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DECOMPOSITION OF A COMPLETE GRAPH INTO HEXAGONS

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1. The Past.

Decomposition of a complete graph into hexagons.

Let K_v be an non-directed complete graph with v vertices, $K_v(p)$ be the same graph that is decomposed into p-gons with disjoined edges.

A. Rosa [3] has shown that a necessary and sufficient condition for the existence of $K_v(p)$ is:

$v \equiv 1$ or $3 \pmod 6$ for $p = 3$,

$v \equiv 1$ or $5 \pmod{10}$ for $p = 5$,

$v \equiv 1$ or $7 \pmod{14}$ for $p = 7$.

Moreover, it is well known that if p is prime power, then the necessary condition for the existence of $K_v(p)$ is

$$(1) \qquad v \equiv 1 \text{ or } p (\text{mod } 2p).$$

If p is not a prime power, then condition (1) is not necessary. The above showed A. Rosa [3] on constructing a K_{51} (15).

It is not known whether the condition (1) is sufficient for the existence of $K_v(p)$ if p is a prime power.

A. Kotzig [1] proved three theorems on the existence of $K_v(p)$.

(i) If $p = 2^r$ for some positive integer r, then the necessary and sufficient condition for the existence of $K_v(p)$ is $v \equiv 1 \pmod{2p}$.

(ii) If $v \equiv 1 \pmod{2p}$, then there exists a $K_v(p)$ for $p \equiv 0 \pmod 4$.

(iii) If $(v, p) = 1$ and $p \equiv 0 \pmod 4$, then the necessary and sufficient condition for the existence of $K_v(p)$ is $v \equiv 1 \pmod{2p}$.

Moreover, he constructed K_{33} (12) and K_{25} (20). This proves that if $(v,p) \neq 1$, $p \equiv 0 \pmod 4$, then the condition $v \equiv 1 \pmod{2p}$ is not necessary.

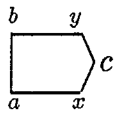
Let v be the number of vertices in a complete graph. V be the set of these vertices, $\cdot$ be a binary operation as follows: $a \cdot a = a$ for each $a \in V$, $a \cdot b = b \cdot a = c$, $b \neq a$, where

$\in K_v(3)$.

It is easy to see that $(V, \cdot)$ is a quasi-group and $v \equiv 1$ or $3 \pmod 6$ is a necessary and sufficient condition for the existence of K_v (3). Moreover, $(V, \cdot)$ is a quasi-group if and only if the above condition holds. Notice that $(V, \cdot)$ is a Steiner triple system.

Similarly, we define an operation as follows $a \times a = a$ for each $a \in V$, $a \times b = b \times a = c$, where $b \neq a$ and

(5)



$\in K_v$.

C.C. Lindner and D.R. Stinson [2] showed thet the condition $v \equiv 1$ or $5 \pmod{10}$ / the condition for the existence $K_v(5)$ / is necessary and sufficient for $(V, \otimes)$ to be a quasi-group. They proved, moreover, that a K_v (5) that was constructed by is not only a quasigroup, but also a Steiner-system $B(v, 5, 2)$.

The aim of this paper is to show that:

I. The necessary condition for the existence of $K_v(p)$ is

a/ v-odd integer, and

b/ $2p/v(v-1)$.

II. The necessary and sufficient condition for the existence of a $K_v(6)$ is $v \equiv 1$ or 9 (mod 12).

If p is even, then $K_v(p)$ do not form Steiner's system, since λ is not integer $\left(\lambda = \frac{p-1}{2} \notin N\right)$. When we were able to construct such $K_v(p)$ condition (B): that every pair of vertices was contained in $\frac{p-2}{2}$ or $\frac{p}{2}$ polygenes then we would assume that λ is a rational number equal to $\frac{p-1}{2}$. This would generalize λ to rational number.

The operation $\oplus$ can be defined in such a way that $(\oplus V)$ forms a quasi-group: $a \oplus a = a$ for each $a \in V$, $a \oplus b = c$, $a \neq b$ and where c belongs to the p-gen containing the edges $\overline{ab}$ and edges $\overline{bc}$.

We notice that $a \oplus b \neq b \oplus a$;

Proof. of I. Let us fix an arbitrary vertex A and let us put edges between A and all of the remaining vertices. If we claim that the remaining additional edges are edge as of closed polygons, then the degree of A is an even number. Since it is a complete graph the cardinality of its vertices must be odd. In such graph there is $\frac{v(v-1)}{2}$ edges and we divide this graph into disjoint p-gons. Hence $\frac{v(v-1)}{2}$ divides p.

Proof. of II. Condition I is equivalent to $v \equiv 1$ or 9 (mod 12) if $p = 6$; To prove II it is sufficient to show. That the above condition is sufficient for the existence of K_v (6). The proof will be done for $v \equiv 1$ (mod 12) and $v \equiv 9$ (mod 12) separately.

Let $v \equiv 1$ (mod 12). First of all we construct a K_{13} (6). Let us number the vertices of a graph from 0 through 12 and write down these blocks whose elements are vertices of corresponding hexagons. These blocks have the properties α-for each $(x, y) \in V$ there exists a unique sequence $[b_1, \ldots, b_6]$ such that $x = b_i$, $y = b_j$ and $|i - j| = 1$ or $|i - j| = 5$

$$B_1 = (1,3,5,7,9,11),\ B_2 = (2,4,6,8,10,12),\ B_3 = (1,7,2,8,3,10),$$

$$B_4 = (3,9,4,10,11,12)\ B_5 = (5,11,6,12,1,2,5),\ B_6 = (0,3,11,2,9,1),$$

$$B_7 = (0,7,3,6,1,5),\ B_8 = (0,11,7,10,5,9),\ B_9 = (0,4,12,9,10,2),$$

$$B_{10} = (0,8,4,3,2,6),\ B_{11} = (0,12,8,5,6,10),\ B_{12} = (7,8,11,4,5,12),$$

$$B_{13} = (1,8,9,6,7,4).$$

It is easy to check that the above blocks satisfy condition α, and condition β.

A construction of a K_{25} (6) will be shown now. Let $V = \{0,1,\ldots,24\}$, $V_1 = \{0,1,\ldots,12\}$ $V_2 = \{0,13,14,\ldots,24\}$. Construct K_{13} (6) on V_1 and V_2 separately and denote by them K_{13}^1 (6) and K_{13}^2 (6) respectively. To get $K_{25}(6)$ we need blocks containing elements of V_1 and V_2 at the same time.

$$L_1 = (1,13,2,14,3,15),\ L_2 = (1,17,2,18,3,19),\ L_3 = (1,21,2,22,3,23)$$

$$L_4 = (1,14,4,13,3,16),\ L_5 = (1,18,4,17,3,20), L_6 = (1,22,4,21,3,24)$$

$$L_7 = (5,13,6,16,4,15),\ L_8 = (5,17,6,20,4,19),\ L_9(5,21,6,24,4,23),$$

$$L_{10} = (5,14,6,15,2,16),\ L_{11} = (5,18,6,19,2,20),\ L_{12} = (5,22,6,21,2,24).$$

The remaining blocks of the family will be obtained by replacing an element i in each of the above 12 blocks by $i+6$ $i = 1,2,\ldots,6$; It is easy to check that all the blocks satisfy condition α.

Let now, $v = 12k+1$, $V = \{0,1,2,...,12k\}$, $V_i = \{0,(i-1)12+1,...,12i\}$ $i = 1,2,...,k$. Construct K_{13}^i (6) on V_i, $i = 1,\ldots,k$. For each pair $(V_i,V_j) i \neq j$, $i,j = 1,\ldots,k$ construct blocks $L_1,\ldots,L_{24}$ in the same way as for $v = 25$. All these blocks form K_{12k+1} (6).

These construction shows that $v \equiv 1 \pmod{12}$ is the sufficient condition for the existence of a decomposition of the complete graph into hexagons with disjoint edges.

Then we shall show the same for $v \equiv 9 \pmod{12}$. We distinguish two cases: a) $v \equiv 9 \pmod{24}$, b) $v \equiv 21 \pmod{24}$.

Case a) We construct K_9 (6) first. Let $V = \{0,1,\ldots,8\}$.

$$B_1 = (1,0,2,3,4,6),\ B_2 = (3,0,4,5,6,8),\ B_3(5,0,6,7,8,2),$$

$$B_4 = (7,0,8,1,2,4)\ B_5 = (1,3,6,2,7,5),\ B_6 = (1,7,3,5,8,4).$$

It is easy to check that the blocks $B_1,\ldots,B_6$ form K_9 (6) with condition β.

Let now $v = 33$ $V = \{0,1,...,32\}$, $V_0 = \{0,1,...,8\}$, $V_1 = \{0,9,10,...,16\}$, $V_2 = \{0,17,18,...,24\}$, $V_3 = \{0,25,26,...,32\}$.

To get $K_v(6)$ we construct K_9 (6) on V_i, $i = 0, 1, 2, 3$ according to the above tretbrad. The remaining blocks will be obtained as follows. Let

$$L_1 = (1,9,2,17,3,25),\ L_2 = (1,10,2,18,3,26),\ L_3 = (1,11,2,19,3,27),$$

$$L_4 = (1,12,2,20,3,28),\ L_5 = (1,13,3,9,4,17),\ L_6 = (1,14,3,10,4,18),$$

$$L_7 = (1,15,3,11,4,19),\ L_8 = (1,16,3,12,4,20),\ L_9 = (1,21,2,25,4,29),$$

$$L_{10} = (1,22,2,26,4,30),\ L_{11} = (1,23,2,27,4,31),\ L_{12} = (1,24,2,28,4,32),$$

$$L_{13} = (2,13,4,21,3,29),\ L_{14} = (2,14,4,22,3,30),\ L_{15} = (2,15,4,23,3,31),$$

$$L_{16} = (2,16,4,24,3,32).$$

The next 16 blocks are formed by replacing an element i by an element $i+4$, $i = 1, 2, 3, 4$.

The remaining blocks are:

$$L_{33} = (9,17,10,21,11,25),\ L_{34} = (9,21,12,25,18,29),$$

$$L_{35} = (9,18,10,22,11,26),\ L_{36} = (9,22,12,26,18,30),$$

$$L_{37} = (9,19,10,23,11,27),\ L_{38} = (9,23,12,27,18,31),$$

$$L_{39} = (9,20,10,24,11,28),\ L_{40} = (9,24,12,28,18,32),$$

$$L_{41} = (11,17,13,21,14,29),\ L_{42} = (19,25,23,27,24,31),$$

$$L_{43} = (11,18,13,22,14,30),\ L_{44} = (19,26,23,28,24,32),$$

$$L_{45} = (11,19,13,23,14,31),\ L_{46} = (20,25,24,29,23,31),$$

$$L_{47} = (11,20,13,24,14,32),\ L_{48} = (20,26,24,30,23,32),$$

$$L_{40} = (12,17,14,25,22,29),\ L_{50} = (15,17,16,25,21,29),$$

$$L_{51} = (12,18,14,26,22,30),\ L_{52} = (15,18,16,26,21,30),$$

$$L_{53} = (12,19,14,27,22,31),\ L_{54} = (15,19,16,27,21,31),$$

$$L_{55} = (12,20,14,28,22,32),\ L_{56} = (15,20,16,28,21,32),$$

$$L_{57} = (15,21,16,29,17,26),\ L_{58} = (10,25,13,29,19,27),$$

$$L_{59} = (15,22,16,30,17,26),\ L_{60} = (10,26,13,30,19,28),$$

$$L_{61} = (15,23,16,31,17,27),\ L_{62} = (10,29,20,27,13,31),$$

$$L_{63} = (15,24,16,32,17,28),\ L_{64} = (10,30,20,28,13,32).$$

All the blocks B_j^i for $(i = 0,1,2,3,\ j = 1,2,3,4,5,6)$, $L_k(k = 1,\dots,64)$ give K_{33} (6).

Let now $v = 57$, $V = \{0,1,\dots,56\}$, $V_i = \{0,8i+1,\dots,8(i+1)\}$, $i = 0,1,2,3,4,5,6$. On V_i we construct K_9 (6) in the same way as for $v = 9$. Denote $W_1(V_1,V_2,V_3)$, $W_2 = (V_4,V_5,V_6)$. In the same way as for $v = 33$ we construct blocks with elements V_0 and W_1, V_1 and V_2, V_1 and V_3, V_2 and V_3 and analogously V_0 and W_2, V_4 and V_5, V_4 and V_6, V_5 and V_6. It remains to construct blocks elements of W_1 and W_2:

$$N_1 = (9,33,10,37,11,41),\ N_2 = (9,37,12,33,13,45),$$

$$N_3 = (9,34,10,38,11,42),\ N_4 = (9,38,12,34,13,46),$$

$$N_5 = (9,35,10,39,11,43),\ N_6 = (9,39,12,35,13,47),$$

$$N_7 = (9,36,10,49,1144),\ N_8 = (9,40,12,36,13,48),$$

$$N_9 = (9,49,14,41,16,53), N_{10} = (10,45,16,33,11,53),$$

$$N_{11} = (9,50,14,42,16,54),\ N_{12} = (10,46,16,34,11,54),$$

$$N_{13} = (9,51,14,43,16,55),\ N_{14} = (10,47,16,35,11,55),$$

$$N_{15} = (9,52,14,44,16,56),\ N_{16} = (10,48,16,36,11,56),$$

$$N_{17} = (10,41,12,45,15,49),\ N_{18} = (11,45,14,53,13,49),$$

$$N_{19} = (10,42,12,46,15,50),\ N_{20} = (11,46,14,54,13,50),$$

$$N_{21} = (10,43,12,47,15,51),\ N_{22} = (11,47,14,55,13,51),$$

$$N_{23} = (10,44,12,48,15,52),\ N_{24} = (11,48,14,56,13,52),$$

$$N_{25} = (13,37,14,33,15,41),\ N_{26} = (12,49,16,37,15,53),$$

$$N_{27} = (13,38,14,34,15,42),\ N_{28} = (12,50,16,38,15,54),$$

$$N_{29} = (13,39,14,35,15,43),\ N_{30} = (12,51,16,39,15,55),$$

$N_{31} = (13, 40, 14, 36, 15, 44)$, $N_{32} = (12, 52, 16, 40, 15, 56)$.

Replacing in $N_1, \ldots, N_{32}$ instead an element i the element $i + 8$, where $i = 9, 10, \ldots, 16$ we get blocks $N_{33}, \ldots, N_{64}$. Replacing in $N_1, \ldots, N_{32}$ instead an element i the element $i + 16$, where $i = 9, \ldots, 16$ we get blocks $N_{65}, \ldots, N_{96}$.

Now the construction in the general case. Let $v = 24k + 9$, $k > 2$, and $V = \{0, 1, \ldots, 24k + 8\}$, $V_i = \{0, 8i + 1, \ldots, 8(i + 1)\}$, for $i = 0, 1, \ldots, 3k$. On V_i we construct K_9 (6) according to the method shown above. Then we form:

$$W = (V_{3(j-1)+1}, V_{3(j-1)+2}, V_{3 \cdot j}) \cdot (j = 1, \ldots, k)$$

and we construct blocks containing elements of V_0 and W_j for each j separately and blocks containing elements only of W^j i.e. elements of $V_{3(j-1)+1}$ and elements of $V_{3(j-1)+2}$, or elements of $V_{3(j-1)+1}$ and V_{3j} or elements of $V_{3(j-1)+2}$ and elements of V_{3j} according to the method described for $v = 33$ / bloks $L_1, \ldots, L_{64}$ / and in this way we get $64k$ blocks.

It remains to construct blocks containing elements of W^i and W^j $(i \neq j)$, $(i, j = 1, 2, \ldots, k)$. To do so we form all the unordered pairs (w^i, W^j). Then we construct blocks that contain elements of W^i and W^j according to the method for $v = 57$ (W^1, W^2) It is easy to check that all the blocks form K_{24k+9} (6).

Let $v \equiv 21 \pmod{24}$; First we construct K_{21} (6) as follows. Let

$$V = \{0, 1, \ldots, 20\}, \; V_0 = \{0, 1, \ldots, 8\},$$

$$V_1 = \{0, 9, \ldots, 16\}, \; V_* = \{0, 17, 18, 19, 20\},$$

On V_0 and V_1 we construct K_9 (6) according to the method described above.

Denote by C_1 a family of blocks containing elements of V_0 and V_1 or elements of V_0 and V_*:

$$C_1 = (1, 9, 2, 11, 3, 13), \; C_2 = (1, 10, 2, 12, 3, 14), \; C_3 = (1, 11, 4, 17, 7, 15),$$

$$C_4 = (1, 12, 4, 18, 7, 16), \; C_5 = (1, 17, 2, 13, 7, 19), \; C_6 = (1, 18, 2, 14, 7, 20),$$

$$C_7 = (2, 15, 3, 9, 8, 19), \; C_8 = (3, 18, 8, 12, 5, 20), \; C_9 = (4, 9, 6, 15, 8, 13),$$

$$C_{10} = (4, 10, 6, 16, 8, 14),\ C_{11} = (4, 15, 5, 17, 6, 19),\ C_{12} = (4, 16, 5, 18, 6, 20),$$

$$C_{13} = (5, 9, 7, 11, 6, 13),\ C_{14} = (5, 10, 7, 12, 6, 14),$$

By D we denote a family of blocks containing elements of V_1 and V_* or elements of V_* only.

$$D_1 = (0, 18, 14, 20, 15, 17),\ D_2 = (0, 20, 9, 17, 11, 19),$$

$$D_3 = (17, 18, 9, 19, 20, 12),\ D_4 = (17, 19, 10, 20, 18, 13),$$

$$D_5 = (17, 20, 11, 18, 19, 14),\ D_6 = (17, 10, 18, 15, 19, 16),$$

$$D_7 = (18, 12, 19, 13, 20, 16).$$

Blocks of K_9 (6) constructed on V_0 or V_1 together with families of blocks C or D form K_{21} (6).

Giving the whole construction could take too much time.

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