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MAXIMAL PARTIAL DESIGNS AND CONFIGURATIONS

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The purpose of this lecture is to survey and bring together several old and new results concerning a wide variety of combinatorial problems having a common underlying feature.

Typically, the situation is as follows. We are given a finite set F of objects called *figures*, and a symmetric irreflexive relation R on F (the compatibility rule) which says when are two figures *compatible*. An (F, R)-configuration or simply a *configuration* is a set of pairwise compatible figures. A configuration C is *maximal* if there is no $f \in F$, $f \notin C$ such that $f \cup C$ is also a configuration (i.e. maximality is here with respect to set inclusion).

(More generally, the compatibility rule R is a function from a subset of the power set of F into $\{0, 1\}$ but all our examples will be of the simpler type above.)

The *size* of a configuration is the number of its figures.

We will be interested mainly in the possible sizes of maximal (F, R) configurations, i.e. in the *spectrum* $Sp(F, R)$ defined by

$Sp(F, R) = \{m$: there exists a maximal (F, R)-configuration of size m$\}$.

We may imagine a procedure under which one tries to build maximal configurations of given type in a naive way: given any configuration, enlarge it by adjoining to it another figure if possible, and so on, until you get stuck. The elements of the spectrum represent sizes of all possible outcomes of such a process. Naive as it is, there are several «real-life» situations in which procedures of this sort are actually used!

Our first example is a problem that has recently been solved completely.

1. Maximal sets of 1-factors.

The figures here are 1-factors of the complete graph K_{2n} on a given set of vertices; two such 1-factors are compatible if they are edge-disjoint.

Let $M(2n)$ be the spectrum for maximal sets of 1-factors of K_{2n}, i.e.

$M(2n) = \{m$: there exists a maximal set of (edge-disjoint) 1-factors in $K_{2n}\}$. One gets as a corollary of Dirac's theorem (cf. [3]) that

$$M(2n) \subseteq \{n, n+1, \dots, 2n-1\}.$$

Trivially, $2n-2 \notin M(2n)$ since the complement of the union of $2n-2$ 1-factors is itself a 1-factor. Furthermore, $n \notin M(2n)$ if n is even [9]. On the other hand, if k is odd and $k \in \{n, n+1, \dots, 2n-1\}$ then $k \in M(2n)$, as shown by the following simple direct construction.

Take the set $Z_k \cup \{a_i : i = 1, 2, \dots, 2n-k\}$ as the set of vertices of K_{2n}, and define the following 1-factor F:

$$F = \{\{a_1, 0\}, \{a_2, 1\}, \{a_3, k-1\}, \dots, \left\{a_{2n-k-1}, \tfrac{1}{2}(2n-k-1)\right\},$$

$$\left\{a_{2n-k}, k - \tfrac{1}{2}(2n-k-1)\right\},$$

$$\left\{\tfrac{1}{2}(2n-k-1)+1, k - \tfrac{1}{2}(2n-k-1)\right\},$$

$$\{\tfrac{1}{2}(2n-k-1)+2, k-\tfrac{1}{2}(2n-k-1)-2\}\ ,\dots,$$

$$\{\tfrac{1}{2}(k-1), \tfrac{1}{2}(k+1)\}$$

(the edges in the second row are used only when $k \neq n$). Developing F modulo k yields a maximal set of 1-factors, since the complement of the union of these 1-factors contains an odd component K_{2n-k}.

It turns out that the case of k even is much more difficult. It is shown in [29] that for k even, $k \in M(2n)$ if and only if $(4n+4)/3 \leq k \leq 2n-4$. Explicitly, we have for small values of n: $M(4) = \{3\}$, $M(6) = \{3,5\}$, $M(8) = \{5,7\}$, $M(10) = \{5,7,9\}$, $M(12) = \{7,9,11\}$, $M(14) = \{7,9,11,13\}$, $M(16) = \{9,11,12,13,15\}, \dots, M(30) = \{15,17,19,21,22,23,24,25,27\}$ and so on.

Thus the spectrum for maximal sets of 1-factors is completely determined. This is certainly not the case for maximal *perfect* sets of 1-factors. In this variation of the above problem, two 1-factors are compatible if they are edge-disjoint and their union is a hamiltonian cycle. Let $M_{perf}(2n)$ be the spectrum of maximal perfect sets of 1-factors. It is not even known whether $2n-1 \in M_{perf}(2n)$ for all n. To determine this is equivalent to determining whether there exists a perfect 1-factorization of K_{2n} for every n, a difficult unsolved problem. It is easily verified that $M_{perf}(4) = \{3\}$, $M_{perf}(6) = \{3,5\}$, $M_{perf}(8) = \{5,7\}$. Further, $M_{perf}(10) \supseteq \{5,9\}$; examining the list of all perfect sets of three 1-factors of K_{10} from [27], one sees that none of these sets is maximal, thus $3 \notin M_{perf}(10)$. Also, $n \in M_{perf}(2n)$ whenever n is an odd prime but not much else seems to be known about $M_{perf}(2n)$.

The following problem is a natural continuation of the previous one.

2. Maximal sets of 2-factors.

The figures are 2-factors of the complete graph on a given set of n vertices; two 2-factors are compatible if they are edge-disjoint.

Let $M^{(2)}(n) = \{m :$ there exists a maximal set of m (edge-disjoint) 2-factors of $K_n\}$. It follows directly from Petersen's theorem (cf. [3])

that for n odd,

$$M^{(2)}(n) = \left\{ \frac{1}{2}(n-1) \right\}.$$

The situation is somewhat more involved for n even. This is due to the fact that for odd d, there exist regular graphs of degree d without proper regular factors. König [24] calls such graphs primitive. An obvious extension of König's example for $d = 3$ yields a primitive graph of odd degree $d (d > 1)$ with $(d+1)^2$ vertices. This is the minimum number of vertices a primitive graph of odd degree d can have as shown recently by Dean Hoffman [20]. This implies that the spectrum $M^{(2)}(n)$ for n even is the following interval:

$$M^{(2)}(n) = \left\{ \left\lceil \frac{n-\sqrt{n}}{2} \right\rceil, \left\lceil \frac{n-\sqrt{n}}{2} \right\rceil + 1, \ldots, \frac{1}{2}(n-2) \right\}.$$

In the next two examples figures are still 2-factors but of a restricted type.

3. Maximal sets of Hamiltonian cycles.

The figures are connected 2-factors of K_n, i.e. Hamiltonian cycles; two Hamiltonian cycles are compatible if they are edge-disjoint.

Let $MH(n) = \{m :$ there exists a maximal set of Hamiltonian cycles of $K_n\}$. Denote

$$Dir(n) = \left\{ \left\lceil \frac{1}{4}(n+3) \right\rceil, \left\lceil \frac{1}{4}(n+3) \right\rceil + 1, \ldots, \left\lfloor \frac{1}{2}(n-1) \right\rfloor \right\}.$$

It follows directly from Dirac's theorem and a result of Nash-Williams (cf. [3]) that $MH(n) \subseteq Dir(n)$. We would like to show that, actually, equality takes place here. Consider the following.

Let n be even, $n = 2k$, and let m be a positive integer, $2m \leq k$. Let G be a regular graph of degree $2k - 4m$ with $2k - 2m$ vertices, and let $H = \bar{K}_{2m} \nabla G$.

Similarly, let n be odd, $n = 2k+1$, m be a positive integer, $2m + 1 \leq k$, and let G be a regular graph of degree $2k - 4m - 1$ with

$2k - 2m$ vertices, and let $H = \bar{K}_{2m+1} \nabla G$ (here ∇ denotes the join, cf. [3]).

In order to show that $MH(n) = Dir(n)$, it clearly suffices to show that the graph H, with G suitably chosen, has a Hamiltonian decomposition. Inded, the complement $\bar{H}$ of H is disconnected, and so the set of Hamiltonian cycles in any Hamiltonian decomposition of H is maximal. The corresponding proof that H has a Hamiltonian decomposition for G suitably chosen is given in [20]. Thus here we have another example of a completely determined spectrum.

4. Maximal sets of Δ-factors.

The figures are Δ-factors of K_n (i.e. 2-factors whose each component is a triangle); two Δ-factors are compatible if they are edge-disjoint. Of course, this requires $n \equiv 0 \pmod 3$.

Let $\Delta(n) = \{m :$ there exists a maximal set of Δ-factors of $K_n\}$.

It follows from a theorem of Corrádi and Hajnal [8] that

$$\Delta(n) \subseteq \left\{ \lceil n/6 \rceil, \lceil n/6 \rceil + 1, \ldots, \left[\frac{1}{2}(n-1) \right] \right\}.$$

A forthcoming paper by R. Rees, W.D. Wallis and myself [28] contains several results towards determining the spectrum $\Delta(n)$ of maximal sets of Δ-factors. At present, however, $\Delta(n)$ is far from being determined completely.

The next four problems deal with latin squares and rectangles.

5. Maximal partial latin squares.

The figures are elements of $N \times N \times N$, i.e. ordered triples from a set N of n elements; two such triples are compatible if they agree in at most one coordinate. We can take for $N = \{1, 2, \ldots, n\}$. It is somewhat more convenient to think of a partial latin square as an $n \times n$ array whose cells are either empty or contain an element of

N such that each element occurs in at most one cell of each row or column. A partial latin square is then maximal if no further nonempty cells can be filled without violating this condition.

Let $ML(n)$ be the spectrum of maximal partial latin squares of order n, i.e.

$ML(n) = \{m$: there exists a maximal partial latin square of order n with exactly m nonempty cells$\}$.

It is easily seen that if $m \in ML(n)$ then $m \geq \left\lceil \frac{1}{2}n^2 \right\rceil$. Trivially, $n^2 - 1 \notin ML(n)$ since a partial latin square whose all cells but one are filled cannot be maximal. Horák, Širáň and myself have shown that if $m \in \left\{ \left\lceil \frac{1}{2}n^2 \right\rceil, \ldots, n^2 \right\}$ and $m \equiv n^2 \pmod 2$, or if $m \in \left\{ \left\lceil \frac{1}{2}n^2 \right\rceil + n + 1, \ldots, n^2 \right\}$ and $m \not\equiv n^2 \pmod 2$, then $m \in ML(n)$. Moroever, if $m \in \left\{ \left\lceil \frac{1}{2}n^2 \right\rceil, \ldots, \left\lceil \frac{1}{2}n^2 \right\rceil + \left\lceil \frac{1}{2}n \right\rceil - 1 \right\}$, and $m \not\equiv n^2 \pmod 2$ then $m \notin ML(n)$. Thus the spectrum $ML(n)$ for maximal partial latin squares would be completely determined if the case of $m \in \left\{ \left\lceil \frac{1}{2}n^2 \right\rceil + \left\lceil \frac{1}{2}n \right\rceil + 1, \ldots, \left\lceil \frac{1}{2}n^2 \right\rceil + n - 1 \right\}$, $m \not\equiv n^2 \pmod 2$ could be settled. We conjecture that in this case actually $m \notin ML(n)$ but have so far been unable to prove it.

6. Row-maximal latin rectangles.

Here the figures are permutations of degree n; two such permutations are compatible if they are discordant (i.e. do not agree in any position). In 1945, M. Hall proved [19] that for $r < n$, any $r \times n$ latin rectangle can be extended to an $(r+1) \times n$ latin rectangle. It follows that the spectrum of row-maximal latin rectangles, i.e.

$MLR(n) = \{r$: there exists a row-maximal $r \times n$ latin rectangle$\}$ consists of a single element, namely n.

No such simple answer can be expected in any of the next two examples.

7. Row-maximal orthogonal latin rectangles.

The figures are pairs of permutations of degree n. Two pairs (P_1, P_1'), (P_2, P_2') are compatible if (P_1, P_2) and (P_1', P_2') are both discordant, and the two $2 \times n$ latin rectangles $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ and $\begin{pmatrix} P_1' \\ P_2' \end{pmatrix}$ are orthogonal.

Denote by $MOR(r, n)$ a pair of row-maximal orthogonal latin $r \times n$ rectangles. Let $MOR(n) = \{r : \text{there exists a } MOR(r, n)\}$. For small values of n, we have

$$MOR(1) = MOR(2) = \{1\}, MOR(3) = \{3\}, MOR(4) = \{3, 4\},$$

$$MOR(5) = \{3, 5\}, MOR(6) = \{3, 4, 5\}, MOR(7) = \{3, 4, 5, 6, 7\},$$

$$MOR(8) = \{3, 4, 5, 6, 7, 8\}.$$

Horák, Kreher, Širáň and myself [21] have obtained several partial results towards settling the following conjecture.

Conjecture. For $n \geq 7$, $MOR(n) = \{r : n/3 < r \leq n\}$.

In particular, it is shown in [21] that $MOR(r, n)$ exists if $n \geq 7$ and

(i) $n/3 < r \leq \frac{1}{2}n$ except possibly when $(r, n) = (6, 12)$

(ii) $(7/9)n \leq r \leq n$

(iii) $(2n - 1)/3 \leq r \leq n - 1$, r odd

(iv) $(3/7)n < r < (3/5)n, r \equiv 3 \pmod 6$, $n \equiv 1 \pmod 2$

(v) $(3/5)n < r \leq (3/4)n, r \equiv 3 \pmod 6$, $n \equiv 0 \pmod 2$

(vi) $\frac{1}{2}n \leq r \leq (3n + 2)/4, r \equiv 0 \pmod 2$, $n \equiv 0 \pmod 2$.

On the other hand, there exists no $MOR(r, n)$ for $r \leq \frac{1}{4}n$.

Several recursive constructions were also found. These, together with the above results, suffice to show, for instance, that $\{r : 11 \leq r \leq 30\} \subseteq MOR(30)$; the set on the left coincides with the conjectured spectrum. But in general, quite a few unsettled cases remain.

8. Maximal sets of mutually orthogonal latin squares.

The figures are latin squares of order n on N; two latin squares are compatible if they are orthogonal. Let $L(n)$ be the spectrum of maximal sets of mutually orthogonal latin squares ($MOLS$), i.e.

$L(n) = \{r :$ there exists a maximal set of r $MOLS$ of order $n\}$.

To determine $L(n)$ in its entirety is, of course, hopeless at present, as this would include solving several famous open problems, such as the one about the existence of finite projective planes order n. Even max $L(n)$ remains undetermined for all values of n other than prime powers or $n = 6$. Nevertheless, progress on this question is very desirable.

Clearly, $L(n) \subseteq \{1, 2, \ldots, n-1\}$. A theorem of Bruck [4] implies that for $n > 4$, $n-2 \notin L(n)$, $n-3 \notin L(n)$. On the other hand, $1 \in L(n)$ for $n \not\equiv 3 \pmod 4$. If q is a prime power then $q^2 - q - 1 \in L(q^2)$ [11]. Drake [14] has determined $L(n)$ for small values of n:

$$L(3) = \{2\}, L(4) = \{1,3\}, L(5) = \{1,4\}, L(6) = \{1\}, L(7) = \{1,2,6\}.$$

He also showed $\{1,2,3,7\} \subseteq L(8)$ but it is still an open question whether $4 \in L(8)$. For many additional results on maximal sets of $MOLS$ see [10] and [14] and the references therein.

For many problems of this kind that can be formulated for block designs, consider just the following two.

9. Maximal partial Steiner triple systems.

The figures are 3-subsets (triples) of a given v-set; two triples are compatible if they intersect in at most one element.

Let $P(v)$ be the spectrum for maximal partial Steiner triple systems (STS), i.e.

$P(v) = \{m$: there exists a maximal partial STS of order v with exactly m triples$\}$.

The largest element of $P(v)$ was determined already in 1840's by Kirkman [23] (and since then repeatedly by many others):

$$\max P(v) = \begin{cases} v(v-1)/6 & \text{if } v \equiv 1 \text{ or } 3 \pmod 6 \\ [v(v-1)-8]/6 & \text{if } v \equiv 5 \pmod 6 \\ & \\ v(v-2)/6 & \text{if } v \equiv 0 \text{ or } 2 \pmod 6 \\ [v(v-2)-2]/6 & \text{if } v \equiv 4 \pmod 6 \end{cases}$$

But it was only in 1974 that the smallest element of $P(v)$ was determined by Novák [26]: $\min P(v) = (v^2 + \delta_v)/12$ where

$$\delta_v = \begin{cases} -2v+36 & \text{if } v \equiv 0 \text{ or } 8 \pmod{12} \\ -1 & \text{if } v \equiv 1 \text{ or } 5 \pmod{12} \\ -2v & \text{if } v \equiv 2 \text{ or } 6 \pmod{12} \\ 3 & \text{if } v \equiv 3 \pmod{12} \\ & \\ -2v+4 & \text{if } v \equiv 4 \pmod{12} \\ 11 & \text{if } v \equiv 7 \text{ or } 11 \pmod{12} \\ 15 & \text{if } v \equiv 9 \pmod{12} \\ -2v+16 & \text{if } v \equiv 10 \pmod{12} \end{cases}$$

The spectrum $P(v)$ for odd v was completely determined by Severn [30].

Let $R(v)$ be the interval $\{\min P(v), \max P(v)\}$. Severn has shown that

$$P(v) = \begin{cases} R(v) \backslash \{\max P(v) - 1\} & \text{if } v \equiv 1 \text{ or } 3 \pmod 6 \\ R(v) & \text{if } v \equiv 5 \pmod 6. \end{cases}$$

For even v, the spectrum $P(v)$ has «almost» been determined by Severn [30] but some open cases remain. More precisely, Severn proved that

(i) if v is even, and if $m \in R(v)$ is such that $m \equiv \frac{1}{2}(v-2) \pmod 2$ then $m \in P(v)$

(ii) if $m \in R(v)$ and $m \not\equiv \frac{1}{2}(v-2)$ and $m \leq \min P(v)+h(v)$ or $m \geq \min P(v)+s(v)$ (where $h(v)$ and $s(v)$ are given below) then $m \in P(v)$.

But if $m \in R(v)$, $m \not\equiv \frac{1}{2}(v-2) \pmod 2$ and $\min P(v)+h(v) < m < \min P(v)+s(v)$ then it is still undecided whether $m \in P(v)$. Severn conjectured that in this case actually $m \notin P(v)$.

$v \equiv t \pmod{12}$

t	$h(v)$	$s(v)$	t	$h(v)$	$s(v)$
0	$(v-18)/6$	$(2v-18)/6$	6	$v/6$	$(2v+6)/6$
2	$(v+4)/6$	$(2v+2)/6$	8	$(v-14)/6$	$(2v-10)/6$
4	$(v+2)/6$	$(2v-2)/6$	10	$(v-4)/6$	$(2v-2)/6$

10. Maximal sets of disjoint Steiner triple systems.

The figures are Steiner triple systems on a given v-set; they are compatible if they are disjoint, i.e. have no triple in common. Here, of course, $v \equiv 1$ or $3 \pmod 6$.

Let $DS(v) = \{m$: there exists a maximal set of m pairwise disjoint $STS(v)'s\}$. It is well known that $DS(7) = \{2\}$. Lu [25] has shown that for $v > 7$, $\max DS(v) = v-2$, except possibly for the six orders $v = 141, 283, 501, 789, 1501, 2365$. The only other general results are: (1) for $v \geq 7$, $1 \notin DS(v)$ [32]; (2) $v-4 \in DS(v)$ for $v = 5.2^i - 1$, $i \geq 1$; (3) $v-5 \in DS(v)$ for $v = 2^{i+2}-1, 5.2^i - 1$, $i \geq 1$ [6].

Cooper has determined $DS(9)$ [7] (follows also from [6]): $DS(9) = \{4,5,7\}$. He also determined the isomorphism classes of all maximal sets of disjoint $STS(9)$'s.

Let us conclude our examples with one of a different kind.

11. Row-maximal Room rectangles.

The figures are pairs (f, α) where f is a 1-factor of K_{2n} on a given

$(2n)$-set N and α is an injection from f into $\{1, 2, \ldots, 2n-1\}$. Two figures (f_1, α_1), (f_2, α_2) are compatible if $\alpha_1^{-1}(i) \cap \alpha_2^{-1}(i) = \phi$ whenever $\alpha_1^{-1}(i) \neq \phi$ and $\alpha_2^{-1}(i) \neq \phi$. Less formally, the figures are rows with $2n-1$ cells of which $n-1$ are empty such that the n nonempty cells contain a partition of N into 2-subsets, and two such rows are compatible if no element occurs in any of the $2n-1$ columns more than once.

Here we have the following result:

A row-maximal Room $(r, 2n)$-rectangle (i.e. one with r rows and $2n$ elements) exists if

(i) $(r, 2n) = (1, 4)$

(ii) $n \leq r \leq 2n$ except when $(r, 2n) = (2, 4), (3, 4)$ or $(5, 6)$.

Indeed, (i) is trivial while (ii) follows directly from the existence result for Howell designs $H(r, 2n)$ [1,31]. The only difference is that although $H(5, 8)$ does not exist, a row-maximal Room (5,8)-rectangle does:

12	34	56	78	–	–	–
–	–	–	13	24	57	68
67	–	14	–	58	23	–
–	15	–	26	–	48	37
38	–	27	–	16	–	45

Examples of other problems of this kind that have been discussed in the literature include maximal sets of orthogonal Hamiltonian circuits [22], maximal sets of orthogonal Hamiltonian decompositions [22], maximal sets of disjoint 1-factorizations [2,5], maximal sets of orthogonal 1-factorizations [2], maximal k-cliques [13, 15, 16, 17], maximal partial projective planes [11], maximal sets of orthogonal Steiner triple systems [18] - this list is by no means exhaustive, and could easily be extended further.

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