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ASSOCIATION AND OTHER SCHEMES RELATED TO SHARMA-KAUSHIK CLASS OF DISTANCES OVER FINITE RINGS

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Delsarte's association schemes in coding theory arise by considering the set of relations in terms of the Hamming metric. It is possible to define a whole class of metrics over a alphabet set which is a finite ring, by considering Sharma-Kaushik partitions.

The paper considers four different ways of defining relation sets in terms of the whole class of Sharma-Kaushik distances. These are directly in terms of the distance corresponding a SK-partition, the other notions called "nomination", "discrimination" and "composition". Intersection numbers, extended intersection numbers for the four different types of schemes over the n-vectors of finite ring, which are not always association schemes, has been studied. Since Hamming and the other used metric, viz. the Lee metric of Coding theory, arise from two special Sharma-Kaushik partitions, the study has potential for wider uses.

1. Introduction.

Bose and Shimamato (1952) first introduced the association schemes in the study of PBIBD's. These are essentially elegant algebraic structures with interesting combinatorial properties, and have found applications in the study of permutation groups, graphs and coding theory.

Delsarte (1973), considering association schemes in coding, slightly generalised the original concept of association schemes. In terms of this widely followed definition, the Bose and Shimamato schemes are termed as simmetric association schemes. An important motivation for Delsarte's work on association schemes of coding theory can be seen as the success of defining the set of relations in terms of Hamming distance, which is a key concept in random error correcting codes. Apart from Hamming distance, another distance employed is the Lee distance. Also Sharma and Kaushik (1977, 1979, 1986) introduced a very general way of defining distances in terms of the partitions of the alphabet set, when it is a finite ring. The Hamming and the Lee distances are the two special distances of the class of the general class of distances, that can be introduced. In the setting of the whole class of Sharma-Kaushik distances, Sharma and Sookoo (1988) examined the schemes when the set of relations is defined in terms Sharma-Kaushik ($SK-$) distances. The general nature of this study allows to introduce sets of relations what are termed as "nomination" and "discrimination" of vectors. In This paper this study is pursued further.

In Section 2, we list various definitions and notions. In Section 3, we obtain connections between the relations that are defined in terms of SK-partitions under the notions of "distance", "nomination", and "discrimination". In Section 4, the intersection numers and extended intersection numbers of the distance-nomination, and discrimination-schemes have been studied. Thereafter in Section 5, the notion of "spectrum" in the study of Delsarte has been adopted as "composition" and schemes with relations in terms of "composition" have been studied, together with their relations with other earlier studied schemes.

2. Definitions and notations.

The definition of the association scheme that we take here is due to Delsarte (1973). For other definitions included here, one may refer to Sharma and Kaushik (1986) and Sharma and Sookoo (1988).

Association Scheme.

DEFINITION. *Given a set X with at least two elements, and a set of relations $R = \{R_0, R_1, \ldots, R_N\}$, where N is a positive integer, (X, R) is*

Weight of an element with respect to an SK-partition P; Given an SK-partition P an element $a \in F_q$ is assigned the weight i if $a \in B_i$. We write this as

$$W_p(a) = i, \quad \text{if} \quad a \in B_i,\ i = 0, 1, \ldots, m-1.$$

Further let F_q^n represent the direct product of the n copies of F_q, and P, as above be an SK-partition of F_q, the *weight of an element* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in F_q^n$ with respect to P is given by

$$W_p(\mathbf{x}) = \sum_{i=1}^{n} W_p(x_i).$$

Next, if $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is anothere element of F_q^n, then *distance between vectors* x and y with respect to partition P is given by

$$d_p(\mathbf{x} - \mathbf{y}) = W_p(\mathbf{x} - \mathbf{y}).$$

That d_p is a metric over F_q^n is proved in Sharma and Kaushik (1977) and it follows from the conditions mentioned in defining an SK-partition. What may be pointed out here is that the Hamming and Lee metrics follows from the following two SK-partitions, Viz.,

$$P_H = \{B_0, B_1\} \quad \text{where} \quad B_0 = \{0\}, \quad \text{and} \quad B_1 = \{1, 2, \ldots, q-1\},$$

and

$$P_L = \{B_0, B_1, \ldots, B_{q-1}\}, \quad \text{where} \quad B_0 = \{0\} \quad \textit{and} \quad B_i = \{i, q-i\},$$

respectively. The Hamming and the Lee metrics shall be denoted by d_H and d_L.

Next, we define and briefly summarise results on SK-distance, nomination and discrimination schemes:

$SK - P$-Distance scheme $S(n, q, P)$.

DEFINITION. *Given $X = F_q^n$ an SK-partition P, the SK-Distance scheme corresponding to the partition P, is given by*

$$S(n, q, P) = (X, R^{d,n,P}),$$

where

$$R^{d,n,P} = \{R_0^{d,n,P}, R_1^{d,n,P}, \ldots, R_{n(m-1)}^{d,n,P}\}$$

and

$$R_i^{d,n,P} = \{(x,y) \in X^2 / d_p(\mathbf{x}-\mathbf{y}) = i\},\ i = 0, 1, \ldots, n(m-1).$$

It may bee seen that if $P = P_H$, the Hamming partition of F_q, then

$$S(n,n,P) = H(n,q),$$

the Hamming association scheme, studied by Delsarte.

It has been shown in Sharma and Sookoo (1988), that $S(n,q,P)$ is in general not an association scheme, and the necessary and sufficient condition of it to be an association scheme is that $|B_j \cap (B_i + x)|$ is constant as x varies over B_k, for i, j, k fixed indices. Also, refer Sharma and Sookoo (1988), $S(1,q,P_L)$ is an association scheme.

To define the other two schemes, we need to first define the following two notions:

Nomination of a vector.

DEFINITION. *The nomination* $\eta(\mathbf{x}, P)$ *of a vector* $\mathbf{x} = (x_1, \ldots, x_n)$ *in* X *corresponding to a* SK*-partition* P*, is the* n*-vector*

$$\eta(\mathbf{x}, P) = (i_1, i_2, \ldots, i_n),$$

if $\mathbf{x} \in B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n}$.

Discrimination of a vector.

DEFINITION. *The discrimination* $\delta(\mathbf{x}, P)$ *of a vector* $\mathbf{x}$ *(as above), corresponding to the* SK*-partition* P *of* $m(\ 2)$ *partitioning sets, is the* m*-vector*

$$\delta(\mathbf{x}, P) = (\delta_0, \delta_1, \ldots, \delta_{m-1})$$

where

$\delta_i =$ *the number of times an entry from* B_i *occurs in* $\mathbf{x}$.

We may now define the following schemes:

SK − P-Nomination scheme $S_{nom}(n,q,P)$.

DEFINITION. *Given* $X = F_q^n$, *an SK-partition P,*

$$S_{nom}(n,q,P) = (X, R^{\eta,n,P}),$$

where

$$R^{\eta,n,P} = \{R^{\eta,n,P}_{(i_1,i_2,\ldots,i_n)} / i_1,i_2,\ldots,i_n \in \{0,1,\ldots,m-1\}$$

and

$$R^{\eta,n,P}_{(i_1,i_2,\ldots,i_n)} = \{(\mathbf{x},\mathbf{y}) \in X^2 / \eta_P(\mathbf{x}-\mathbf{y}) = (i_1,i_2,\ldots,i_n)\}$$

It has been proved in Sharma and Sookoo (1988) that $S_{nom}(n,q,P)$ is an association scheme if $|B_j \cap (B_i + x)|$ is constant for all $x \in B_k$, and fixed i,j,k. This condition hold for all q and n when $P = P_H$, and $S_{nom}(n,q,P)$ is an association schemes, not identified earlier.

SK − P-Discrimination Scheme $S_{dis}(n,q,P)$.

DEFINITION. *Given* $X = F_q^n$ *and an SK-partition P,*

$$S_{dis}(n,q,P) = (X, R^{\delta,n,P})$$

where

$$R^{\delta,n,P} = \{R^{\delta,n,P}_{(0,1,\ldots,m-1)} / \delta_0,\delta_1,\ldots,\delta_{m-1} = 0,1,\ldots,n$$

and

$$\delta_0 + \delta_1 + \ldots + \delta_{m-1} = n,$$

and

$$R^{\delta,n,P} = \{(\mathbf{x},\mathbf{y}) \in X^2 / \delta_P(\mathbf{x}-\mathbf{y}) = (\delta_0,\delta_1,\ldots,\delta_{m-1})\}.$$

In the next section, we first study connections between the sets of relations defined in terms of distance, d_p, nomination η_P, and the discrimination δ_P, corresponding to the SK-partition P of F_q.

3. Connections between relation sets arising from distance, nomination and discrimination.

The three sets of results are as follows:

I. *Connection between discrimination and nomination relations*: For an SK-partition P of F_q, as defined before with m partitioning subsets, we have

$$R^{\delta,n,P}_{\delta_0,\delta_1,\ldots,\delta_{m-1})} = \{(\mathbf{x},\mathbf{y}) \in x^2 | \delta_p(\mathbf{x},\mathbf{y}) = (\delta_0, \delta_1, \ldots, \delta_{m-1})\}$$

$$= \{(\mathbf{x},\mathbf{y}) \in x^2 | (\mathbf{x},\mathbf{y}) \in B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n}$$

$$\text{and } \delta_t \text{ of } i_1, i_2, \ldots, i_n = t, (t = 0, 1, \ldots, m-1)\}$$

$$= \bigcup_{i_1,i_2,\ldots,i_n} \{(\mathbf{x},\mathbf{y}) \in x^2 | (\mathbf{x} - \mathbf{y}) \in B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n}\}$$

where in taking the union over $i_1, i_2, \ldots, i_n$, the conditions over these numbers would be

$$\delta_t \text{ of } i_1, i_2, \ldots, i_n = t \ (t = 0, 1, \ldots, m-1) = \bigcup_{i_1,i_2,\ldots,i_n} R^{\eta,n,P}_{(i_1,i_2,\ldots,i_n)}$$

where in taking the union over $i_1, i_2, \ldots, i_n$ the conditions over these numbers would be δ_t of $i_1, i_2, \ldots, i_n = t$, $(t = 0, 1, \ldots, m-1)$.

The next two relations examine similar situations between distance- and nomination-, and distance- and discrimination- relations.

II. *Relation between distance and nomination relation*: We have, straightforwardly,

$$R^{d,n,P}_i = \{(\mathbf{x},\mathbf{y}) \in X^2 | d_P(x,y) = i\}, \text{ where } X = F^n_q$$

$$= \{(\mathbf{x},\mathbf{y}) \in X^2 | W_p(\mathbf{x} - \mathbf{y}) = i\}$$

$$= \{(\mathbf{x},\mathbf{y}) \in X^2 | (\mathbf{x} - \mathbf{y}) \in B_{i_1} \times B_{i_2} \times \ldots \times Bi_n,$$

$$\text{for any } i_1, i_2, \ldots, i_n \ni i_1 + i_2 + \ldots + i_n = i\}$$

$$= \bigcup_{i_1,i_2,\ldots,i_n} R^{\eta,n,P}_{i_1,i_2,\ldots i_n}$$

where in taking the union over $i_1, i_2, \ldots, i_n$, the condition over these numbers would be $i_1 + i_2 + \ldots + i_n = i$.

III. *Relation between distance and discrimination relations:* Once again, as before

$$R_i^{d,n,P} = \{(\mathbf{x},\mathbf{y}) \in X^2 | W_p(\mathbf{x},\mathbf{y}) = i\}$$
$$= \bigcup_{\delta_0,\delta_1,\ldots,\delta_{m-1}} R^{\delta,n,P}_{(\delta_0,\delta_1,\ldots,\delta_{m-1})}$$

where in taking the union over $\delta_0, \delta_1, \ldots \delta_{m-1}$, the condition over these numbers would be

$$\delta_1 + 2\delta_2 + 3\delta_3 + \ldots + (m-1)\delta_{m-1} = i$$

since the weight of any element of X having discrimination $(\delta_0, \delta_1, \ldots, \delta_{m-1})$ is

$$\delta_1 + 2\delta_2 + 3\delta_3 + \ldots + (m-1)\delta_{m-1}.$$

4. The intersection numbers and extended intersection numbers of the distance, nomination and discrimination schemes.

Given an association scheme, refer Delsarte (1973), its intersection numbers can be defined. We define, first intersection numbers for the schemes studied in the previous section, and prove some results on these numbers. Later we define extended intersection numbers and study them.

Intersection Numbers of the Distance Scheme.

DEFINITION. *Given an SK-partition $P = B_0, B_1, \ldots, B_{m-1}$ of $F_q = 0, 1, \ldots, q-1$, let $X = F_q^n$, where n 1, and let $(X, R^{d,n,P})$ be the distance scheme. For* $\mathbf{x}$, $\mathbf{y}$ *X and numbers $i, j = 0, 1, \ldots, n(m-1)$, the intersection number $C_{i,j}^{d,n,P}$ $(\mathbf{x},\mathbf{y})$ is defined by*

$$C_{i,j}^{d,n,P}(\mathbf{x},\mathbf{y}) = |\{\mathbf{z} \in X | (\mathbf{x},\mathbf{z}) \in R_i, (\mathbf{z},\mathbf{y}) \in R_j\}|.$$

The Intersection Numbers of the Nomination Scheme.

DEFINITION. *Let P, F_q and X be as above, and let $(X, R^{\eta,n,P})$ be the Nomination scheme over X. Given* $\mathbf{x}, \mathbf{y} \in X$ *and n-tuples*

$$\mathbf{i} = (i_1, i_2, \ldots, i_n)(i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1)$$

and

$$\mathbf{j} = (j_1, j_2, \ldots, j_n)(j_1, j_2, \ldots, j_n = 0, 1, \ldots, m-1)$$

the intersection number $C_{\mathbf{i},\mathbf{j}}^{\eta,n,P}(\mathbf{x},\mathbf{y})$ *is defined by*

$$C_{\mathbf{i},\mathbf{j}}^{\eta,n,P}(\mathbf{x},\mathbf{y}) = |\{\mathbf{z} \in X | (\mathbf{x},\mathbf{z}) \in R_{\mathbf{i}}^{\eta,n,P}, (\mathbf{z},\mathbf{y}) \in R_{\mathbf{j}}^{\eta,n,P}|.$$

The Intersection Numbers of the Discrimination Scheme.

DEFINITION. *Let P, F_q and X be as in the previous two definitions and let $(X, R^{\delta,n,P})$ be the Discrimination Scheme over X. Given m-tuples*

$$\mathbf{i} = (i_0, i_1, \ldots, i_{m-1})(i_0, i_1, \ldots, i_{m-1} \geq 0, i_0 + i_1 + \ldots + i_{m-1} = n)$$

and

$$\mathbf{j} = (j_0, j_1, \ldots, j_{m-1})(j_0, j_1, \ldots, j_{m-1} \geq, 0, j_0 + j_1 + \ldots + j_{m-1} = n)$$

the intersection number $C_{\mathbf{i},\mathbf{j}}^{\delta,n,P}$ $(\mathbf{x},\mathbf{y})$ *is defined by*

$$C_{\mathbf{i},\mathbf{j}}^{\delta,n,P}(\mathbf{x},\mathbf{y}) = |\{\mathbf{z} \in X | (\mathbf{x},\mathbf{z}) \in R_{\mathbf{i}}^{\delta,n\mathbf{P}}, (\mathbf{z}-\mathbf{y}) \in R_{\mathbf{j}}^{\delta,n,P}\}|$$

We need the following lemma in order to evaluate these intersection numbers.

LEMMA 1. *Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of $F_q = \{0, 1, \ldots, q-1\}$. For any $x, y \in F_q$, and $i, j = 0, 1, \ldots, m-1$.*

$$|\{z \in F_q | (x-z) \in B_i, (z-y) \in B_j\}| = |(x+B_i) \cap (y+B_j)|.$$

Proof.

$$|\{z \in F_q | (x-z) \in B_i, (z-y) \in B_j\}| = |\{z \in F_q | z \in x+B_i, z \in y+B_j\}|$$
$$= |(x+B_i) \cap (y+B_j)|.$$

We next evaluate $C_{i,j}^{d,n,P}(\mathbf{x},\mathbf{y})$

THEOREM 1. *Given an SK- partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of $F_q = \{0, 1, \ldots, q-1\}$, let $X = F_q^n (n \geq 1)$ and let $(X, R^{d,n,P})$ be the distance* *scheme over X. Given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in X and numbers $i, j = 0, 1, \ldots, m-1$, the intersection number $C_{i,j}^{d,n,P}(\mathbf{x}, \mathbf{y})$ is equal to*

$$\sum_{\substack{i_1, i_2, \ldots, i_n \\ j_1, j_2, \ldots, j_n}} \prod_{g=1}^{n} |(x_g + B_{i_g}) \cap (y_g + B_{j_g})|$$

where in taking the sum over $i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n$ the conditions over these numbers would be

$$i_1 + i_2 + \ldots + i_n = i$$

$$j_1 + j_2 + \ldots + j_n = j$$

Proof. Given $\mathbf{z} \in X$, let $\mathbf{z} = (z_1, z_2, \ldots, z_n)$; $z_1, z_2, \ldots, z_n \in F_q$.

$$C_{i,j}^{d,n,P}(\mathbf{x}, \mathbf{y}) |\{\mathbf{z} \in X | (\mathbf{x}, \mathbf{z}) \in R_i^{d,n,P}, (\mathbf{z}, \mathbf{y}) \in R_j^{d,n,P}\}|$$

$$= \sum_{\substack{i_1, i_2, \ldots, i_n \\ j_1, j_2, \ldots, j_n}} |\{\mathbf{z} \in X | x_g - z_g \in B_{i_g}, z_g - y_g \in B_{j_g},\ g = 1, 2, \ldots, n\}|$$

where in taking the sum over $i_1, i_2, \ldots, i_n$, $j_1, j_2, \ldots, j_n$, the conditions over these numbers would be

$$i_1 + i_2 + \ldots + i_n = i$$

$$j_1 + j_2 + \ldots + j_n = j$$

$$\sum_{\substack{i_1, i_2, \ldots, i_n \\ j_1, j_2, \ldots, j_n}} \prod_{g=1}^{n} |\{z_g \in F_q | x_g - z_g \in B_{i_g}, z_g - y_g \in B_{j_g}\}|$$

where in taking the sum over $i_1, i_2, \ldots, i_n$, $j_1, j_2, \ldots, j_n$, the conditions over these numbers would be

$$i_i + i_2 + \ldots + i_n = i$$

$$j_1 + j_2 + \ldots + j_n = j$$

$$= \sum_{\substack{i_1, i_2, \ldots, i_n \\ j_1, j_2, \ldots, j_n}} \prod_{g=1}^{n} |(x_g + B_{i_g})(y_g + B_{j_g})\}|$$

where in taking the sum over $i_1, i_2, \ldots, i_n$; $j_1, j_2, \ldots, j_n$ the conditions over these numbers would be

$$i_1 + i_2 + \ldots + i_n = i$$

$$j_1 + j_2 + \ldots + j_n = j$$

from Lemma 1.

This establishes the proof.

For the Nomination Scheme, we have a similar result to the foregoing. It is stated as follows:

THEOREM 2. *Let P, F_q and X be as in the previous theorem and let $(X, R^{\eta,n,P})$ be the Nomination scheme over X. Given elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in X, and n-tuples*

$$\mathbf{i} = (i_1, i_2, \ldots, i_n), (i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1)$$

and

$$j = (j_1, j_2, \ldots, j_n)(j_1, j_2, \ldots, j_n = 0, 1, \ldots, m-1)$$

the intersection number $c_{\mathbf{i},\mathbf{j}}^{\eta,n,P}(\mathbf{x},\mathbf{y})$ is equal to

$$\prod_{g=1}^{n} |(x_g + B_{i_g}) \cap (y_g + B_{j_g})|$$

Proof. Given $\mathbf{z} \in x^n$, let $\mathbf{z} = (z_1, z_2, \ldots, \ldots, z_n)$; $z_1, z_2, \ldots, z_n \in F_q$

$$\begin{aligned} c_{\mathbf{i},\mathbf{j}}^{\eta,n,P}(\mathbf{x},\mathbf{y}) &= |\{\mathbf{z} \in X | (\mathbf{x},\mathbf{z}) \in R_{\mathbf{i}}^{\eta,n,P}, (\mathbf{z},\mathbf{y}) \in R_{j}^{\eta,n,P}\}| \\ &= |\{\mathbf{z} \in X | (\mathbf{x}-\mathbf{z}) \in B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n}, \\ &\qquad (\mathbf{z}-\mathbf{y}) \in B_{j_1} \times B_{j_2} \times \ldots \times B_{j_n}\}| \\ &= |\{\mathbf{z} \in X | x_g - z_g \in B_{i_g}, z_g - y_g \in B_{j_g}, g = 1, 2, \ldots, n\}| \\ &= \prod_{g=1}^{n} |\{z_g \in F_q | x_g - z_g \in B_{i_g}, Z_g - y_g \in B_{j_g}\}| \\ &= \prod_{g=1}^{n} |(x_g + B_{i+g}) \cap (y_g + B_{j_g})| \end{aligned}$$

from Lemma 1.

The intersection numbers of the Discrimination Scheme can be obtained easily, as we show in the following theorem.

THEOREM 3. *Let P, F_q and X be as in the two previous theorems and let $(X, R^{\delta,n,P})$ be the Discrimination Scheme over X. Given elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in X, and m-tuples*

$$\mathbf{i} = (i_0, i_1, \ldots, i_{m-1}), (i_0, i_1, \ldots, i_{m-1} \geq 0, i_0 + i_1 + \ldots + i_{m-1} = n)$$

and

$$\mathbf{j} = (j_0, j_1, \ldots, j_{m-1}), (j_0, j_1, \ldots, j_{m-1} \geq 0, j_0 + j_1 + \ldots + j_{m-1} = n)$$

the intersection number $c_{\mathbf{i},\mathbf{j}}^{\delta,n,P}$ $(\mathbf{x}, \mathbf{y})$ is equal to

$$\sum_{\substack{a_0, a_1, \ldots, a_n \\ b_0, b_1, \ldots, b_n}} \prod_{g=1}^{n} |(x_g + B_{a_g}) \cap (y_g + B_{b_g}|,$$

where in taking the sum over $a_0, a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_n$, the conditions over these numbers would be

$$i_e \textit{ of } a_0, a_1, \ldots, a_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$j_e \textit{ of } b_0, b_1, \ldots, b_n = e, \ (e = 0, 1, \ldots, m-1).$$

Proof. Given $\mathbf{z} \in X$, let $\mathbf{z} = (z_1, z_2, \ldots, z_n)$, $(z_1, z_2, \ldots, z_n \in F_q)$

$$\begin{aligned} c_{\mathbf{i},\mathbf{j}}^{\delta,n,P}(\mathbf{x}, \mathbf{y}) &= |\{\mathbf{z} \in X | (\mathbf{x}, \mathbf{z}) \in R_{\mathbf{i}}^{\delta,n,P}, (\mathbf{z}, \mathbf{y}) \in R_{\mathbf{j}}^{\delta,n,P}\}| \\ &= \sum_{\substack{a_0, a_1, \ldots, a_n \\ b_1, b_1, \ldots, b_n}} |\{\mathbf{z} \in X | x_g - z_g \in B_{a_g}, z_g - y_g \in B_{b_g}, \ g = 0, 1, \ldots, n\}| \end{aligned}$$

where in taking the sum over $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$, the conditions over these numbers would be

$$i_e \text{ of } a_0, a_1, \ldots, a_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$j_e \text{ of } b_0, b_1, \ldots, b_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$= \sum_{\substack{a_0,a_1,\ldots,a_n \\ b_0,b_1,\ldots,b_n}} |\{z_g \in F_q | x_g - z_g \in B_{a_g}, z_g - y_g \in B_{b_g},\ g = 0,1,\ldots,n\}|$$

where in taking the sum over $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$, the conditions over these numbers would be

$$i_e \text{ of } a_0, a_1, \ldots, a_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$j_e \text{ of } b_0, b_1, \ldots, b_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$= \sum_{\substack{a_0,a_1,\ldots,a_n \\ b_0,b_1,\ldots,b_n}} \prod_{g=1}^{n} |(x_g + B_{a_g}) \cap (y_g + B_{b_g})|$$

where in taking the sum over $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$, the conditions over these numbers would be

$$i_e \text{ of } a_0, a_1, \ldots, a_n = e, \ (e = 0, 1, \ldots, m-1)$$

$$j_e \text{ of } b_0, b_1, \ldots, b_n = e, \ (e = 0, 1, \ldots, m-1)$$

Hence the proof.

Next we investigate numbers which are defined in a similar way to the intersection numbers, and which we call extended intersection numbers of the scheme under consideration. The definition of an extended number follows:

Extended Intersection Numbers.

DEFINITION. *Let X be a non-empty set with at least two elements, and let $R = \{R_I | I \in D\}$ be a set of relations over X, where D is some suitable set of subscripts. Given elements $I_1, I_2, \ldots, I_{h+1} \in D$ and $\mathbf{x}, \mathbf{y} \in X$, we define the extended intersection number $C_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x},\mathbf{y})$ of (X, R) in the following way*

$$C_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x},\mathbf{y}) = |\{(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_h) \in X^h | (\mathbf{x}, \mathbf{z}_1) \in R_{I_1},$$

$$= (\mathbf{z}_1, \mathbf{z}_2) \in R_{I_2}, \ldots, (\mathbf{z}_h, \mathbf{y}) \in R_{I_{h+1}}\}|$$

We investigate the extended intersection numbers of the distance, nomination and discrimination schemes below. The first result in this direction is for the distance scheme.

THEOREM 4. *Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of $F_q = \{0, 1, \ldots, q-1\}$ and let $X = F_q^n$. Also let $R_I^{d,n,P}(I = 0, 1, \ldots, n(m-1))$ be the distance relations over X and $(X, R^{d,n,P})$ the distance scheme over X^2. If $\mathbf{x}, \mathbf{y} \in X$, $I_1, I_2, \ldots, I_{h+1} = 0, 1, \ldots, n(m-1)$ and $C^{d,n,P}_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x}, \mathbf{y})$ is an extended intersection number of $(X, R^{d,n,P})$, then*

$$C^{d,n,P}_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x}, \mathbf{y}) = |\{(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{h+1}) \in X^{h+1} | \mathbf{x} - \mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{h+1},$$

and

$$w_p(\mathbf{b}_\alpha) = I_\alpha(\alpha = 1, 2, \ldots, h+1)\}|$$

Proof.

$$\begin{aligned} &C^{d,n,P}_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x}, \mathbf{y}) \\ &= |\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_h) \in X^h | (\mathbf{x}, \mathbf{z}_1) \in R^{d,n,P}_{I_1}, (\mathbf{z}_1, \mathbf{z}_2) \in R^{d,n,P}_{I_2}, \\ &\qquad \ldots, (\mathbf{z}_h, \mathbf{y}) \in R^{d,n,P}_{I_{h+1}}\}| \\ &= |\{(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_h) \in X^h | \mathbf{x} - \mathbf{z}_1 = \mathbf{b}_1 \\ &\qquad \mathbf{z}_1 - \mathbf{z}_2 = \mathbf{b}_2 \\ &\qquad \vdots \\ &\qquad z_h - \mathbf{y} = \mathbf{b}_{h+1} \end{aligned}$$

as $\mathbf{b}_\alpha(\alpha = 1, 2, \ldots, h+1)$ varies over

$X \ni w_p(\mathbf{b}_\alpha) = I_\alpha\}| = |\{(b_1, b_2, \ldots, \mathbf{b}_{h+1}) \in X^{h+1} | \mathbf{x} - \mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{h+1},$

and $w_p(\mathbf{b}_\alpha) = I_\alpha(\alpha = 1, 2, \ldots, h+1)\}|$.

We next prove a similar result to the above for the nomination scheme.

THEOREM 5. *Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ be an SK-partition of $F_q = \{0, 1, \ldots, q-1\}$, $X = F_q^n (n \geq 1)$ and let*

$$\mathbf{I}_\alpha = (i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha_n})$$

$$(I_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha_n} = 0, 1, \ldots, m-1; \alpha = 1, 2, \ldots, h+1)$$

be $h+1$ n-tuples), Also let $(X, R^{\eta,n,P})$ be the nomination scheme over X, and let $R^{\eta,n,P}_{\mathbf{I}_\alpha}$ $(\alpha = 1, 2, \ldots, h+1)$ be the nomination relations.

Given $\mathbf{x}\mathbf{y} \in X$, the extended intersection number. $C^{\eta,n,P}_{\mathbf{I}_1,I_2,\ldots,\mathbf{I}_{h+1}}(\mathbf{x},\mathbf{y})$ of $(X, R^{\eta,n,P})$ is equal to

$$|\{(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_{h+1}) \in X^{h+1}|\mathbf{x}-\mathbf{y} = \mathbf{b}_1+\mathbf{b}_2+\ldots+\mathbf{b}_{h+1},$$

and $\eta_P(\mathbf{b}_\alpha) = \mathbf{I}_\alpha(\alpha = 1,2,\ldots,h+1)\}|$

Proof.

$$\begin{aligned}
&C^{\eta,n,P}_{I_1,I_2,\ldots,I_{h+1}}(\mathbf{x},\mathbf{y})\\
&= |\{\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_h) \in X^h|(\mathbf{x},\mathbf{z}_1) \in R^{\eta,n,P}_{I_1}, (\mathbf{z}_1,\mathbf{z}_2) \in R^{\eta,n,P}_{I_2},\\
&\qquad \ldots, (\mathbf{z}_h,\mathbf{y}) \in R^{\eta,n,P}_{I_{h+1}}\}|\\
&= |\{(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_h) \in X^h|\mathbf{x}-\mathbf{z}_1 = \mathbf{b}_1\\
&\qquad \mathbf{z}_1-\mathbf{z}_2 = \mathbf{b}_2\\
&\qquad \vdots\\
&\qquad z_h - \mathbf{y} = \mathbf{b}_{h+1}
\end{aligned}$$

as $\mathbf{b}_\alpha(\alpha = 1,2,\ldots,h+1)$ varies over X

$$\ni \eta_p(\mathbf{b}_\alpha) = \mathbf{I}_\alpha\}|$$

$$= |\{(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_{h+1}) \in X^{h+1}|\mathbf{x}-\mathbf{y} = \mathbf{b}_1+\mathbf{b}_2,\ldots,\mathbf{b}_{h+1}$$

and

$$\eta_P(\mathbf{b}_\alpha) = \mathbf{I}_\alpha(\alpha = 1,2,\ldots,h+1)\}|$$

This completes the proof.

The extended intersection numbers of the discrimination scheme can be obtained by a method similar to the one used in the above theorem to obtain the extended intersection numbers of the nomination scheme. This method is shown in the next theorem.

THEOREM 6. *Let* $P = \{B_0, B_1, \ldots, B_{m-1}\}$ *be an SK-partition of* $F_q = \{0,1,\ldots,q-1\}$, $X = F_q^n(n \geq 1)$ *and let*

$$\mathbf{I}_\alpha = (i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha(m-1)}) \cdot (i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha(m-1)} =$$
$$= 0,1,\ldots,n; \alpha = 1,2,\ldots,h+1)$$

be $(h+1)$ n-tuples. Also let $(X, \mathbb{R}^{\delta,n,P})$ be the discrimination scheme over X and let $R_{I_\alpha}^{\delta,n,P}$ $(\alpha = 1, 2, \ldots, h+1)$ be discrimination relations.

Given $\mathbf{x}, y, \in X$, the extended intersection number $C_{\mathbf{I}_1, I_2, \ldots, \mathbf{I}_{h+1}}^{\delta,n,P}(\mathbf{x}, \mathbf{y})$ of $(X, R^{\delta,n,P})$ is equal to

$$|\{(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{h+1}) \in X^{h+1} | \mathbf{x} - \mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{h+1}$$

and

$$\delta_P(\mathbf{b}_\alpha) = \mathbf{I}_\alpha (\alpha = 1, 2, \ldots, h+1)\}|$$

Proof.

$$\begin{aligned} & C_{I_1, I_2, \ldots, I_{h+1}}^{\delta,n,P}(\mathbf{x}, \mathbf{y}) \\ &= |\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_h) \in X^h | (\mathbf{x}, \mathbf{z}_1) \in R_{I_1}^{\delta,n,P}, (\mathbf{z}_1, \mathbf{z}_2) \in R_{I_2}^{\delta,n,P}, \\ &\qquad \ldots, (\mathbf{z}_h, \mathbf{y}) \in R_{I_{h+1}}^{\delta,n,P}\}| \\ &= |\{(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_h) \in X^h | \mathbf{x} - \mathbf{z}_1 = \mathbf{b}_1 \\ &\qquad \mathbf{z}_1 - \mathbf{z}_2 = \mathbf{b}_2 \\ &\qquad \vdots \\ &\qquad z_h - \mathbf{y} = \mathbf{b}_{h+1} \end{aligned}$$

as $\mathbf{b}_\alpha (\alpha = 1, 2, \ldots, h+1)$ varies over X

$$\ni \delta_p(\mathbf{b}_\alpha) = \mathbf{I}_\alpha\}|$$

$$= |\{(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{h+1}) \in X^{h+1} | \mathbf{x} - \mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{h+1},$$

and

$$\delta_P(\mathbf{b}_\alpha) = I_\alpha (\alpha = 1, 2, \ldots, h+1)\}|$$

We next investigate a scheme defined with respect to the concept composition. We refer to this scheme as the composition scheme.

5. The composition scheme.

An association scheme called the spectral scheme, refer to Delsarte (1973), was defined in terms of the concept spectrum. We refer to spectrum as composition (c.f. Mac Williams and Sloane (1978)), so naturally we call the spectral scheme the composition scheme.

In this section, we define the Composition Relations and the Composition Scheme and show the relationships between these relations and the Distance, Nomination and Discrimination relations. We can also prove that the Composition Scheme is an association scheme. We evaluate its intersection numbers and extended intersection numbers.

First, we give the definition of the composition (c.f. Mac-Williams and Sloane (1978)) of a vector in $X = F_q^n$, where F_q is the ring of integers modulo q.

Composition of a Vector in $X = F_q^n$.

DEFINITION. *Given the ring* $F_q = \{0, 1, \ldots, q-1\}$ *of vectors modulo* q*, let* $X = F_q^n$*,* $n \geq 1$*. The composition of a vector* $\mathbf{v} \in X$ *is*

$$comp\ (\mathbf{v}) = s(\mathbf{v}) = (s_0, s_1, \ldots, s_{q-1})$$

where $s_i = s_i(\mathbf{v})$ *=number of co-ordinates of* $\mathbf{v}$ *equal to* $i \in F_q$ $(i = 0, 1, \ldots, q-1)$

Relations can be defined in terms of composition, as shown below.

Composition Relations.

DEFINITION. *Given the natural numbers* $t_0, t_1, \ldots, t_{q-1}$ *satisfying the condition* $t_0 + t_1 + \ldots + t_{q-1} = n$*, we define the Composition Relation* $R^{c,n}_{t_0,t_1,\ldots,t_{q-1}}$ *over* $X = F_q^n, (n \geq 1)$ *as follows:*

$$R^{c,n}_{(t_0,t_1,\ldots,t_{q-1})} = \{(\mathbf{x}, \mathbf{y}) \in X^2 |\ comp\ (\mathbf{x} - \mathbf{y}) = (t_0, t_1, \ldots, t_{q-1})\}.$$

We need the following notation in order to define the Composition Scheme.

Notation: For $n \geq 1$, let

$$T_n = \{(t_0, t_1, \ldots, t_{q-i}) | t_0 + t_1 + \ldots + t_{q-1} = n$$

where t_i is a natural number $(i = 0, 1, \ldots, q-1)\}$ and let $M_n = |T_n|$.

DEFINITION. *Let*

$$R^{c,n} = \{R^{c,n}_{(t_0, t_1, \ldots, t_{q-1}} | (t_0, t_1, \ldots, t_{q-1}) \in T_n\}.$$

$(X, R^{c,n})$ is called the Composition Scheme over $X = F_q^n$.

We investigate the relationship between the Composition Relations and those we introduced earlier, starting with Distance Relations.

Relations Between Composition and Distance Relations.

Given an SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of $F_q = \{0, 1, \ldots, q-1\}$, let $X = F_q^n$, $n \geq 1$. We have

$$\begin{aligned} R_i^{d,n,P} &= \{(\mathbf{x}, \mathbf{y}) \in X^2 | w_p(\mathbf{x} - \mathbf{y}) = i\} \\ &= \{(\mathbf{x}, \mathbf{y}) \in X^2 | \text{ comp } (\mathbf{x} - \mathbf{y}) = (t_0, t_1, \ldots, t_{q-1}), \\ &\qquad \text{where } (t_1 w_p(1) + t_2 w_p(2) + \\ &\qquad \ldots + t_{q-1} w_p(q-1) = i)\} \\ &\bigcup_{t_0, t_1, \ldots, t_{q-1}} R^{c,n}_{(t_0, t_1, \ldots, t_{q-1}}) \end{aligned}$$

where in taking the union over $t_0, t_1, \ldots, t_{q-1}$ the condition satisfied by these numbers would be

$$t_1 w_p(1) + t_2 w_p(2) + \ldots + t_{q-1} w_p(q-1) = i$$

We next show the relationship between Composition Relations and Nomination Relations over $X = F_q^n$ with respect to an SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of F_q.

Relationship Between Composition and Nomination Relations.

Given an n-tuple $(i_1, i_2, \ldots, i_n)$ of integers

$$(i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1)$$

whe have

$$\bigcup_{(i_i, i_2, \ldots, i_n)} R^{\eta, n, P}_{(i_1, i_2, \ldots, i_n)}$$

where the union is taken over all permutations of $(i_1, i_2, \ldots, i_n)$

$$= \bigcup_{(i_1, i_2, \ldots, i_n)} \{(\mathbf{x}, \mathbf{y}) \in X^2 | (\mathbf{x} - \mathbf{y}) \in B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n}$$

where the union is taken over all permutations of $(i_1, i_2, \ldots, i_n)$

$$= \bigcup_{(t_0, t_1, \ldots, t_{q-1})} \{(\mathbf{x}, \mathbf{y}) \in x^2 | \text{ comp } (\mathbf{x} - \mathbf{y}) = (t_0, t_1, \ldots, t_{q-1})\}$$

where the union is taken over all permutations of all compositions $(t_0, t_1, \ldots, t_{q-1})$ of elements of $B_{i_1} \times B_{i_2} \times \ldots \times B_{i_n} =$

$$= \bigcup_{(t_0, t_1, \ldots, t_{q-1})} R^c_{t_0, t_1, \ldots, t_{q-1}}$$

where the union is taken over all permutations of all compositions $(t_0, t_1, \ldots, t_{q-1})$ of elements of $B_{i_1} \times B_{i_2} \times \cdots \times B_{i_n}$.

For an SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of $F_q = \{0, 1, \ldots, m-1\}$, we can relate the Discrimination Relations over $X = F_q^n (n \geq 1)$ to the Composition Relations as shown below.

Relationship Between Composition and Discrimination Relations.

For an $(m-1)$-tuple $(\delta_0, \delta_1, \ldots, \delta_{m-1})$ of positive integers $\delta_0, \delta_1, \ldots,$ $\delta_{m-1} = 0, 1, \ldots, n,$

$$R^{\delta, n, P}_{(\delta_0, \delta_1, \ldots, \delta_{m-1})} = \{(\mathbf{x}, \mathbf{y}) \in X^2 | \delta_p(\mathbf{x} - \mathbf{y}) = (\delta_0, \delta_1, \ldots, \delta_{m-1})\}$$

$$= \bigcup_{(t_0, t_1, \ldots, t_{q-1}}) \in D_{(\delta_0, \delta_1, \ldots, \delta_{m-1})} \{(\mathbf{x}, \mathbf{y}) \in x^2 | \text{comp}(\mathbf{x} - \mathbf{y}) = (t_0, t_1, \ldots, t_{q-1})\}$$

where the union is taken over all elements $t_0, t_1, \ldots, t_{q-1})$ of the set $D_{(\delta_0,\delta_1,\ldots,\delta_{m-1})}$ of all compositions of elements of X having discrimination $(\delta_0, \delta_1, \ldots, \delta_{m-1})$

$$= \bigcup_{(t_0,t_1,\ldots,t_{q-1})} R^{c,n}_{t_0,t_1,\ldots,t_{q-1}}$$

where the union is taken ove rall elements $(t_0, t_1, \ldots, t_{q-1})$ of the set $D_{(\delta_0,\delta+1,\ldots,\delta_{m-1})}$ of all composition of elements of X having discrimination $(\delta_0, \delta_1, \ldots, \delta_{m-1})$.

Next, we define and evaluate the intersection numbers and extended intersection numbers of the composition scheme.

Intersection Numbers of the Composition Scheme.

DEFINITION *Given q-tuples* $\mathbf{i} = (i_0, i_1, \ldots, i_{q-1})$ $\mathbf{j} = (j_0, j_1, \ldots, j_{q-1})$ *and* $\mathbf{k} = (k_0, k_1, \ldots, k_{q-1})$*, the intersection number*

$$C^{c,n}_{\mathbf{ijk}}(\mathbf{x},\mathbf{y}) = |\{\mathbf{z} \in X | (\mathbf{x},\mathbf{z}) \in R^{c,n}_{\mathbf{i}}, (\mathbf{z},\mathbf{y}) \in R^{c,n}_{j}\}|$$

for any $(\mathbf{x},\mathbf{y}) \in R^{c,n}_{\mathbf{k}}$.

Extended Intersection Numbers of the Composition Scheme.

DEFINITION *Given elements* $\mathbf{x},\mathbf{y},\in X = F^n_q$ *and q-tuples* $\mathbf{i}_i, \mathbf{i}_2, \ldots,$ $\mathbf{i}_{h+1}$*, the extended intersection number* $C^{c,n}_{i_1,i_2,\ldots,\mathbf{i}_{h+1}}(\mathbf{x},\mathbf{y})$ *is equal to*

$$|\{(\mathbf{z}_1, z_2, \ldots, \mathbf{z}_h) \in x^h | (\mathbf{x},\mathbf{z}_1) \in R^{c,n}_{\mathbf{i}_1}, (\mathbf{z}_1,\mathbf{z}_2) \in R^{c,n}_{\mathbf{i}_2}, \ldots, (\mathbf{z}_h,\mathbf{y}) \in R^{c,n}_{\mathbf{i}_{h+1}}\}|$$

THEOREM 7. *Let* $(X, R^{c,n})$ *be the composition scheme over* $X = F^n_q$*. Given q-tuples* $\mathbf{i} = (i_0, i_1, \ldots, i_{q-1})$*,* $\mathbf{j} = (j_i, j_2, \ldots, j_{q-1})$ *and* $\mathbf{k} = (k_0, k_1, \ldots, k_{q-1})$ *and elements* $\mathbf{x},\mathbf{y} \in X \ni (\mathbf{x},\mathbf{y}) \in R^{c,n}_{\mathbf{k}}$ *the intersection number*

$$C^{c,n}_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{x},\mathbf{y}) = |\{(\mathbf{a},\mathbf{b}) \in X^2 | \mathbf{a},\mathbf{b} \in X, comp(\mathbf{a}+\mathbf{b}) = \mathbf{k},$$

$$comp(\mathbf{a}) = \mathbf{i} \; and \; comp\,(\mathbf{b}) = \mathbf{j}\}|.$$

Proof.

$$
\begin{aligned}
C^{c,n}_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{x},\mathbf{y}) &= |\{\mathbf{z} \in X \mid \text{comp}\,(\mathbf{x}-\mathbf{z}) = \mathbf{i},\ \text{comp}\,(\mathbf{z}-\mathbf{y}) = \mathbf{j}\}| \\
&= |\{\mathbf{z} \in X | \mathbf{a},\mathbf{b} \in X, \mathbf{x}-\mathbf{z} = \mathbf{a} \\
&\qquad \mathbf{z}-\mathbf{y} = \mathbf{b} \\
&\qquad \text{comp}\,(\mathbf{a}) = i \text{ and comp}\,(\mathbf{b}) = j\}| \\
&= |\{(\mathbf{a},\mathbf{b}) \in X^2 | \mathbf{a},\mathbf{b} \in X, \mathbf{x}-\mathbf{y} = \mathbf{a}+\mathbf{b} \\
&\qquad \text{comp}\,(\mathbf{a}) = i \text{ and comp}\,(\mathbf{b}) = j\}|.
\end{aligned}
$$

Now, since $(X, R^{c,n})$ is an association scheme, the above number does not depend on $\mathbf{x}$ and $\mathbf{y}$, provided that $(\mathbf{x},\mathbf{y}) \in R^{c,n}_{\mathbf{k}}$. Hence

$$
\begin{aligned}
C^{c,n}_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{x},\mathbf{y}) = |\{(\mathbf{a},\mathbf{b}) \in X^2 | \mathbf{a},\mathbf{b} \in X, \text{comp}\,(\mathbf{a}+\mathbf{b}) = \mathbf{k}, \\
\text{comp}\,(\mathbf{a}) = \mathbf{i},\ \text{comp}\,(\mathbf{b}) = \mathbf{j}\}|.
\end{aligned}
$$

We present a similar result to the above for the extended intersection numbers of $(x, R^{c,n})$.

THEOREM 8. *Let $(X, R^{c,n})$ be the composition scheme over $X = F^n_q$. Given q-tuples, $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_{h+1}$ and elements $\mathbf{x}, \mathbf{y} \in X$, the extended intersection number*

$$
\begin{aligned}
C^{c,n}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_{h+1}}(\mathbf{x},\mathbf{y}) = &|\{(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{h+1}) \in X^{h+1} | \mathbf{x}-\mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_{h+1}, \\
&\textit{and comp}\,(\mathbf{b}_\alpha) = \mathbf{i}_\alpha (\alpha = 1, 2, \ldots, h+1)\}|.
\end{aligned}
$$

Proof.

$$C^{c,n}_{\mathbf{i}_0,\mathbf{i}_1,\ldots,i_{h+1}}(\mathbf{x},\mathbf{y})$$

$$= |\{(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_h) \in X^h | \begin{matrix} \text{comp}(\mathbf{x}-\mathbf{z}_1)=\mathbf{i}_1, \\ \text{comp}(\mathbf{z}_1-\mathbf{z}_2)=\mathbf{i}_2, \\ \vdots \\ \text{comp}(\mathbf{z}_h-\mathbf{y})=\mathbf{i}_{h+1} \end{matrix}$$

$$|\{(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_{h+1}) \in x^{h+1} | \begin{matrix} \mathbf{x}-\mathbf{z}_1=\mathbf{b}_1 \\ \mathbf{z}_1-\mathbf{z}_2=\mathbf{b}_2, \\ \vdots \\ \mathbf{z}_h-\mathbf{y}=\mathbf{b}_{h+1} \end{matrix}$$

$$\text{comp }(\mathbf{b}_\alpha)=\mathbf{i}_\alpha(\alpha=1,2,\ldots,h+1)\}|$$

$$= |\{(b_1,\mathbf{b}_2,\ldots,\mathbf{b}_{h+1}) \in X^{h+1}|\mathbf{x}-\mathbf{y}=\mathbf{b}_1+\mathbf{b}_2+\ldots+\mathbf{b}_{h+1},$$

$$\text{comp }(\mathbf{b}_\alpha)=\mathbf{i}_\alpha(\alpha=1,2,\ldots,h+1)$$

This concludes the proof.

REFERENCES

[1] Bose R.C., Shimamato T., *Classification and Analysis of Partially Balanced Incomplete Block Designs two Associates,* J. Amer. Stat. Association **47** (1952), 151-184.

[2] Delsarte P., *An Algebraic Approach to the Association Schemes of Coding Theory,* Philips Research Reports Supp. **10** (1973).

[3] MacWilliams F.J., Sloane N.J.A., *The Theory of Error-Correcting Codes,* North Holland publ; Co., Amsterdam (1977).

[4] Bhu Dev Sharma., Manohar Lal Kaushik., *Error Correcting Codes through a New Metric,* 41st Annual Conf. of. Inter. Stat. Inst. New Delhi, (1977) (Abstract in Proceedings).

[5] Bhu Dev Sharma., Manohar Lal Kaushik., *Limited Intensity Random and Burst error Codes with Class-weight Considerations,* Electronische, Infor. und Kyber. **15** (1979), 315-321.

[6] Bhu Dev Sharma., Manohar Lal Kaushik., *Algebra of Sharma and Kaushik's Metric Inducing Partitions of Z_q,* Jr. of Comb. Infor. and Syst. Sci. **11** (1986), 19-32.

[7] Bhu Dev Sharma., Norris Sookoo., *Association Scheme and their Extensions arising from Metric Inducing Partitions of F,* Invited talk at the

International Conference on Comb. Math. and Appli., Indian Stat. Inst. Calcutta (India), December 14-18, (1988). To appaer in proceedings.

[8] Bhu Dev Sharma., Norris Sookoo., *An Association Scheme in General weight structured Coding theory,* under preparation, 1989.

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