerning edge-disjoint decompositions of K_n into copies of G, where G may be a cycle, a path, or just a small graph. Additional properties of such a decomposition are considered, such as nestings, resolvability, i-perfect decompositions and self-complementary decompositions. The use of skew Room frames is demonstrated by obtaining some new results when G is small.

1. Introduction.

In this paper, a survey is presented of some of the recents concerning decompositions of the complete graph K_n into edge-disjoint copies of a graph G. In the process, two of the techniques which have led to these advances are described: namely the use of graphs with holes; and the utilisation of design theoretic structures. Furthermore, the use of these techniques is then demonstrated by obtaining some new results for the decomposition problem when G is a small graph.

When considering graph decompositions there are many quesitions that may be asked (and not many have complete answers yet!), but perhaps the most natural problem is to find the set of values of n for which there exists a decomposition of K_n into edge-disjoint copies of a fixed graph G; we call this set of values the *spectrum* of G-decompositions of K_n. Surprisingly enough, the spectrum for a decomposition of K_n

into relatively uncomplicated graphs, such as for example cycles, has yet to be determined. Discussing the spectrum problem for some obvious choices of G, namely for cycles, paths, and small graphs, is the main problem focussed on in this paper. However several other problems are addressed, including: asking for some sort of resolvability in the decomposition, and requiring that the decomposition gives rise to another decomposition of K_n in some natural way (this includes nesting problems, self-complementary decompositions, and when G is a cycle to require the additional property of being i-perfect). These other problems are defined later in the paper when they are discussed in detail.

To be more precise, we define a G-decomposition of H to be a partition of $E(H)$ (the edge set of H) that has the property that the subgraph induced by each element of the partition is isomorphic to G. (Any of the graph theoretical terms that are not defined here can be found in [12].) For various choices of G and H, such a decomposition has been given other names. For example, using this notation a balanced incomplete block design with block size k, index 1 and on n symbols is simply a K_k-decomposition of K_n. For the sake of uniformity, this notation is adopted throughout the paper, though the more common name will be mentioned when the problem is first considered.

Finally in this section we define the design theoretic structures that are used later in the paper, namely latin squares, orthogonal latin squares, and skew Room squares, each of which may also have holes. Introducing the concept of graphs with holes is delayed until Section 2.

A *latin square* of order n on the symbols $\{1, \ldots, n\}$ is an $n \times n$ array, each cell of which contains exactly one symbol and each symbol occurs exactly once in each row and in each column. Two latin squares L_1 and L_2 are *orthogonal* if for each ordered pair (x, y) of symbols there is exactly one cell (r, c) which contains symbol x in L_1 and symbol y in L_2. L_1 is *self-orthogonal* if it is orthogonal to its transpose.

$L_1 =$

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

$L_2 =$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Fig. 1 - A pair of orthogonal latin squares of order 4. Since L_2 is the transpose of L_1, L_1 is a self-orthogonal latin square.

A skew Room square of order n on the symbols $\{0, 1, \ldots, n\}$ is an $n \times n$ array in which:

(1) each cell contains 0 or 2 symbols;

(2) each unordered pair of symbols occurs in exactly one cell, and each symbol occurs exactly once in each row and in each column;

(3) for $1 \leq i \leq n$ cell (i, i) contains the symbols 0 and i; and

(4) for each $x \neq y$, exactly one of cells (x, y) and (y, x) contains 2 symbols (this is the *skew* property).

01	45	27		36		
	02	56	13		47	
		03	67	24		15
26			04	17	35	
	37			05	12	46
57		14			06	23
34	16		25			07

Fig. 2 - A Skew Room square of order 7.

Introducing holes into these structures in a very general way is likely to cloud the ideas involved, so we restrict our attention to the case where all holes have size 2 (it is then easy to see how holes of varying sizes can be used). To this end, define $h_i = \{2i - 1, 2i\}$ for $1 \leq i \leq n$, let $S = \{1, \ldots, 2n\}$ and let $T(S)$ be the set of 2-element subsets of S. A *latin square with holes of size* 2 and of order $2n$ is a $2n \times 2n$ array in which

(1) for $1 \leq i \leq n$, the cells of $h_i \times h_i$ contain no symbols, and each other cell contains 1 symbol; and

(2) for $1 \leq i \leq n$ each row and column in h_i contains each symbol in $S - h_i$ exactly once.

A latin square with holes of size 2 is *symmetric* if the symbol in cell (x, y) is the symbol in cell (y, x) for all x and y (see Figure 3).

Two latin squares L_1 and L_2 with holes of size 2 are *orthogonal* if for all $(x,y) \in S \times S$ but $\{x,y\} \not\subseteq h_i$ for all i, there is exactly one cell (r,c) which contains symbol x in L_1 and symbol y in L_2 (see Figure 4).

A *skew Room frame with holes of size* 2 of order $2n$ is a $2n \times 2n$ array in which:

		8	5	4	7	6	3
		6	7	8	3	4	5
8	6			7	2	5	1
5	7			1	8	2	6
4	8	7	1			3	2
7	3	2	8			1	4
6	4	5	2	3	1		
3	5	1	6	2	4		

Fig. 3 - A symmetric latin square of order 8 with holes of size 2.

$L_1=$

		7	8	4	3	6	5
		5	6	8	7	4	3
8	6			7	2	5	1
7	5			1	8	2	6
3	7	8	2			1	4
4	8	1	7			3	2
5	3	6	1	2	4		
6	4	2	5	3	1		

$L_2=$

		5	6	7	8	3	4
		8	7	3	4	6	5
5	8			1	7	2	6
6	7			8	2	5	1
7	3	1	8			4	2
8	4	7	2			1	3
3	6	2	5	4	1		
4	5	6	1	2	3		

Fig. 4 - A pair orthogonal latin squares of order 8 with holes of size 2, one of which is symemtric.

			6 9		8 10		3 5	4 7	
		6 10		7 9		4 5			3 8
5 10					2 7		1 9	6 8	
	5 9			1 8		2 10			6 7
8 9		1 7					4 10	2 3	
	7 10		2 8				3 9		1 4
4 6		2 9		3 10				1 5	
	3 6		1 10		4 9				2 5
	4 8		5 7		1 3		2 6		
3 7		5 8		2 4		1 6			

Fig. 5 - A skew Room frame of order 10 with holes of size 2.

(1) the cells in $h_i \times h_i$ contain no symbols and each other cell contains 0 or 2 symbols;

(2) each unordered pair of symbols except for those in h_i for some i occurs in exactly one cell, and for $1 \leq i \leq n$, the symbols in h_i each occur exactly once in each row and each column except that they do not occur in the rows and columns in h_i; and

(3) for each unordered pair $\{x, y\}$ except for those equal to h_i for some i, exactly one of cells (x, y) and (y, x) contains 2 symbols (this is the *skew* property).

We now proceed to consider G-decomposition problems of H in the cases where G is a cycle, a path and some small graphs in turn. Throughout what follows, let C_m be a cycle of length m, and let P_m a path of length m (so P_m contains m edges).

2. Cycle-decompositions.

2.1. The spectrum problem.

After block designs perhaps the next most natural decomposition problem to be studied is to find the spectrum of cycle-decompositions of K_n. Indeed early results on this problem date back to 1965 and 1966 when the spectrum for m-cycle decompositions of K_{2xm+1} was found by Kotzig [22] when $m \equiv 0 \pmod 4$ and by Rosa [31] for the remaining congruence classes; Rosa [32] also found decompositions into troils of lenght m of K_{2xm+m} when m is odd. However, almost certainly the following necessary conditions for the existence of an m-cycle decomposition of K_n are also sufficient:

(N1) if $n > 1$ then $n \geq m$;

(N2) n is odd (each vertex has even degree); and

(N3) $2m$ divides $n(n-1)$ (m divides $|E(K_n)|$).

Therefore when m is not a prime power, there are many values of n which satisfy the necessary conditions but are not of the form of the early results of Kotzig and Rosa. Notice that for a fixed value of m, $N1$, $N2$ and $N3$ simply require that n lies in certain congruence classes modulo $2m$; call congruence classes modulo $2m$ that contain integers which satisfy $N1$, $N2$ and $N3$ *admissable* congruence classes. For this reason we will often express n as $n = 2mx + a$ where a is an integer satisfying $1 \leq a \leq 2m - 1$ that lies in an admissable congruence class.

There is strong evidence now that indeed $N1$, $N2$ and $N3$ are sufficient conditions for the existence of an m-cycle decomposition of K_n (these decompositions are also called *balanced circuit designs* and *m-cycle systems*). In a series of paper from 1975 to 1980, several results were published showing that these conditions are sufficient for some small values of m and for $m = 2p^e$ where p is a prime [3, 9, 11].

However, one of the most useful results to emerge did not concern cycle-decompositions of K_n, but cycle-decompositions of $K_{x,y}$ the complete bipartite graph. Sotteau in 1981 proved the following very neat result.

THEOREM 1. [33] *There exists a C_{2t}- decomposition of $K_{x,y}$ if and only if $x \geq t$, $y \geq t$, x and y are even, and $2t$ divides xy.*

Not only does this result completely solve the spectrum problem for cycle-decompositions of $K_{x,y}$ for cycles of even length, but it also leads to the following result for cycle-decompositions of K_n.

THEOREM 2. *If there exists a C_{2t}-decomposition of K_a for some integer $a, 2t < a \leq 6t$ then there exists a C_{2t}-decomposition of $K_{2mx+a} = K_{4tx+a}$ for all $x \geq 1$.*

Proof. The proof is by induction on x, so assume that there exist a C_{2t}-decomposition of K_{4tx+a} on the vertex set $\{0, \ldots, 4tx + a - 1\}$. By the results of Kotzig and Rosa quoted earlier, there exists a C_{2t}-decomposition of K_{4t+1} on the vertex set $\{0, 4tx + a, 4tx + a + 1, \ldots, 4t(x+1) + a - 1\}$. Also, by the result of Sotteau, there exists a C_{2t}-decomposition of $K_{4tx+a-1,4t}$ on the vertex sets $\{1, \ldots, 4tx + a - 1\}$ and $\{4tx + a, \ldots, 4t(x+1) + a - 1\}$. Combining these three cycle decompositions gives a C_{2t}-decomposition of $K_{4t(x+1)} + a$ as required. □

What Theorem 2 actually says is that if you can find a C_{2t}-decomposition on the smallest number of vertices in an admissible congruence class (modulo $4t$) then you can find a C_{2t}-decomposition on n vertices for all n in that congruence class modulo $4t$. So for even cycles, solving the spectrum problem becomes a job of finding the smallest decomposition in each admissable congruence class.

The equivalent result to Theorem 2 for odd cycles (cycles of odd length) did not appear until this year, but now we also have the following result.

THEOREM 3. [20] *If there exists a C_{2t+1}-decomposition of K_a for some integer $a, 2t < a \leq 6t$ then there exists a C_{2t+1}-decomposition of $K_{2mx+a} = K_{2(2t+1)x+a}$ for all $x \geq 1$.*

Clearly a proof of Theorem 3 needs some new ideas, since the proof of Theorem 2 relies heavily on Theorem 1. Of course there is no hope of obtaining the result equivalent to Theorem 1 for odd cycles: bipartite graphs have no odd cycles! In fact two ideas were needed.

The first idea was to use a symmetric latin square of order $2x$ with holes of size 2 (see Figure 3) to obtain a C_{2t+1}-decomposition of the complete x-partite graph with $4t+2$ vertices in each part (we shall denote this graph by K^x_{4t+2}). This can be done as follows. Let L_1 be a latin square of order $2x$ with holes of size 2 on the symbols $\{1, \ldots, 2x\}$. Label the vertices of K^x_{4t+2} with the elements of $\{0, 1, \ldots, 2t\} \times \{1, 2, \ldots, 2x\}$ so that for $1 \leq i \leq x$, the vertices in the *ith* part are the vertices in $\{0, 1, \ldots, 2t\} \times \{2i-1, 2i\}$. Let $c(y, z; i)$ denote the $2t+1$ cycle depicted in Figure 6, where $y \cdot z$ in the symbol in cell (y, z) of L.

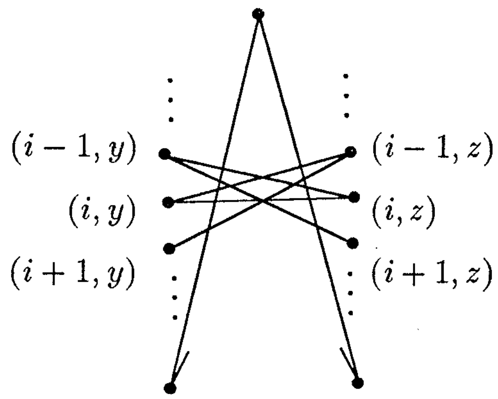


Fig. 6 - The $(2t+1)$ - cycle $c(y, z; i)$.

Formally $c(y, z; i) = (c_0, c_1, \ldots, c_{2t})$ where for $0 \leq j \leq t-1$,

$$c_j = \begin{cases} (i + j/2, y) & \text{if } j \text{ is even,} \\ (i - (j+1)/2, z) & \text{if } j \text{ is odd,} \end{cases}$$

$$c_{2t-j} = \begin{cases} (i + j/2, z) & \text{if } j \text{ is even,} \\ (i - (j+1)/2, y) & \text{if } j \text{ is odd, and} \end{cases}$$

$$c_t = (i + ((-1)^t(2t+1) - 1)/4, y \cdot z).$$

It is natural to think of the vertices arranged in a $(2t+1) \times 2x$ array, the vertices on level l being the vertices with first coordinate l, the vertices in *column* y being the vertices with second coordinate y. Then with this notation it is clear that for $0 \leq l \leq 2t$, the $(2t+1)$-cycles $c(y, z; l)$ partition almost all of the edges between vertices in column y and vertices in column z, missing only the edges joining such vertices that are exactly t levels apart; these cycles also include the edges joining vertices in column y and z to vertices in column $y \cdot z$ which are exactly t levels apart. Now the reason that we choose L_1 to be a latin square with holes of size 2 becomes apparent: if we write $y = 2w$ (or $2w - 1$) then as z ranges ever the elements of $\{1, \ldots, 2x\} - \{2w - 1, 2w\}, y \cdot z$ ranges over $\{1, \ldots, 2x\} - \{2w - 1, 2w\}$ as well and so the cycles $c(y, z; l)$ include all edges from each vertex in column $y = 2w$ (or $2w - 1$) to each other vertex exactly t levels apart except for those vertices in column $2w - 1$ (or $2w$); but of course vertices in columns $2w - 1$ and $2w$ lie within one part of K^x_{4t+2} and so are not adjacent vertices anyway.

The second idea is to use graphs with a hole. In fact one could describe K^x_{4t+2} as a complete graph with x holes of size $4t + 2$, corresponding to the x parts each containng $4t + 2$ vertices. Since we have just produced a C_{2t+1}-decomposition of K^x_{4t+2}, in a sense we have only the holes to fill with small decompositions. The one remaining problem is that to prove Theorem 3 we have $n = (4t + 2)x + a$, so a additional vertices are to be included. The remaining ingredient we need is clearly to find a C_{2t+1}-decomposition of the graph formed from K_{4t+2+a} by removing the edges joining pairs of the a added vertices; one might call this graph K_{4t+2+a} *with a hole of size* a, though for convenience we denote this graph by $K_{4t+2+a} - K_a$. Once this ingredient is found, Theorem 3 follows since we can add a vertices to a C_{2t+1}-decomposition of K^x_{4t+2}, then on the vertices in each of the x parts together with the a new vertices place a C_{2t+1}-decomposition of $K_{4t+2+a} - K_a$, the hole being the a added vertices, and finally on the a new vertices place the C_{2t+1}-cycle decomposition of K_a whose existence is postulated in the theorem. It turns out that this ingredient can be found [20], mainly using a complicated application of difference methods, a technique which was used in the early Kotzig and Rosa papers and is by now well known.

We now summarize known results (at least to the author) concerning the spectrum problem. The necessary conditions $N1$, $N2$ and $N3$ have been shown to be sufficient for the existence of a C_m-decomposition of K_n when

(a) $m = p^r$ for some prime p,

(b) $m = 2p^r$ for some prime p,

(c) $m \leq 31$ and m is odd, and

(d) $m \leq 18$ and m is even.

Furthermore, a C_m-decomposition of K_n is known to exist if

(e) $n \equiv 1 \pmod{2m}$, and if

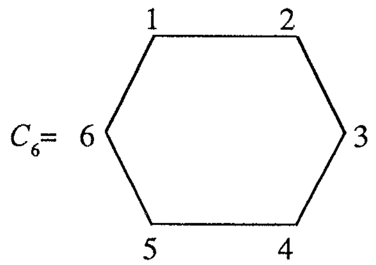
(f) $n \equiv m \pmod{2m}$ and m is odd.

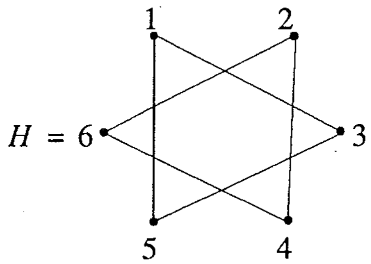
Of course this leaves many problems to be considered. However since Theorems 2 and 3 show that increasing 31 in (c) and 18 in (d) simply requires finding a C_m-decomposition of K_n for the smallest value of n in each admissable congruence class, I think a much more interesting problem is to find more results of the form of (e) and (f). I would expect that such results would still heavily rely on Theorems 2 and 3.

2.2. *Other problems.*

Many other questions can be asked about C_m-decompositions of K_n by requiring the decomposition to have additional structure. For example, one might ask that the m-cycles can be themselves partitioned into 2-factors, in which case the decomposition is called *resolvable*, or perhaps into 2-regular graphs on $n-1$ vertices, in which case the decomposition is called *almost resolvable*. The spectrum problem for resolvable C_m-decompositions of K_n (otherwise known as the Oberwolfach problem to those in this area of mathematics) has been completely solved: since the vertices must have even degree, n must be odd, and since each 2-factor contains say x m-cycles, $xm = n$, so we have $n \equiv m \pmod{2m}$ is a necessary condition; it is also a sufficient condition [2, 28]. The spectrum problem for resolvable C_m-decompositions of $K_n - F$ for some 1-factor F has also been settled, the necessary and sufficient conditions being $n \equiv 0 \pmod{m}$ if m is even and $n \equiv 0 \pmod{2m}$ if m is odd, but there is no resolvable C_3-decomposition of K_6 or of K_{12} [1, 2, 21, 23]. The spectrum problem for almost resolvable C_m-decompositions of K_n is easy to settle: n must be 1! However it is natural to consider almost resolvable C_m-decompositions of $2K_n$ (in $2K_n$ there are exactly 2 edges between each pair of vertices) in which case the spectrum has been shown to be $n \equiv 1 \pmod{m}$ whenever m is odd [18] and whenever m is even [13].

A second additional structure that may be required of a C_m-decomposition of K_n is the property of being i-perfect. A C_m-decomposition of K_n is *i-perfect* when: if each cycle C in the decomposition is replaced by the graph H formed by joining vertices that are distance i apart in C then another cycle-decomposition of K_n results. So if $m = 5$ and $i = 2$ then H is also a 5-cycle. However if $m = 6$ and $i = 2$ then H is the union of two vertex disjoint 3-cycles:





Clearly any C_m-decomposition of K_n is 1-perfect, and if it is i-perfect then it is also $(m-i)$-perfect, so we can assume that $2 \le i \le \lfloor (m-1)/2 \rfloor$. C_m-decompositions of K_n have been considered when m is 5, 6 and 7. When m is 5 or 6 we can assume that $i = 2$: 2-perfect C_5-decompositions of K_n exist for all $n \equiv 1$ or 5 (mod 10), $n \ne 15$ [26]; 2-perfect C_6-decompositions of K_n exist for all $n \equiv 1$ or 9 (mod 12), $n \ne 9$ except possibly when $n \in \{45, 57\}$ [24]. Also, there exist 2-perfect C_7-decompositions of K_n for all $n \equiv 1$ or 7 (mod 14) except possibly when $m \in \{21, 85\}$ [27]; this construction is a good example of using a pair of orthogonal latin squares with holes of size 2, one of the squares also being symmetric (see Figure 4). Notice that after replacing each cycle C in a 2-perfect C_7-decomposition of K_n with the 7-cycle formed by joining vertices distance 2 apart in C, the result is a 3-perfect C_7-decomposition of K_n, and vice versa; therefore the spectra for 2-perfect and 3-perfect C_7-decompositions of K_n are equal.

One might also ask for a C_7-decomposition of K_n that is both 2-perfect and 3-perfect. More generally, a C_m-decomposition of K_n that is i-perfect for $1 \leq i \leq \lfloor (m-1)/2 \rfloor$ is called a *Steiner* C_m-decomposition of K_n. The spectrum for Steiner C_7-decompositions of K_n is far from settled as are the spectra for i-perfect C_m decompositions for $m \geq 8$, so much work remains to be done!

3. Path-decompositions.

3.1. Various problems.

All the problems that were discussed in the previous section concerning cycle-decompositions have counterparts for path-decompositions. For example, the spectrum problem for P_m-decompositions of $\lambda K_n (\lambda K_n$ is the complete multigraph with exactly λ edges between each pair of vertices) has been completely solved. Tarsi [36] has shown that such a decomposition exists if and only if $\lambda n(n-1) \equiv 0 \pmod{2m}$ and $n \geq m+1$. As with cycle-decompositions, the spectrum problem for resolvable P_m-decompositions of λK_n has been completely solved (where, of course, a P_m-decomposition of λK_n is *resolvable* if the paths themselves can be partitioned into sets of vertex-disjoint paths which together span λK_n): there exists a resolvable P_m-decomposition of λK_n if and only if $n \equiv 0 \pmod{m+1}$ and $\lambda(m+1)(n-1) \equiv 0 \pmod{2m}$ [6, 19].

More recently, more general kinds of graph decompositions have been considered in conjunction with the notion of resolvability. For example Rees [29] has considered the problem of decomposing K_n into copies of P_1 and C_3 in such a way that these graphs can be partitioned into sets, each of which consists of vertex-disjoint P_1's or of vertex-disjoint C_3's, which span K_n; a natural name for such a resolution is a (P_1, C_3)-factorization of K_n. Rees showed that (P_1, C_3)-factorizations of K_n which contain t 1-factors exist if and only if $n \equiv 0 \pmod 6$, t is odd, $n-1 \geq t \geq 1$ and if $t = 1$ then n is not 6 or 12. Now Bermond, Heinrich and Yu [7] have found necessary and sufficient conditions for the existence of (P_1, P_m)-factorizations of λK_n. Many interesting problems of this type remain unsolved, such as the generalization of Rees' result, to find (P_1, C_m)-factorizations of K_n; necessary and sufficient conditions have been found when m is even [7].

3.2. Self-complementary P_3-decompositions.

Finally in this section, we consider a problem that is similar in nature to the 2-perfect C_m-decompositions considered in Section 2. Clearly, one could generalize 2-perfect C_m-decompositions of K_n to G-decompositions of K_n with the added restriction that replacing each copy of G with a closely related graph H results in an H-decomposition of K_n. This more general setting shows, for example, the similarity of 2-perfect C_m-decompositions to the well-studied nesting problem for Steiner triple systems [14, 34]: a nesting of a Steiner triple system fits this description exactly with $G = K_3$ and H being the complement of G in K_4 (so $H = K_{1,3}$). So now define a *self-complementary P_3-decomposition* of K_n to be a P_3-decomposition of K_n with the added restriction that replacing each copy P of P_3 with the complement of P in $V(P)$ results in another P_3-decomposition of K_n.

EXAMPLE 3.1. For $0 \leq i \leq 6$, the paths $G_i = (i, 1+i, 4+i, 2+i)$, reducing each element modulo 7, form a P_3-decomposition of K_7 on the vertex set $\{0, 1, \ldots, 6\}$ The complement of G_i in $V(G_i)$ is the path $H_i = (1+i, 2+i, i, 4+i)$, and the paths H_i for $0 \leq i \leq 6$ also form a P_3-decomposition of K_7. Therefore $G_0, \ldots, G_6$ form a *self-complementary* P_3-decomposition of K_7.

It has been proved [17] that the spectrum for self-complementary P_3-decompositions of K_n is $n \equiv 1 \pmod 3$. This result can be proved very neatly by making use of a self-orthogonal latin square L (see Figure 1) of order $t = (n-1)/3$. Let the vertex set of K_n be $\{\infty\} \cup (\{0, 1, 2\} \times \{0, 1, \ldots, t-1\})$. Then the following paths form a self-complementary P_3-decomposition of K_{3t+1}:

$$\{(\infty, (0,i), (1,i), (2,i)), ((0,i), (2,i), \infty, (1,i)) | 0 \leq i \leq t-1\} \cup$$

$$\{(l+1, i \cdot j), (l, i), (l, j), (l+1, j \cdot i) | 0 \leq l \leq 2, 0 \leq i < j \leq t-1\}$$

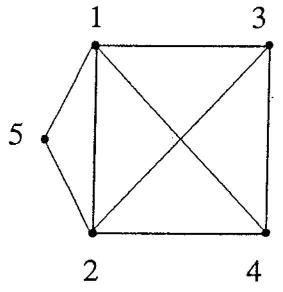
where $i \cdot j$ is the symbol in cell (i, j) of L, and all first and second coordinates are to be reduced modulo 3 and t respectively. As when considering cycles, we should think of the vertices as arranged with t vertices on each of 3 levels. Then using a latin square for our underlying structure is exactly what is needed to include edges $\{(l, x), (l+1, u)\}$ in a P_3: simply find the cell in row x that contains symbol u, say cell (x, y); then one of the defined paths, namely $((l+1, x\cdot y = u), (l, x), (l, y), (l+1, y\cdot x))$, contains the required edge. The self-orthogonality of the underlying structure is exactly what is needed to include edges $\{(l, u), (l, v)\}$ in the complement of a P_3: there exist x and y such that $x\cdot y = u$ and $y\cdot x = v$; then one of the defined paths is $((l, x\cdot y), (l-1, x), (l-1, y), (l, y\cdot x))$, so its complement is $((l-1, y), (l, x\cdot y = u),\ (l, y\cdot x = v),\ (l-1, x))$, which contains the required edge.

This construction for self-complementary P_3-decompositions of K_n together with a self-orthogonal latin squares of order t, containing a hole of size n (these are called incomplete self-orthogonal latin squares) have been used to completely solve an embedding problem for such decompositions [30].

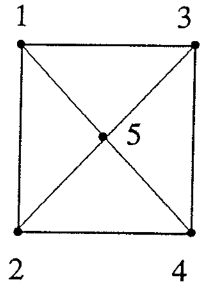
Of course there still remain many interesting unsolved questions that one might ask and are of this form, such as finding graph decompositions of K_n for other self-complementary graphs G, so that the complements of each copy of G in the decomposition also form a graph decomposition of K_n.

4. Graph decompositions for small graphs.

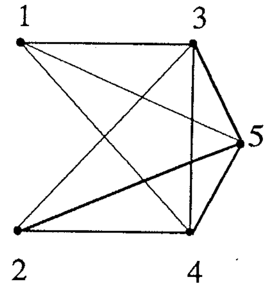
In the context of this survey, it is natural to ask for which small graphs G do there exist G-decompositions of K_n? This problem has been completely settled for all graphs with at most 4 vertices [10]. Then a major assualt was launched in 1980 on the graphs with 5 vertices where the problem was very nearly solved. There remain several graphs for which there are a few values of n for which it is not known whether or not decompositions of K_n exist (see [8] for details). However, more interestingly, there are also three graphs which have infinitely many values of n for which decompositions of K_n have not yet been found (of course, that's not quite true since one can use Wilson's result [38] to show that such decompositions exist except for possibly finitely many values of n). Those three graphs are



G_{20},



G_{21} and



$G_{22}=K_5\text{-}e$

(the notation is borrowed from [8]).

It is known that [8]:

1. there exists a G_{20}-decomposition of K_n for $n \in \{17, 33, 49, 97, 113, 117\}$ and necessary conditions for such decompositions to exist are that $n \equiv 0$ or 1 (mod 16);
2. there exists a G_{21}-decomposition of K_n for all $n \equiv 1$ (mod 16) and for $n = 64$, there is no such decomposition when $n = 16$, and necessary conditions for such decompositions to exist are that $n \equiv 0$ or 1 (mod 16); and
3. there exists a G_{22}-decomposition of K_n for $n = 19$, there do not exist such decompositions for $n \in \{9, 10, 18\}$, and necessary conditions for such decompositions to exist are that $n \equiv 0$ or 1 (mod 9).

We now demonstrate the use of skew Room frames with holes of size 2 (see Figure 5) in constructing graph decompositions. This construction arose out of discussions with Dean Hoffman as we constructed G-decompositions of K_n where G has at most 6 edges.

THEOREM 4. *There exists a G_{20}-decomposition of K_n for all $n \equiv 1$ (mod 16) except possibly $n = 65$.*

Proof. Let $n = 8(2x) + 1$ and let the vertex set of K_{16x+1} be $\{\infty\} \cup (\{0, 1, \ldots, 7\} \times \{1, 2, \ldots, 2x\})$.

As in the proof of Theorem 2.3, we proceed as follows: first decompose K_{16}^{x} (the complete x-partite graph with 16 vertices in each part) into copies of G_{20}, the vertices in the parts being $\{0, 1, \ldots, 7\} \times \{2i-1, 2i\}$ for $1 \le i \le x$; then decompose the edges in these x holes together with the edges incident with ∞ into copies of G_{20}.

Clearly the second step is easy, since we know there exists a G_{20}-decomposition of K_{17}: place a copy of such a decomposition on the x copies of K_{17} with vertex sets $\{\infty\} \cup (\{0,1,\ldots,7\} \times \{2i-1,2i\})$ for $1 \leq i \leq x$.

The first part is more complicated. We begin with our underlying structure. Let R be a skew Room frame with holes of size 2 (see Figure 5). Let L be a symmetric latin square with holes of size 2 (see Figure 3). Denote the vertex-labelled copy of G_{20} depicted earlier by $g_{20}(3,4;1,2;5)$. Then the following set forms a G_{20}-decomposition of K_{16}^{x}:

$$\{g_{20}((l+5r),(l+1,c);(l,y),(l,z);(l+2,y\cdot z))|0 \leq$$

$$\leq l \leq 7, 1 \leq r \leq 2x, 1 \leq c \leq 2x\}$$

where y and z are the symbols in cell (r,c) of R, where $y \cdot z$ is the symbol in cell (y,z) of L, and where the first coordinates are reduced modulo 8. Less formally, if we again think of the vertices as being arranged on 8 levels with $2x$ vertices on each level, we can depict these copies of G_{20} as in Figure 7.

The interesting thing to note in this construction is how the skew property is used (see (3) in the definition of skew Room frames). Suppose we want to find the copy of G_{20} that contains a pair of vertices that are 4 levels apart, say (l,a) and $(l+4,b)$ (the number 4 is relevant since it is exactly half the total number of levels used). By the skew property, exactly one of the cells (a,b) or (b,a) contains a pair of symbols say x and y. Then the edge (l,a) and $(l+4,b)$ is in

$$g_{20}((l,a),(l+4,b);(l+3,x),(l+3,y);(l+5,x\cdot y))$$

if (a,b) contains x and y, and in

$$g_{20}((l,a),(l+4,b);(l-1,x),(l-1,y);(l+1,x\cdot y))$$

if (b,a) contains x and y.

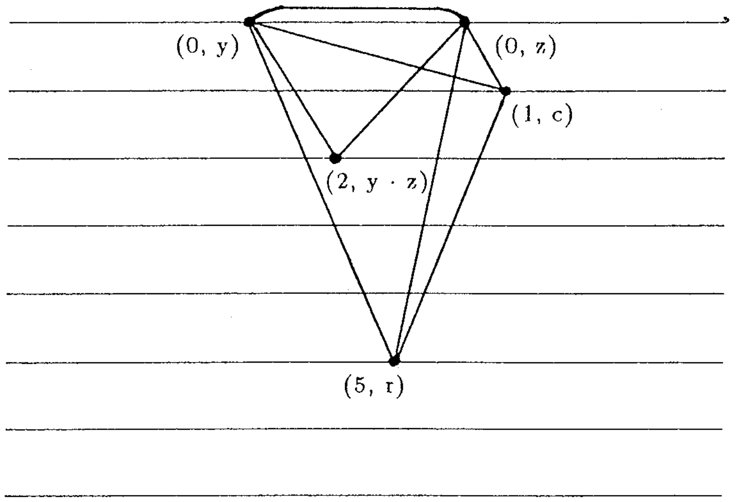


Fig. 7 - $g_{20}((5,r),(1,c);(0,y),(0,z);(2,y\cdot z))$

Why do we use holes of size 2 in the underlying structure? These holes of size 2 manifest themselves in the K_n as holes of size 2 times the number of levels, that is holes of size 16. On these 16 vertices together with ∞ we can place a G_{20}-decomposition of K_{17}. If we used underlying structure with holes of size 4 then we would have $4 \times 8 = 32$ vertices in each hole of the K_{16x+1}, so again with ∞ we could place on the 33 vertices a G_{20}-decomposition of K_{33} which is also known to exist. Clearly if the underlying structure is to contain holes of size s, then we need to know the existence of a G_{20}-decomposition of K_{8s+1}. So choosing $s = 1$ (that is, using a Room square for the underlying structure, see Figure 2) would not work since there is no G_{20}-decomposition of K_9.

Since there exist skew Room frames with holes of size 2 and/or 4 for all even orders except 6, 8 and 12 [25], and since there exist G_{20}-decompositions of K_{8s+1} for $s \in \{2,4,6,12\}$, this construction produces a G_{20}-decomposition of K_n for all $n \equiv 1 \pmod{16}$ except perhaps $n = 65$.□

Clearly this construction also works when $n \equiv 0 \pmod{16}$ except that at the moment, we have no G_{20}-decomposition of K_{8s+1} for each $s \in \{2,4,6,8,12\}$; in fact we don't have a G_{20}-decomposition for any $s \in \{2,4,6,8,12\}$! We can state this result in the following way.

LEMMA 1. *If there exists a G_{20}-decomposition of K_{8s+1} for each*

$s \in \{2,4,6,8,12\}$ *then there exists a* G_{20}*-decomposition of* K_n *for all* $n \equiv 0$ *(mod 16).*

This result may not be as good as it seems, as it depends heavily on the existence of a G_{20}-decomposition of K_{16}, which may not exist.

Finally in this section we note that we can make some progress towards settling the spectrum problem for G_{22}; If we denote by $g_{22}(a,b;c,d,e)$ the graph $K_5 - \{a,b\}$ on the vertex set $\{a,b,c,d,e\}$ and if R is a skew Room frame with holes of size 2 and of order $2x$, then

$$\{g_{22}((l+1,r),(l+5,r);(l,y),(l,z),(l+3,c))|0 \le l \le 8, 1 \le r \le 2x, 1 \le c \le 2x\}$$

where y and z are the symbols in cell (r,c) of R and where the first coordinates are reduced modulo 9, is a G_{22}-decomposition of K_{18}^{x} on the vertex set $\{0,1,\ldots,8\} \times \{1,2,\ldots,2x\}$ (see Figure 8).

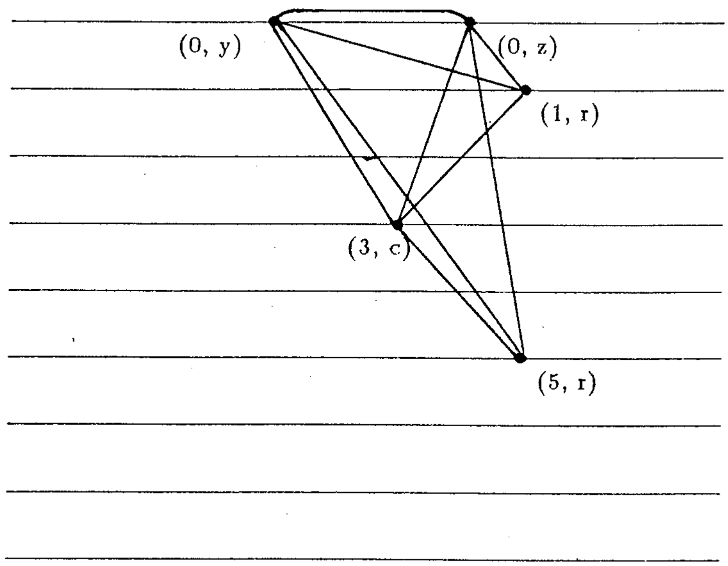


Fig. 8 - $g_{22}((1,r),(5,r);(0,y),(0,z),(3,c))$.

Again the holes of K_{18}^{x} together with ∞ can each be filled with a G_{22}-decomposition of K_{19} (which is known to exist) to produce a G_{22}-decomposition of K_n for all $n \equiv 1$ (mod 18) except possibly when $n = 9s+1$ and $s \in \{4,6,8,12,44,46,52,54,56,60,68,76\}$ (that is, s is an order for which no skew Room frame with holes of size 2 is known).

5. Conclusion.

In this survey, we have touched upon some of the recent results concerning graph decompositions. However, this is by no means an exhaustive coverage. For example, after much activity the spectrum problem for $K_{1,m}$-decompositions (also known as star or claw decompositions) of λK_n [35] and of K_n^x [37] have now been completely settled. Also, we have dealt exclusively with undirected graphs, though the equivalent questions for directed graph decompositions, naturally enough, have also been asked and answered with some success. For example, the spectrum problem for decomposing the complete directed multigraph into directed stars has been solved [15, 16]. There are several results concerning directed cycle decompositions [9, 11],and concerning the equivalent directed graph problem to that of Steiner C_m-decompositions (such directed cycle decompositions are called perfect Mendelsohn designs [4, 5]). Also, a good application of using the cycle in Figure 6 is to direct the cycle and then, using a pair of orthogonal latin squares as underlying structure, produce directed cycle systems that have a directed nesting [25]. There are many more such results, but they are beyond the scope of this paper.

The two techniques described here will undoubtedly bring further advances. Using holes in graphs is very powerful; I expect that, for example finding a G_{22}-decomposition of $K_{27} - K_9$ (that is K_{27} with a hole of size 9) or of $K_{28} - K_{10}$ would essentially settle the existence problem of G_{22}-decompositions of K_n when $n \equiv 9 \pmod{18}$ or $n \equiv 10$ (mod 18) respectively, though perhaps finding such decompositions may be difficult. Using latin squares and related design theoretic structures has a wide range of applications, mainly because of the flexibility one has in finding the additional properties of such structures that each problem demands.

6. Acknowledgment.

This research was supported by NSF grant DMS-9024645 and NSA grant MDA-904-89-H-2016.

REFERENCES

[1] Alspach B., Haggkvist R., *Some observations on the Oberwolfach problem,* J. Graph. Theory, **9** (1985), 117-187.

[2] Alspach B., Schellenberg P., Stinson D.R., Wagner D., *The Oberwolfach problem and factors of uniform odd length cycles,* J. Combinatoria Theory (A), **52** (1989), 20-43.

[3] Alspach B., Varma N., *Decomposing complete graphs into cycles of length* $2p^e$*,* Annals of Discrete Math., **9** (1980), 155-162.

[4] Bennett F. E., *Conjugate orhtogonal latin squares and Mendelsohn designs,* Ars Combinatoria, **19** (1985), 51-62.

[5] Bennett F.E., Phelps K.T., Rodger C.A., Zhu L., *Constructions of perfect Mendelsohn designs,* Discrete Math., to appear.

[6] Bermond J.C., Heinrich K., Min-Li Yu., *Existence of resolvable path designs,* European J. of Combinatorics, to appear.

[7] Bermond J.C., Heinrich K., Min-Li Yu., *On resolvable mixed path designs,* submitted.

[8] Bermond J.C., Huang C., Rosa A;, Sotteau D., *Decompositions of complete graphs into isomorphic subgraphs with five vertices,* Ars Combinatoria, **10** (1980), 211-254.

[9] Bermond J.C., Huang C., Sotteau D., *Balanced cycle and circuit designs: even case,* Ars Combinatoria, **5** (1978), 293-318.

[10] Bermond J.C., Schonheim J., *G-decompositions of* K_n*, where G has four vertices or less,* Discrete Math., **19** (1977), 113-120.

[11] Bermond J.C., Sotteau D., *Cycle and circuit designs: odd case,* Proc. colloq. Oberhof Illmenau (1978), 11-32.

[12] Bondy J.A., Murty U.S.R., *Graph Theory with Applications,* North Holland, 1976.

[13] Burling J., Heinrich K., *Near 2-factorizations of* $2K_n$*: cycles of even length,* Graphs and Combinatorics **5** (1989), 213-222.

[14] Colbourn C.J., Colbourn M.J., *Nested triple systems,* Ars combinatoria, **16** (1983), 27-34.

[15] Colbourn C.J., Hoffman D.G., Rodger C.A., *Directed star decompositions of the complete directed graph.*

[16] Colbourn C.J., Hoffman D.G., Rodger C.A., *Directed stardecompositions of directed multigraphs,* submitted.

[17] Granville A., Moisiadis A., Rees R., *On complementary decompositions of the complete graph,* Graphs and Combinatorics, **5** (1989), 57-62.

[18] Heinrich K., Lindner C.C., Rodger C.A., *Almost resolvable decompositions of* $2K_n$ *into cycles of odd length,* J. Combinatorial Theory (A), **49** (1988), 218-232.

[19] Hell P., Rosa A., *Graph decompositions, handcuffed prisoners and balanced P-designs,* Discrete Math., **2** (1972), 229-252.

[20] Hoffman D.G., Lindner C.C., Rodger C.A., *On the construction of odd cycle*

systems, J. Graph Theory, **13** (1989), 417-426.

[21] Hoffman D.G., Schellenberg P., *The existence of C_k-factorizations of $K_{2n} - F$,* Discrete Math., to appear.

[22] Kotzig A., *On decompositions of complete graphs into 4k-gons,* Mat.-Fyz. Cas, **15** (1965), 227-233.

[23] Kotzig A., Rosa A., *Nearly Kirkman triple systems,* Proc. 5th Southeastern Conf. on Combinatorics, Graph Theory and Computing, (1974), 607-614.

[24] Lindner C.C., Phelps K.T., Rodger C.A., *2-perfect 6-cycle systems,* J. Combinatorial Theory (A) **57** (1991), 76-85.

[25] Lindner C.C., Rodger C.A., Stinson D.R., *Nestings of directed cycle systems,* Ars Combinatoria, to appear.

[26] Linder C.C., Stinson D.R., *Steiner pentagon systems,* Discrete Math., **52** (1984), 67-74.

[27] Manduchi E., *2-perfect heptagon systems,* Ars Combinatoria, to appear.

[28] Ray-Chadhuri D.K., Wilson R.M., *Solution of Kirkman's school-girl problem,* Proc. Sympos. pure Math., Amer. Math. Soc., **19** (1971), 187-204.

[29] Rees R., *Uniformly resolvable paiwise bbalanced designs with block sizes two and three,* J. Combinatorial Theory (A), **45** (1987), 207-225.

[30] Rees R., Rodger C.A., *Subdesigns in complementary path decompositions and incomplete two-fold designs with block size four,* submitted.

[31] Rosa A., *On cyclic decompositions of the compete graph into $(4m+2)$-gons,* Mat.-Fyz. Cas, **16** (1966), 349-352.

[32] Rosa A., *On the cyclic decompositions of the complete graph into polygons with an odd number of edges,* Casopis Pest. Math., **91** (1966), 53-63.

[33] Sotteau D., *Decompositions of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length 2k,* J. Combinatorial Theory (B) **29** (1981), 75-81.

[34] Stinson D.R., *The spectrum of nested Steiner triple systems,* Graphs and Combinatorics, **1** (1985), 189-191.

[35] Tarsi M., *Decompositions of complete multigraphs into stars,* Discrete Math., **26** (1979), 273-278.

[36] Tarsi M., *Decomposition of the complete multigraph into simple paths: non-balanced handcuffed designs,* J. Combinatorial Theory (A), **34** (1983), 60-70.

[37] Ushio K., Tazawa S., Yamamoto S., *On claw-decompositions of a complete multipartite graph,* Hiroshima Math. Journal, **8** (1978), 207-210.

[38] Wilson R.M., *Decompositions of complete graphs into subgraphs isomorphic to a given graph,* Cong. Num., **15** (1976), 647-659.

Department of Algebra
Combinatorics and Analysis
Auburn University Alabama
USA 36849-5307