DECOMPOSITION OF A COMPLETE GRAPH INTO HEXAGONS
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1. The Past.

Decomposition of a complete graph into hexagons.

Let \( K_v \) be a non-directed complete graph with \( v \) vertices, \( K_v(p) \) be the same graph that is decomposed into \( p \)-gons with disjoined edges.

A. Rosa [3] has shown that a necessary and sufficient condition for the existence of \( K_v(p) \) is:

\[
\begin{align*}
v &\equiv 1 \text{ or } 3 \pmod{6} \text{ for } p = 3, \\
v &\equiv 1 \text{ or } 5 \pmod{10} \text{ for } p = 5, \\
v &\equiv 1 \text{ or } 7 \pmod{14} \text{ for } p = 7.
\end{align*}
\]

Moreover, it is well known that if \( p \) is prime power, then the necessary condition for the existence of \( K_v(p) \) is

\[
(1) \quad v \equiv 1 \text{ or } p \pmod{2p}.
\]

If \( p \) is not a prime power, then condition (1) is not necessary. The above showed A. Rosa [3] on constructing a \( K_{51} \) (15).

It is not known whether the condition (1) is sufficient for the existence of \( K_v(p) \) if \( p \) is a prime power.
A. Kotzig [1] proved three theorems on the existence of $K_v(p)$.

(i) If $p = 2^r$ for some positive integer $r$, then the necessary and sufficient condition for the existence of $K_v(p)$ is $v \equiv 1 \pmod{2p}$.

(ii) If $v \equiv 1 \pmod{2p}$, then there exists a $K_v(p)$ for $p \equiv 0 \pmod{4}$.

(iii) If $(v, p) = 1$ and $p \equiv 0 \pmod{4}$, then the necessary and sufficient condition for the existence of $K_v(p)$ is $v \equiv 1 \pmod{2p}$.

Moreover, he constructed $K_{33}$ (12) and $K_{25}$ (20). This proves that if $(v, p) \neq 1$, $p \equiv 0 \pmod{4}$, then the condition $v \equiv 1 \pmod{2p}$ is not necessary.

Let $v$ be the number of vertices in a complete graph. $V$ be the set of these vertices, $\cdot$ be a binary operation as follows: $a \cdot a = a$ for each $a \in V$, $a \cdot b = b \cdot a = c$, $b \neq a$, where

$$
\begin{array}{c}
\begin{array}{c}
\triangle
\end{array}
\end{array}
\in K_v(3).
$$

It is easy to see that $(V, \cdot)$ is a quasi-group and $v \equiv 1$ or 3 (mod 6) is a necessary and sufficient condition for the existence of $K_v(3)$. Moreover, $(V, \cdot)$ is a quasi-group if and only if the above condition holds. Notice that $(V, \cdot)$ is a Steiner triple system.

Similarly, we define an operation as follows $a \times a = a$ for each $a \in V$, $a \times b = b \times a = c$, where $b \neq a$ and

$$
\begin{array}{c}
\begin{array}{c}
\triangle
\end{array}
\end{array}
\in K_v.
$$

C.C. Lindner and D.R. Stinson [2] showed that the condition $v \equiv 1$ or 5 (mod 10) / the condition for the existence $K_v(5)$ / is necessary and sufficient for $(V, \circ)$ to be a quasi-group. They proved, moreover, that a $K_v(5)$ that was constructed by is not only a quasigroup, but also a Steiner-system $B(v, 5, 2)$.

The aim of this paper is to show that:

I. The necessary condition for the existence of $K_v(p)$ is

a/ $v$-odd integer, and

b/ $2p/v(v - 1)$. 

II. The necessary and sufficient condition for the existence of a $K_v(6)$ is $v \equiv 1$ or $9 \pmod{12}$.

If $p$ is even, then $K_v(p)$ do not form Steiner's system, since $\lambda$ is not integer $\left( \lambda = \frac{p-1}{2} \notin N \right)$. When we were able to construct such $K_v(p)$ condition (B): that every pair of vertices was contained in $\frac{p-2}{2}$ or $\frac{p}{2}$ polygenes then we would assume that $\lambda$ is a rational number equal to $\frac{p-1}{2}$. This would generalize $\lambda$ to rational number.

The operation $\oplus$ can be defined in such a way that $(\oplus V)$ forms a quasi-group: $a \oplus a = a$ for each $a \in V$, $a \oplus b = c$, $a \neq b$ and where $c$ belongs to the $p$-gen containing the edges $\overline{ab}$ and edges $\overline{bc}$.

We notice that $a \oplus b \neq b \oplus a$;

Proof. of I. Let us fix an arbitrary vertex $A$ and let us put edges between $A$ and all of the remaining vertices. If we claim that the remaining additional edges are edge as of closed polygons, then the degree of $A$ is an even number. Since it is a complete graph the cardinality of its vertices must be odd. In such graph there is $\frac{v(v-1)}{2}$ edges and we divide this graph into disjoint $p$-gons. Hence $\frac{v(v-1)}{2}$ divides $p$.

Proof. of II. Condition I is equivalent to $v \equiv 1$ or $9 \pmod{12}$ if $p = 6$; To prove II it is sufficient to show. That the above condition is sufficient for the existence of $K_v(6)$. The proof will be done for $v \equiv 1 \pmod{12}$ and $v \equiv 9 \pmod{12}$ separately.

Let $v \equiv 1 \pmod{12}$. First of all we construct a $K_{13}(6)$. Let us number the vertices of a graph from 0 through 12 and write down these blocks whose elements are vertices of corresponding hexagons. These blocks have the properties $\alpha$-for each $(x, y) \in V$ there exists a unique sequence $[b_1, \ldots, b_6]$ such that $x = b_i$, $y = b_j$ and $|i - j| = 1$ or $|i - j| = 5$

$B_1 = (1, 3, 5, 7, 9, 11)$, $B_2 = (2, 4, 6, 8, 10, 12)$, $B_3 = (1, 7, 2, 8, 3, 10)$,

$B_4 = (3, 9, 4, 10, 11, 12)$, $B_5 = (5, 11, 6, 12, 1, 2, 5)$, $B_6 = (0, 3, 11, 2, 9, 1)$,

$B_7 = (0, 7, 3, 6, 1, 5)$, $B_8 = (0, 11, 7, 10, 5, 9)$, $B_9 = (0, 4, 12, 9, 10, 2)$,

$B_{10} = (0, 8, 4, 3, 2, 6)$, $B_{11} = (0, 12, 8, 5, 6, 10)$, $B_{12} = (7, 8, 11, 4, 5, 12)$,
$B_{13} = (1, 8, 9, 6, 7, 4)$. 

It is easy to check that the above blocks satisfy condition $\alpha$, and condition $\beta$.

A construction of a $K_{25}$ (6) will be shown now. Let $V = \{0, 1, \ldots, 24\}$, $V_1 = \{0, 1, \ldots, 12\}$ $V_2 = \{0, 13, 14, \ldots, 24\}$. Construct $K_{13}$ (6) on $V_1$ and $V_2$ separately and denote by them $K_{13}^1$ (6) and $K_{13}^2$ (6) respectively. To get $K_{25}(6)$ we need blocks containing elements of $V_1$ and $V_2$ at the same time.

$L_1 = (1, 13, 2, 14, 3, 15), \ L_2 = (1, 17, 2, 18, 3, 19), \ L_3 = (1, 21, 2, 22, 3, 23)$

$L_4 = (1, 14, 4, 13, 3, 16), \ L_5 = (1, 18, 4, 17, 3, 20), \ L_6 = (1, 22, 4, 21, 3, 24)$

$L_7 = (5, 13, 6, 16, 4, 15), \ L_8 = (5, 17, 6, 20, 4, 19), \ L_9(5, 21, 6, 24, 4, 23)$

$L_{10} = (5, 14, 6, 15, 2, 16), \ L_{11} = (5, 18, 6, 19, 2, 20), \ L_{12} = (5, 22, 6, 21, 2, 24)$

The remaining blocks of the family will be obtained by replacing an element $i$ in each of the above 12 blocks by $i + 6$, $i = 1, 2, \ldots, 6$; It is easy to check that all the blocks satisfy condition $\alpha$.

Let now, $v = 12k + 1, \ V = \{0, 1, 2, \ldots, 12k\}$, $V_i = \{0, (i - 1)12 + 1, \ldots, 12i\}$ $i = 1, 2, \ldots, k$. Construct $K_{13}^1$ (6) on $V_i$, $i = 1, \ldots, k$. For each pair $(V_i, V_j) i \neq j, i, j = 1, \ldots, k$ construct blocks $L_1, \ldots, L_{24}$ in the same way as for $v = 25$. All these blocks form $K_{12k+1}$ (6).

These construction shows that $v \equiv 1 \pmod{12}$ is the sufficient condition for the existence of a decomposition of the complete graph into hexagons with disjoint edges.

Then we shall show the same for $v \equiv 9 \pmod{12}$. We distinguish two cases: a) $v \equiv 9 \pmod{24}$, b) $v \equiv 21 \pmod{24}$.

Case a) We construct $K_{9}$ (6) first. Let $V = \{0, 1, \ldots, 8\}$.

$B_1 = (1, 0, 2, 3, 4, 6), \ B_2 = (3, 0, 4, 5, 6, 8), \ B_3(5, 0, 6, 7, 8, 2)$,

$B_4 = (7, 0, 8, 1, 2, 4) \ B_5 = (1, 3, 6, 2, 7, 5), \ B_6 = (1, 7, 3, 5, 8, 4)$.

It is easy to check that the blocks $B_1, \ldots, B_6$ form $K_{9}$ (6) with condition $\beta$.

Let now $v = 33 V = \{0, 1, \ldots, 32\}$, $V_0 = \{0, 1, \ldots, 8\}$, $V_1 = \{0, 9, 10, \ldots, 16\}$, $V_2 = \{0, 17, 18, \ldots, 24\}, \ V_3 = \{0, 25, 26, \ldots, 32\}$. 
To get $K_v(6)$ we construct $K_9(6)$ on $V_i$, $i = 0, 1, 2, 3$ according to the above tretabrad. The remaining blocks will be obtained as follows. Let

$L_1 = (1, 9, 2, 17, 3, 25), \quad L_2 = (1, 10, 2, 18, 3, 26), \quad L_3 = (1, 11, 2, 19, 3, 27),$

$L_4 = (1, 12, 2, 20, 3, 28), \quad L_5 = (1, 13, 3, 9, 4, 17), \quad L_6 = (1, 14, 3, 10, 4, 18),$

$L_7 = (1, 15, 3, 11, 4, 19), \quad L_8 = (1, 16, 3, 12, 4, 20), \quad L_9 = (1, 21, 2, 25, 4, 29),$

$L_{10} = (1, 22, 2, 26, 4, 30), \quad L_{11} = (1, 23, 2, 27, 4, 31), \quad L_{12} = (1, 24, 2, 28, 4, 32),$

$L_{13} = (2, 13, 4, 21, 3, 29), \quad L_{14} = (2, 14, 4, 22, 3, 30), \quad L_{15} = (2, 15, 4, 23, 3, 31),$

$L_{16} = (2, 16, 4, 24, 3, 32).$

The next 16 blocks are formed by replacing an element $i$ by an element $i + 4$, $i = 1, 2, 3, 4$.

The remaining blocks are:

$L_{33} = (9, 17, 10, 21, 11, 25), \quad L_{34} = (9, 21, 12, 25, 18, 29),$

$L_{35} = (9, 18, 10, 22, 11, 26), \quad L_{36} = (9, 22, 12, 26, 18, 30),$

$L_{37} = (9, 19, 10, 23, 11, 27), \quad L_{38} = (9, 23, 12, 27, 18, 31),$

$L_{39} = (9, 20, 10, 24, 11, 28), \quad L_{40} = (9, 24, 12, 28, 18, 32),$

$L_{41} = (11, 17, 13, 21, 14, 29), \quad L_{42} = (19, 25, 23, 27, 24, 31),$

$L_{43} = (11, 18, 13, 22, 14, 30), \quad L_{44} = (19, 26, 23, 28, 24, 32),$

$L_{45} = (11, 19, 13, 23, 14, 31), \quad L_{46} = (20, 25, 24, 29, 23, 31),$

$L_{47} = (11, 20, 13, 24, 14, 32), \quad L_{48} = (20, 26, 24, 30, 23, 32),$

$L_{49} = (12, 17, 14, 25, 22, 29), \quad L_{50} = (15, 17, 16, 25, 21, 29),$

$L_{51} = (12, 18, 14, 26, 22, 30), \quad L_{52} = (15, 18, 16, 26, 21, 30),$

$L_{53} = (12, 19, 14, 27, 22, 31), \quad L_{54} = (15, 19, 16, 27, 21, 31),$

$L_{55} = (12, 20, 14, 28, 22, 32), \quad L_{56} = (15, 20, 16, 28, 21, 32),$

$L_{57} = (15, 21, 16, 29, 17, 26), \quad L_{58} = (10, 25, 13, 29, 19, 27),$
\[ L_{59} = (15, 22, 16, 30, 17, 26), \quad L_{60} = (10, 26, 13, 30, 19, 28), \]
\[ L_{61} = (15, 23, 16, 31, 17, 27), \quad L_{62} = (10, 29, 20, 27, 13, 31), \]
\[ L_{63} = (15, 24, 16, 32, 17, 28), \quad L_{64} = (10, 30, 20, 28, 13, 32). \]

All the blocks \( B^j_i \) for \( i = 0, 1, 2, 3, \ j = 1, 2, 3, 4, 5, 6 \), \( L_k(k = 1, \ldots, 64) \) give \( K_{33} \) (6).

Let now \( v = 57, \ V = \{0, 1, \ldots, 56\}, \ V_i = \{0, 8i + 1, \ldots, 8(i + 1)\}, \ i = 0, 1, 2, 3, 4, 5, 6 \). On \( V_i \) we construct \( K_9 \) (6) in the same way as for \( v = 9 \). Denote \( W_1(V_1, V_2, V_3), \ W_2 = (V_4, V_5, V_6) \). In the same way as for \( v = 33 \) we construct blocks with elements \( V_0 \) and \( W_1, V_1 \) and \( V_2, V_3, V_2 \) and \( V_3 \) and analogously \( V_0 \) and \( W_2, V_4 \) and \( V_5, V_4 \) and \( V_6 \) and \( V_5 \) and \( V_6 \). It remains to construct blocks elements of \( W_1 \) and \( W_2 \):

\[
N_1 = (9, 33, 10, 37, 11, 41), \quad N_2 = (9, 37, 12, 33, 13, 45),
\]
\[
N_3 = (9, 34, 10, 38, 11, 42), \quad N_4 = (9, 38, 12, 34, 13, 46),
\]
\[
N_5 = (9, 35, 10, 39, 11, 43), \quad N_6 = (9, 39, 12, 35, 13, 47),
\]
\[
N_7 = (9, 36, 10, 49, 1144), \quad N_8 = (9, 40, 12, 36, 13, 48),
\]
\[
N_9 = (9, 49, 14, 41, 16, 53), \quad N_{10} = (10, 45, 16, 33, 11, 53),
\]
\[
N_{11} = (9, 50, 14, 42, 16, 54), \quad N_{12} = (10, 46, 16, 34, 11, 54),
\]
\[
N_{13} = (9, 51, 14, 43, 16, 55), \quad N_{14} = (10, 47, 16, 35, 11, 55),
\]
\[
N_{15} = (9, 52, 14, 44, 16, 56), \quad N_{16} = (10, 48, 16, 36, 11, 56),
\]
\[
N_{17} = (10, 41, 12, 45, 15, 49), \quad N_{18} = (11, 45, 14, 53, 13, 49),
\]
\[
N_{19} = (10, 42, 12, 46, 15, 50), \quad N_{20} = (11, 46, 14, 54, 13, 50),
\]
\[
N_{21} = (10, 43, 12, 47, 15, 51), \quad N_{22} = (11, 47, 14, 55, 13, 51),
\]
\[
N_{23} = (10, 44, 12, 48, 15, 52), \quad N_{24} = (11, 48, 14, 56, 13, 52),
\]
\[
N_{25} = (13, 37, 14, 33, 15, 41), \quad N_{26} = (12, 49, 16, 37, 15, 53),
\]
\[
N_{27} = (13, 38, 14, 34, 15, 42), \quad N_{28} = (12, 50, 16, 38, 15, 54),
\]
\[
N_{29} = (13, 39, 14, 35, 15, 43), \quad N_{30} = (12, 51, 16, 39, 15, 55),
\]

$N_{31} = (13, 40, 14, 36, 15, 44), \ N_{32} = (12, 52, 16, 40, 15, 56)$.

Replacing in $N_1, \ldots, N_{32}$ instead an element $i$ the element $i + 8$, where $i = 9, 10, \ldots, 16$ we get blocks $N_{33}, \ldots, N_{64}$. Replacing in $N_1, \ldots, N_{32}$ instead an element $i$ the element $i + 16$, where $i = 9, \ldots, 16$ we get blocks $N_{65}, \ldots, N_{96}$.

Now the construction in the general case. Let $v = 24k + 9, \ k > 2,$ and $V = \{0, 1, \ldots, 24k + 8\}, \ V_i = \{0, 8i + 1, \ldots, 8(i + 1)\}$, for $i = 0, 1, \ldots, 3k$. On $V_i$ we construct $K_9$ (6) according to the method shown above. Then we form:

$$W = (V_{2(j-1)+1}, V_{2(j-1)+2}, V_{3j}) \cdot (j = 1, \ldots, k)$$

and we construct blocks containing elements of $V_0$ and $W_j$ for each $j$ separately and blocks containing elements only of $W_j$ i.e. elements of $V_{3(j-1)+1}$ and elements of $V_{3(j-1)+2}$, or elements of $V_{3(j-1)+1}$ and $V_{3j}$ or elements of $V_{3(j-1)+2}$ and elements of $V_{3j}$ according to the method described for $v = 33$ / bloks $L_1, \ldots, L_{64}$ / and in this way we get $64k$ blocks.

It remains to construct blocks containing elements of $W^i$ and $W^j$ $(i \neq j), \ (i, j = 1, 2, \ldots, k)$. To do so we form all the unordered pairs $(w^i, W^j)$. Then we construct blocks that contain elements of $W^i$ and $W^j$ according to the method for $v = 57$ ($W^1, W^2$) It is easy to check that all the blocks form $K_{24k+9}$ (6).

Let $v \equiv 21 \pmod{24}$; First we construct $K_{21}$ (6) as follows. Let

$$V = \{0, 1, \ldots, 20\}, \ V_0 = \{0, 1, \ldots, 8\},$$

$$V_1 = \{0, 9, \ldots, 16\}, \ V_2 = \{0, 17, 18, 19, 20\},$$

On $V_0$ and $V_1$ we construct $K_9$ (6) according to the method described above.

Denote by $C_1$ a family of blocks containing elements of $V_0$ and $V_1$ or elements of $V_0$ and $V_2$:

$$C_1 = (1, 9, 2, 11, 3, 13), \ C_2 = (1, 10, 2, 12, 3, 14), \ C_3 = (1, 11, 4, 17, 7, 15),$$

$$C_4 = (1, 12, 4, 18, 7, 16), \ C_5 = (1, 17, 2, 13, 7, 19), \ C_6 = (1, 18, 2, 14, 7, 20),$$

$$C_7 = (2, 15, 3, 9, 8, 19), \ C_8 = (3, 18, 8, 12, 5, 20), \ C_9 = (4, 9, 6, 15, 8, 13),$$
$C_{10} = (4, 10, 6, 16, 8, 14), \ C_{11} = (4, 15, 5, 17, 6, 19), \ C_{12} = (4, 16, 5, 18, 6, 20),$

$C_{13} = (5, 9, 7, 11, 6, 13), \ C_{14} = (5, 10, 7, 12, 6, 14),$

By $D$ we denote a family of blocks containing elements of $V_1$ and $V_*$ or elements of $V_*$ only.

$D_1 = (0, 18, 14, 20, 15, 17), \ D_2 = (0, 20, 9, 17, 11, 19),$

$D_3 = (17, 18, 9, 19, 20, 12), \ D_4 = (17, 19, 10, 20, 18, 13),$

$D_5 = (17, 20, 11, 18, 19, 14), \ D_6 = (17, 10, 18, 15, 19, 16),$

$D_7 = (18, 12, 19, 13, 20, 16).$

Blocks of $K_9$ (6) constructed on $V_0$ or $V_1$ together with families of blocks $C$ or $D$ form $K_{21}$ (6).

Giving the whole construction could take too much time.

REFERENCES

