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# PROBLEMS AND RESULTS ON GRAPH AND HYPERGRAPH COLORINGS

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Colorings have always attracted much attention in graph theory, therefore it seems almost hopeless to give a comprehensive list of the open problems of this topic. Two nice collections of some important questions have been compiled by Toft [26], [27], also providing relevant references for the historical background of the problems discussed there.

In the present note the reader will find a set of about fifty graph and hypergraph coloring problems (divided into 12 groups) which have been raised during current research, or posed many years ago but seem to have been forgotten although they are really attractive. Most of those questions are selected from areas which, in spite of their definite interest, have not yet been extensively investigated, and therefore some of them might be more easily solvable than the «classic» problems on vertex and edge colorings.

## Notation.

We use the standard terminology of graph theory, therefore we only give the most important definitions and notation here. Some more particular notions will be introduced in the text.

For a finite undirected graph $G = (V, E)$ with vertex set $V$ and edge set $E$, denote by $\alpha(G)$, $\omega(G)$, and $\chi(G)$ the *independence number*, the *clique number*, and the *chromatic number* of $G$, respectively (i.e., $\alpha(G)$ is the maximum cardinality of an independent set [= vertex set consisting of pairwise non-adjacent vertices], $\omega(G)$ is the maximum number of pairwise adjacent vertices, and $\chi(G)$ is the minimum number of independent sets covering V). The vertex set and the edge set of $G$ sometimes will be denoted by $V(G)$ and $E(G)$, respectively.

In a hypergraph $\mathcal{H}$, a vertex set $S$ is independent if it does not contain any edge of $\mathcal{H}$, and $T$ is a *transversal* if it meets all edges of $\mathcal{H}$ (i.e., if the complement of $T$ is independent). The *transversal number* $\tau(\mathcal{H})$ is the smallest cardinality of a transversal. The chromatic number $\chi(H)$ of $H$ is defined analogously to that of graphs; the hypergraph is 2-colorable if it has a transversal whose complement also is a transversal.

Recall that a graph $G$ is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$. A $G' = (V', E')$ is a *subgraph* of $G$ if $V' \subseteq V$ and $E' \subseteq \{e \in E | e \subseteq V'\}$; $G'$ is *induced* by $V'$ if $E' = \{e \in E | e \subseteq V'\}$.

## 1. Orientations vs. colorings.

Two of the most famous sufficient conditions insuring k-colorability of graphs (which happen to be necessary, too) point out relations between the chromatic number and certain orientations of the edge set. The theorem of Gallai and Roy ([12], [23]) deals with directed *paths*, while Minty's theorem [20] puts a restriction on the orientations of edges on the *cycles* of the graph. Recently a common generalization of those two important results has been found ([34]), and it has been pointed out that, in an algorithmic sense as well, Minty's and Gallai and Roy's assumptions are *polynomially equivalent* to $k$-colorability. (The first of the two assumptions leads to a coloring algorithm with *linear* running time!) It is not clear, however, whether or not heuristics for finding an orientation without 'long' directed paths are easier than determining the chromatic number itself.

*Problem* 1.1. Design fast (polynomial or sub-exponential) algorithms that find an orientation of a graph, in which the length of the longest directed path is «not very far» from the chromatic number.

Here 'not very far' means a modest requirement; any improvement on the performance ratio of graph coloring algorithms would be warmly welcome. The accuracy of approximation to chromatic number (or to the smallest possible length of a longest directed path) may depend on the running time.

It follows from the results of [34] that if $G$ contains no cycle of length $\equiv 1 \pmod{k}$, then $\chi(G) \leq k$. (This fact has recently been observed by Dean and Toft, too [8].) For this reason it is natural to pose the following question.

*Problem* 1.2. Let $p, q, r$ $(p \geq q > r \geq 0)$ be given integers. Find simple properties $\mathcal{P}$ insuring that every graph with property $\mathcal{P}$ and having chromatic number $> p$ contains a cycle of length congruent to $r \pmod{q}$.

Of course, the case $q = 2$ is trivial. For $q = 3$, Dean and Toft have proved that every $K_4$-free graph of chromatic number greater than 3 contains cycles with lengths of 0, 1 and 2 (mod 3).

A more general question is as follows.

*Problem* 1.3. Describe (reasonably small) sets $R$ of natural numbers with the property that every graph $G$ with $\chi(G) > p$ contains a cycle of length $r$ for some $r \in R$.

The previous two questions may be combined, looking for *sets* of residue classes satisfying the cycle-length property in graphs of large chromatic number.

## 2. Perfect-graph recognition.

There are fast recognition algorithms for many particular classes of graphs (planar graphs, chordal graphs, etc.). However, no positive result is known for perfect graphs. By Lovász's famous theorem [17] the question is equivalent to the following one.

*Problem* 2.1. How many steps are needed to check $\alpha(G')\omega(G') \geq |V(G')|$ for all induced subgraphs $G'$ of $G$?

An $n$-element set has $2^n$ subsets, so that for the first sight «perfectness» means that many requirements. If the Strong Perfect Graph Conjecture is true, however, then it leads to a considerable simplification, as observed in [37], since the problem reduces to the following one.

*Problem* 2.2. How many steps are needed to decide whether or not a graph on n vertices contains an odd cycle of length $\geq 5$ as an induced subgraph?

For this latter problem a stronger upper bound of $O((2-c)^n)$ has been proved in [37], for some constant $c > 0$. The proof is based on a hypergraph-theoretic observation concerning minimal transveral sets of finite set systems. The results in [37] are not sharp, however, and some improvement could be achieved (in the algorithmic sense, too) if the answer to the next question were known.

*Problem* 2.3. Let $r \geq 3$ be a given integer. Find $c_r = \inf \{c|$ every $r$-uniform hypergraph on n vertices has $\leq O(c^n)$ transversals minimal under inclusion$\}$.

By its connection with perfect graphs, the most interesting particular case is $r = 3$, which does not seem to be very difficult. (For $r = 2$, Moon and Moser [21] proved that $c_2 = 3^{1/3}$.) It is worth mentioning that preparing the *list* of all transversals usually requires just $p(n)$ times more steps than the *number* of transversals (where $p(n)$ is a polynomial of the number $n$ of vertices), i.e. almost the whole difficulty is concentrated in estimating the number of minimal transversals.

During a discussion with V. Chvátal we found that the following «extremal» variant of Problem 2.2 may be of some interest.

*Problem* 2.4. How many induced odd cycles (or, induced cycles) can a graph on n vertices have?

Chvátal and I observed that the hypergraph method yields $O(c^n)$ for some $c < 2$ as an upper bound, and also that $3^{n/3}$ is a lower bound in some graphs. It may be the case that the maximum number of induced (odd) cycles grows with the powers of $c = 3^{1/3}$, but at the moment we do not have a proof for it.

So far we have considered upper bounds. Unfortunately, nothing essential is known from the other side.

*Problem* 2.5. Find non-trivial *lower* bounds for the perfect-graph recognition problem.

## 3. Neighborhood-perfect graphs.

The neighborhood number of a graph was first defined in [24]; we shall use the slightly different definition introduced in [16], however, because it leads to a more natural formulation of the problem in the present context.

The *neighborhood* $N(x)$ of a vertex $x$ in a graph $G = (V, E)$ is the set of *edges* $e \in E$ such that $x \in e$ or both vertices of $e$ are adjacent to $x$. The *neighborhood number* $\rho_N(G)$ is the smallest number of neighborhoods whose union is $E$; the *neighborhood-independence number* $\alpha_N(G)$ is the largest number of edges such that no pair of them belongs to the neighborhood of any vertex.

It follows immediately from the definitions that $\rho_N(G) \geq \alpha_N(G)$ holds for all graphs $G$. Call $G$ *neighborhood-perfect* if $\rho_N(G') = \alpha_N(G')$ in every induced subgraph $G'$ of $G$. The following problem was raised in [16].

*Problem* 3.1. Prove that every neighborhood-perfect graph is perfect.

A supporting evidence is provided by the Strong Perfect Graph Conjecture, because neither the odd cycles of lengths $\geq 5$ nor their complements are neighborhood-perfect.

A more general question is as follows.

*Problem* 3.2. Find a structural characterization of neighborhood-perfect graphs.

Certainly, this problem is interesting for some reasonably large subclasses of graphs as well. Neighborhood-perfect *chordal* graphs have been characterized in [16]; in particular, it follows that all 'strongly chordal' graphs (and, consequently, all interval graphs) are neighborhood-perfect.

From the algorithmic point of view, $\rho_N(G)$ and $\alpha_N(G)$ are $NP$-hard to find [5], but there are fast (linear-time) algorithms for them in interval graphs [16] and more generally in strongly chordal graphs [5].

It would also be worth investigating which of the other classes of graphs, not necessarily subclasses of perfect graphs, admit fast algorithms to find the neighborhood (independence) number.

## 4. Decompositions into perfect subgraphs.

Graph decompositions may be viewed as special types of colorings. From the rich literature we pick a particular problem, raised recently in [35], which has a more explicit relation with the previously discussed topics. (Some further questions concerning decompositions will be mentioned in later sections.)

A decomposition of $G$ consists of $t$ graphs $G_1 = (V_1, E_1), \dots, G_t = (V_t, E_t)$ (for some $t$), where each $G_i$ is a subgraph of $G$. We assume that $G_1 \cup \dots \cup G_t = G$, i.e. $V_1 \cup \dots \cup V_t = V$ and $E_1 \cup \dots \cup E_t = E$ hold. Various types of requirements can be considered, for example the following ones:

(1) Each $G_i$ is perfect.

(2) Each $G_i$ is perfect, and $E_i \cap E_j = \emptyset$ for $1 \le i < j \le t$.

(3) Each $V_i$ induces a perfect subgraph in $G$.

(4) Each $V_i$ induces a perfect subgraph in $G$, and $E_i \cap E_j = \emptyset$ for $1 \le i < j \le t$.

For $k = 1, \dots, 4$ and for a given graph $G$, denote by $f_k(G)$ the smallest natural number $t$ for which there are subgraphs $G_1, \dots, G_t$ of $G$ satisfying assumption ($k$). We put

$$f_k(n) = \max\{f_k(G) \mid |V(G)| = n\}.$$

*Problem* 4.1. Determine $f_k(n)$ for $1 \le k \le 4$.

One can also investigate the *weighted* version of the problem. For $k = 1, \dots, 4$, denote by $g_k(G)$ the minimum value of $|V_1| + \dots + |V_t|$ under the assumption that the $G_i$ satisfy $(k)$, and put

$$g_k(n) = \max\{g_k(G) \mid |V(G)| = n\}.$$

*Problem* 4.2. Determine $g_k(n)$ for $1 \le k \le 4$.

The estimates on $f_k(n)$ and $g_k(n)$ (proved in [35]) are not sharp, usually there is a factor of $\log n$ (or at least $\log\log n$) in the ratio of the upper and lower bounds. Those 8 functions mean 8 problems; among them the following one seems to be the most natural to ask.

*Problem* 4.3. Does every graph on $n$ vertices have an edge decomposition into $(c \log n)/\log\log n$ perfect subgraphs, for some constant $c$?

A probabilistic argument, also applying an extremal set-theoretic result of [31], yields that one cannot expect less than $(c' \log n)/\log\log n$ subgraphs in a decomposition, for some positive constant $c'$. Surprisingly, even though random graphs are applied, this lower bound is shown to be true for about the *half* of graphs only.

*Problem* 4.4. Is $f_1(G) \ge (c' \log n)/\log\log n$ for almost all graphs $G$ on $n$ vertices, as $n \to \infty$?

Also, it is not clear how much stronger the restrictions (2) and (4) are when compared to (1) and (3), respectively.

*Problem* 4.5. Give estimates on $h_{k+1}(G) - h_k(G)$ and $h_{k+1}(G)/h_k(G)$, in terms of $n := |V(G)|$, where $h \in \{f, g\}$ and $k = 1$ and 3. Can the difference tend to infinity (and how fast) if $n$ gets large? Is there a constant upper bound for their ratio?

## 5. Ramsey-type questions on local $k$-colorings.

An edge coloring of a (hyper)graph is a *local $k$-coloring* if for each vertex $x$, there are at most $k$ distinct colors occurring on the set of edges containing $x$. (When just $k$ colors are used in the coloring, we simply call it a $k$-coloring.) For $r$-uniform hypergraphs, and for $1 \leq j \leq r-1$, $j$*-local* $k$-colorings can be defined in a similar way; in this case the requirement is that at most $k$ colors should occur on the edges containing $J$, for each $j$-element set $J$ of vertices (i.e., «1-local» =«local».) Those concepts were introduced in the subsequent papers [14], [13], and [28].

Here we focus on the Ramsey-type problems involving local colorings. Let $G$ be a graph (or $r$-uniform hypergraph), $k$ and $j$ integers, $k \geq 2$, $r > j \geq 1$. Denote by $r(G,k)$ and $r(G,k,j\text{–loc})$ the Ramsey number and the $j$-local Ramsey number for $k$ colors, respectively, that is the smallest $n$ such that every $k$-coloring or $j$-local $k$-coloring of the complete graph (complete $r$-uniform hypergraph) on $n$ vertices contains a monochromatic sub(hyper)graph isomorphic to $G$.

*Problem* 5.1. Find necessary and/or sufficient conditions for $r(G,k) = r(G,k,j\text{–loc})$.

The case $j = 1$ has been studied to a certain extent. It was shown in [4] that equality holds when, instead of a single (hyper)graph $G$, the class of all $t$-colorable $r$-uniform hypergraphs is taken, for some $t$ and $r$. (Then both Ramsey numbers are $(r-1)t^k + 1$.) Moreover, a theorem of [14] states that in case of $r = k = 2$, equality holds for complete graphs of arbitrarily given size (and also for triangles when $k = 3$). For this reason it is natural to ask the following.

*Problem* 5.2. Is $r(G,k) = r(G,k,1\text{–loc})$ for all $k$ and for every complete ($r$-uniform hyper)graph $G$?

One can ask the same question for $j \geq 2$ as well; in this case, however, the structure of the colorings does not seem to be so strictly determined and perhaps some examples exist for which the local Ramsey number is larger than the Ramsey number. Similarly, we do not know whether or not $n > (r-1)t^k$ implies that every $j$-local $k$-coloring of the complete $r$-uniform hypergraph on $n$ vertices contains a monochromatic subhypergraph $H$ with $k(H) > t$ (for $1 < j < r$).

Another important question is to investigate to what extent local $k$-colorings allow larger structures than k-colorings.

*Problem* 5.3. Estimate $r(G,k,1-\mathrm{loc})/r(G,k)$ and $r(G,k,j-\mathrm{loc})/r(G,k,(j-1)-\mathrm{loc})$ for $1 < j < r$.

A result of [14] states that the first quotient can be arbitrarily large (even for $k$ fixed, say $k = 2$) if no further assumption is put on $G$. On the other hand, it is proved in [28] that for *connected* hypergraphs it remains below a constant that only depends on $k$.

*Problem* 5.4. Given $k \geq 2$, find the smallest constant $c_k$ such that $r(G,k,1\!-\!\mathrm{loc}) \leq c_k r(G,k)$ for every connected (hyper)graph $G$.

It is observed in [14] that $9/8 \leq c_2 \leq 3/2$ holds for *graphs*. Probably this particular case (i.e., $r = k = 2$) is the first one in which the supremum of the ratio of the two Ramsey numbers should be determined, possibly in the «asymptotic» sense, assuming that $|V(G)|$ tends to infinity. A general upper bound $c_k < k^k/k!$ is proved in [28].

*Problem* 5.5. Given $k \geq 2$, find the smallest integer $k'$ such that $r(G,k,1\!-\!\mathrm{loc}) \leq r(G,k')$ for every connected (hyper)graph $G$.

As proved in [28], $k' \leq c^k$ for some constant $c$. Moreover, it is mentioned in [13] (without proof) that $k' \leq k^2$ is valid in the restricted class of connected graphs having at least one triangle. This is the only case, however, in which the exponential upper bound is improved to a polynomial one.

In order to find a sufficient condition insuring bounded ratio of the Ramsey numbers, perhaps the following property will work (cf. [28]). Let $\mathcal{H}$ be an $r$-uniform hypergraph, and let $1 < j < r$. Define the *$j$-intersection graph* of $\mathcal{H}$ as the graph whose vertices are the edges of $\mathcal{H}$, two vertices being adjacent if and only if the corresponding edges of $\mathcal{H}$ share at least $j$ vertices. Call $\mathcal{H}$ *$j$-intersection connected* if its $j$-intersection graph is connected.

*Problem* 5.6. Does there exist a constant $c_k$ such that $r(G,k,j\!-\!\mathrm{loc}) \leq c_k r(G,k)$ for every $j$-intersection connected ($r$-uniform hyper)graph $G$?

The simplest unsolved case is $k = j = 2, r = 3$ (i.e. the class of 3-uniform hypergraphs). As shown in [28], this problem can be reduced to local 2-colorings with just 3 colors.

## 6. 3-chromatic uniform hypergraphs.

In contrast with graphs, it is a difficult problem to tell which hypergraphs are 2-colorable. There are several sufficient conditions to insure 2-colorability (also called «property B» in the literature), but the folowing nice problem is still open.

*Problem* 6.1. Let $4 \leq r \leq 7$. Is every $r$-regular $r$-uniform hypergraph 2-colorable?

Note the König-Hall theorem implies that if $\mathcal{H}$ is $r$-regular and $r$-uniform, then we can delete a vertex from each edge of $\mathcal{H}$ in such a way that the hypergraph obtained is $(r-1)$-regular and $(r-1)$-uniform. Hence, the simplest counterexample might be given for $r = 4$, or the 'easiest' proof of an affirmative answer can be expected for $r = 7$. The Fano-plane (as a 3-uniform hypergraph) is not 2-colorable, this is the reason why $r \geq 4$ is assumed.

Erdös and Lovász [9] proved by a probabilistic argument that every 3-chromatic hypergraph has some vertices of large degree. Along this line, a stronger variant of their «Local Lemma» implies that every 9-regular 9-uniform hypergraph is 2-colorable. The proof of Alon and Bergman [1] for $r = 8$ applies a different approach, using deep algebraic results (Van der Waerden's conjecture concerning the permanent of stochastic matrices and Hadamard's inequality on determinants). Note that 3-colorability has been proved for all $r \geq 3$ by Bollobás and Harris in [3], where further references and related problems can also be found. (The Local Lemma proves 3-colorability when $r \geq 5$.)

There are some further properties which are closely related to the assumption $\chi(\mathcal{H}) = 3$. From now on suppose that $\mathcal{H}$ is *intersecting*, i.e. $H \cap H' \neq \emptyset$ for $H, H' \in \mathcal{H}$. Then, for $r$-uniform $\mathcal{H}, \chi(\mathcal{H}) \geq 3$ implies $\tau(\mathcal{H}) \geq r$. (In fact, $\tau(\mathcal{H}) = r$ since $\mathcal{H}$ is assumed to be intersecting.) A slightly weaker property is when $\mathcal{H}$ is *intersection-critical* (frequently called *$\nu$-critical* in the literature) which means that for every $H \in \mathcal{H}$ and $x \in H$ there is a $H' \in \mathcal{H}$ such that $H \cap H' = \{x\}$. (In words, if we replace any edge by some of its proper subsets, then the hypergraph does not remain intersecting).

The study of the following three extremal problems was initiated in [9] and [18], respectively.

*Problem* 6.2. Determine the minimum number of edges in an intersecting $r$-uniform hypergraph $H$ with $\tau(\mathcal{H}) = r$.

*Problem* 6.3. Determine the maximum number of edges in an intersection-critical $r$-uniform hypergraph.

*Problem* 6.4. Determine the maximum number of non-isolated vertices in an intersection-critical $r$-uniform hypergraph.

For partial results concerning the minimum, see the paper of P. Erdös in this volume. (A basic question would be to decide if the minimum is $O(r)$ or it grows faster.)

For the maximum number $m(r)$ of edges, Erdos and Lovász [9] proved $(e - 1 - o(1))r! \leq m(r) \leq r^r$. Here the gap between the two estimates is a factor exponential in $r$. Recently we have improved the upper bound by a constant factor [36].

Much better estimates are known for the maximum number of *vertices*. The upper and lower bounds (in [29] and [36]) both are of the form $c\binom{2r}{r}$, but the value of $c$ is not the same in the two estimates. Hence, the next problem here would be to show that in fact the maximum is $(c + o(1))\binom{2r}{r}$ for some constant $c$. (From the results of [30] a sharper upper bound follows, but the method cannot reach the best current lower bound given in [29].) Note that the above estimates remain valid if, instead of $r$-uniform hypergraphs, we consider hypergraphs in which every edge has *at most* r vertices.

## 7. Hypergraphs with Helly property.

There are several interesting extremal problems on Helly-type hypergraphs, but here we mention just one of them which, in a sense, is the analogue of Kneser's conjecture.

Call $\mathcal{H}$ a *Helly-hypergraph* if it satisfies the 1-dimensional Helly property, i.e. if the edges of a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ have an empty intersection then $H'$ contains two disjoint edges.

*Problem* 7.1. Let $\mathcal{H}_i$ $(1 \le i \le t)$ be $r$-uniform Helly-hypergraphs on the same $n$-element vertex set $X$, $n \ge r \ge 3$, such that every $r$-element subset of $X$ is an edge of some $\mathcal{H}_i$. Is then $t \ge n - r + 1$?

It follows from the definition that the answer is affirmative for $n < 2r$.

A more general problem is to determine the largest possible number of edges in the union of $t$ Helly-hypergraphs having the same vertex set. References and related questions are discussed in [38].

## 8. Strong chromatic index of graphs.

Here we mention some open problems originated from the quadruple paper [11]. Call two edges $e, e'$ of a graph $G = (V, E)$ *strongly independent* if they are vertex-disjoint and their union $e \cup e'$ induces no further edges in $G$. Denote by $am(G)$ the maximum number of edges no two of which are strongly independent (= antimatching), by $sm(G)$ the maximum number of pairwise strongly independent edges (= strong matching), and by $sq(G)$ the minimum number of strong matchings whose union is $E$ (=the strong chromatic index of $G$).

The first question was raised by Erdös and Nesětřil (private communication), the second one was published in [11].

*Problem* 8.1. Is $sq(G) \le 5d^2/4$ in every graph $G$ of maximum degree $d$?

*Problem* 8.2. Is $sq(G) \le d^2$ in every *bipartite* graph $G$ of maximum degree $d$?

The answer to both questions is known to be affirmative in the case when $sm(G) = 1$, see [6]. (For $d$ odd, the upper bound of $5d^2/4$ can be improved to the sharp one $5d^2/4 - d/2 + 1/4$.)

Let $\mathcal{G}$ be a class of graphs, and denote by $f_{\mathcal{G}}(d)$ the maximum number of edges in a graph $G \in \mathcal{G}$ with $sm(G) = 1$ and maximum degree not exceeding $d$.

*Problem* 8.3. For which classes $\mathcal{G}$ does $am(G) \leq f_{\mathcal{G}}(d)$ hold for all $G \in \mathcal{G}$ such that $G$ has maximum degree at most $d$?

*Problem* 8.4. For which classes $\mathcal{G}$ does $|E(G)| \leq f_{\mathcal{G}}(d) sm(G)$ hold for all $G \in \mathcal{G}$ of maximum degree at most $d$?

It is conjectured in [11] that the above two properties hold when $\mathcal{G}$ is the class of all graphs. The corresponding statements were proved in [11] and [10], respectively, for the class of *bipartite* graphs. (As noted above, in the bipartite case $f_{\mathcal{G}}(d) = d^2$.) We mention that these problems are open even in the case $d = 3$, and also for some very simple subclasses of graphs. Several particular problems of this type are raised in [11].

## 9. Bipartite subgraphs of 4-chromatic transitive graphs.

In [22] the minimum number of edges to be deleted from a «large» $\chi$-critical graph $G$ when making $G$ bipartite is determined. (Critical means that $\chi(G - e) < \chi(G)$ for all edges $e \in E(G)$.) This minimum depends on $\chi(G)$ but is independent of $|V(G)|$ (for large $|V|$). On the other hand, if the automorphism group of $G = (V, E)$ acts transitively on $V$ (or on $E$), then for $\chi(G) \geq 5$ one has to delete at least $|V|^{1/2}$ vertices (or at least $|E|^{1/2}$ edges) in order to obtain a bipartite subgraph of $G$. Of course, such a statement is not true for 3-chromatic graphs in general, since the odd cycles are edge-transitive and $\chi$-critical. The situation in the case $\chi(G) = 4$, however, has not yet been investigated.

*Problem* 9.1. Find the largest integer $f(n)$ such that no vertex-transitive graph $G$ of $n$ vertices with $\chi(G) \geq 4$ contains an induced bipartite subgraph of more than $n - f(n)$ vertices.

*Problem* 9.2. Find the largest integer $g(m)$ such that no edge-transitive graph $G$ of $m$ edges with $\chi(G) \geq 4$ contains a bipartite subgraph of more than $m - g(m)$ edges.

Certainly, the first question to be settled is whether or not $f(n)$ and $g(m)$ tend to infinity with $n$ and $m$, respectively.

A more general problem is to find estimates on the largest number of vertices and edges in $q$-chromatic (induced or not necessarily induced) subgraphs of $p$-chromatic $\chi$-critical transitive graphs ($p > q \geq 2$). As shown in [22], the upper bounds of $|V| - |V|^{1/2}$ and $|E| - |E|^{1/2}$ hold whenever $p > 2q$ and $p > q^2$, respectively, but no estimate is known for smaller values of p. Moreover, it is not known how fast the number of vertices (edges) grows when $p$ is much larger than $2q$ or $q^2$. (Iterating the result of [22] one can obtain some improvement on the lower bound, but probably it is very far from being sharp.)

## 10. Representations.

The following nice combinatorial problem occurred in *algebraic logic*, in the study of representations of symmetric atomic relation algebras. Denote by $v_1, ..., v_n$ the vertices of the complete graph $K_n = (V_n, E_n)$, $n = 1, 2, ...$ Call a natural number $t$ *representable* if there is an edge coloring $f : E_n \to \{1, ..., t\} =: [t]$ for *some* $n$, with the following properties:

(i) $f^{-1}(j)$ is non-empty for all $j \in [t]$.

(ii) $K_n$ contains no monochromatic triangle.

(iii) For any *ordered* triplet $(a, b, c)$ of three integers $a, b, c \in [t]$ (two of which may coincide) and for any *ordered* pair $v_i, v_j \in V_n$ of vertices with $f(v_iv_j) = a$ there is a $v_k \in V_n$ such that $f(v_iv_k) = b$ and $f(v_jv_k) = c$.

*Problem* 10.1. Is every t representable?

The answer was shown to be affirmative for $t \leq 5$ by Comer [7]. Even if some counterexamples occur for some $t$ large, it remains an interesting problem to determine which $t$ *are* representable. For further results in this direction, concerning combinatorial (coloring-) characterizations of algebraic properties, see [32].

## 11. Arithmetic progressions.

The folowing question is the simplest unsolved case of a problem discussed in [2].

*Problem* 11.1. For a natural number $k$, determine the smallest integer $n = n(k)$ with the following property: If the integers $1, \ldots, n$ are colored in such a way that every color occurs at most $k$ times, then there is an arithmetic progression of 3 terms having pairwise distinct colors.

Our best general estimates are $2k < n(k) \leq (4.5 + 0(1))k$. Perhaps the truth is $n(k) = 2k + o(k)$, and most probably $n(k) \leq 3k$ holds for all $k \geq 1$. (For small values of $k$, $n(1) = 3$, $n(2) = 5$, $n(3) = 9$.)

A closely related question is as follows.

*Problem* 11.2. Find upper and lower bounds on $n(k+1) - n(k)$.

In particular, it would be interesting to prove (if true) that the above difference remains under a constant for all $k$.

There are several generalizations and related problems in this topic (for longer arithmetic progressions, for edge-colored complete graphs, etc.); some of them are considered in [2], some others are investigated in a joint work with Alon, Caro, and Rödl (in preparation).

## 12. Chromatic sum.

At the end of this paper we discuss some problems concerning a coloring concept introduced quite recently. In this context, a *coloring* (or $s$-coloring) of a graph $G = (V, E)$ is a mapping $f : V \to \{1, 2, \ldots, s\}$ for some natural number $s$, such that $f(x) \neq f(y)$ whenever $xy \in E$. The weight of a coloring is defined as $w(f) := \Sigma_{x \in V} f(x)$, and the *chromatic sum* $\Sigma(G)$ of $G$ is the smallest possible value of $w(f)$, taken over all colorings of $G$. A coloring $f$ is called *minimal* or *best* (in [33] and [15], respectively) if $w(f) = \Sigma(G)$. We define the *strength* $s(G)$ as the smallest integer $s$ for which $G$ has a minimal $s$-coloring.

Some estimates on $\Sigma(G)$, in terms of $|E|$, have been given in [25]. On the other hand, it was proved in [15] that finding $\Sigma(G)$ is an $NP$-complete problem. Hence, one cannot expect a good characterization of graphs with a given chromatic sum.

*Problem* 12.1. Is it $NP$-complete to determine $s(G)$?

*Problem* 12.2. Design fast algorithms that determine the chromatic sum and the strength of graphs belonging to some important particular classes.

It has been ovserved by Kubicka and Schwenk [15] that even some very simple graphs, such as trees, can have arbitrarily large strength. For every $s \geq 2$ they constructed a tree on $((2+\sqrt{2})^{s-1} - (2-\sqrt{2})^{s-1})/\sqrt{2}$ vertices with strength $s$, and I proved in [33] that their example is the unique smallest tree of given strength (this result is also claimed in [15] but the proof is incorrect there). As a matter of fact, this extremal result follows from a much stronger theorem which expresses a close relation between strength and edge-contractions: For every $s \geq 3$ there are two trees $T_s$ and $R_s$ such that every tree of strength $s$ can be contracted to $T_s$ or $R_s$.

These results show that the concept of chromatic sum leads to a rich area of current research, offering lots of challenging open problems. We first recall an unpublished extremal question of P. Erdös.

*Problem* 12.3. Determine the minimum number of vertices in a graph of given strength and given chromatic number.

Of course, the complete graph $K_n$ has strength $n$ (and its minimal coloring is unique). Hence, the relation between strength and the number of vertices can be linear and also exponential (as in case of trees). It would be interesting to draw the boundaries between «fast-growing» and «slow-growing» classes of graphs. More precisely, for a class $\mathcal{G}$ of graphs denote by $n_{\mathcal{G}}(s)$ the smallest number of vertices in a graph $G \in \mathcal{G}$ with $s(G) = s$.

*Problem* 12.4. For which integer-valued functions $h(x)$ does there exist a «nice» class $\mathcal{G}$ of graphs such that $h(s) = n_{\mathcal{G}}(s)$ for all $s$?

In particular, we would like to see some «separating» graph classes, i.e. largest or smallest classes in which $n_G(s)$ is a linear, polynomial (of given degree), or exponential function of $s$.

Another direction is to compare $s(G)$ and $\chi(G)$.

*Problem* 12.5. Characterize those graphs whose strength is equal to their chromatic number.

A generalization of this question, to characterize those graphs $G$ in which $\chi(G) - s(G)$ is equal to a given integer, seems to be hopelessly difficult. Perhaps the next two problems are easier.

*Problem* 12.6. Characterize those graphs $G$ in which $s(G') = \chi(G')$ for all induced subgraphs $G'$ of $G$.

*Problem* 12.7. Describe the structure of those graphs whose minimal coloring is unique.

We have already mentioned that sometimes edge-contractions are relevant with respect to strength. Below we list a collection of problems of this motivation, taken from [33].

*Problem* 12.8. Describe the basic structural properties of *strength-critical* graphs (= graphs $G$ in which $s(G - e) < s(G)$ for every edge $e$).

*Problem* 12.9. Describe the basic structural properties of *contraction-critical* graphs (= graphs $G$ in which the graph $G_e$ obtained after the contraction of the edge $e$ satisfies $s(G_e) < s(G)$ for every $e \in E(G)$).

*Problem* 12.10. Characterize those graphs which are strength-critical *and* contraction-critical.

*Problem* 12.11. Describe the basic structural properties of *contraction-minimal* graphs (= graphs $G$ in which $s(G_e) > s(G)$ for every edge $e$).

*Problem* 12.12. In which contraction-critical (contraction-minimal) graph $G = (V, E)$ is $G_e$ contraction-minimal (contraction-critical) for all $e \in E(G)$?

From this point of view, odd and even cycles provide a complementary pair of graph classes.

*Problem* 12.13. Do there exist «double-minimal» graphs (i.e. in which the contraction of any pair of edges increases the strength by 2)?

Similarly, one can ask which graphs are «double-critical» with respect to vertex deletion or edge contraction.

The strength and the chromatic sum do not increase when an edge is deleted from the graph. Call an $e \in E(G)$ *weak* if $\Sigma(G-e) = \Sigma(G)$, and call e *critical* if $s(G-e) < s(G)$.

*Problem* 12.14. When is a weak edge critical, and when is a critical edge weak?

Most of these questions seem to be interesting also for some (reasonably rich) particular classes of graphs. Further problems can be posed concerning graphs $G$ which are «vertex-strength-critical», i.e. when the deletion of any vertex decreases $s(G)$. (For example, study the connections among the classes of (vertex)-strength-critical, contraction-critical, and $\chi$-critical graphs.) Moreover, one can investigate the relation between strength and *subcontractions* of a graph (that are obtained by edge contraction and edge deletion).

## REFERENCES

[1] Alon N., Bergman Z., *Every 8-uniform 8-regular hypergraph is 2-colorable.* Graphs and Combinatorics **4** (1988) 303-305.

[2] Alon N., Caro Y. , Tuza Zs., *Sub-Ramsey numbers for arithmetic progressions.* Graphs and Combinatorics **5** (1989), 303-306.

[3] Bollobás B., Harris A. J., *List colourings of graphs.* Graphs and Combinaorics **1** (1985) 115-127.

[4] Caro Y., Tuza Zs., *Hypergraph coverings and local colorings.* J. Combin. Theory, Ser. B, to appear.

[5] Chang G.J., Farber M., Tuza Zs., *Algorithmic aspects of neighborhood numbers.* submitted.

[6] Chung F. R. K., Gyárfás A., Trotter W. T., Jr., Tuza Zs., *The maximum number of edges in $2K_2$-free graphs of bounded degree.* Discrete Math., **81** (1990), 129-135.

[7] Comer S.D., private communication.

[8] Dean N., Toft B., private communication.

[9] Erdös P., Lovász L., *Problems and results on 3-chromatic hypergraphs and some related questions.* in: Infinite and Finite Sets, Proc. Colloq. Math. Soc. János Bolyai, Keszthely (Hungary) 1973, North-Holland, pp. 609-627.

[10] Faudree R. J., Gyárfás A., Schelp R. H., Tuza Zs., *Induced matchings in bipartite graphs.* Discrete Math., **78** (1989) 83-87.

[11] Faudree R. J., Gyárfás A., Schelp R. H., Tuza Zs., *The strong chromatic index of graphs.* Ars Combinatoria **29**-B (1990) 205-211.

[12] Gallai T., *On directed paths and circuits.* in: Theory of Graphs (P. Erdös and G. O. H. Katona, Eds.), Academic Press, 1968, pp. 115-118.

[13] Gyárfás A., Lehel J., Nesetril J., Rödl V., Schelp R. H., Tuza Zs., *Local k-colorings of graphs and hypergraphs.* J. Combin. Theory, Ser. B, **43** (1987) 127-139.

[14] Gyárfás A., Lehel J., Schelp R. H., Tuza Zs., *Ramsey numbers for local colorings.* Graphs and Combinatorics **3** (1987) 267-277.

[15] Kubicka E., Schwenk A. J., *An introduction to chromatic sums.* in: Proc. ACM Computer Science Conference, Louisville (Kentucky) 1989, 39-45.

[16] Lehel J., Tuza Zs., *Neighborhood perfect graphs.* Discrete Math. **61** (1986) 93-101.

[17] Lovász L., *A characterization of perfect graphs.* J. Combinatorial Theory, Ser. B, **13** (1972) 95-98.

[18] Lovász L., *On minimax theorems of combinatorics.* Mat. Lapok **26** (1975) 209-264. (in Hungarian)

[19] Maddux R., *Topics in Relation Algebras.* Ph.D. Thesis, University of California, Berkeley, 1978.

[20] Minty G. J., *A theorem on n-colouring the points of a linear graph.* Amer. Math. Monthly **67** (1962) 623-624.

[21] Moon J. W., Moser L., *On cliques in graphs.* Israel J. Math. **3** (1965) 23-28.

[22] Rödl V. and Tuza Zs., *On color critical graphs.* J. Combin. Theory, Ser. B, **38** (1985) 204-213.

[23] Roy R., *Nombre chromatiqœ et plus longs chemins d'un graphe.* Revue AFIRO **1** (1967) 127-132.

[24] Sampathkumar E., Neeralagi P.S. *The neighbourhood number of a graph.* Indian J. Pure Appl. Math. **16** (1985) 126-132.

[25] Thomassen C., Erdös P., Alavi Y., Malde P. J., Schwenk A. J., *Tight bounds on the chromatic sum of a connected graph.* J. Graph Theory **13** (1989) 353-357.

[26] Toft B., *Graph colouring problems.* Preprints 1987, nr. 2, Institut for

Matematik og Datalogi, Odense Universitet.

[27] Toft B., *75 graph colouring problems.* in: Graph Colouring (R. Nelson and R. J. Wilson, Eds.), Pitman Research Notes 219, Longman, 1990, pp. 9-35.

[28] Truszczyński M., and Tuza Zs., *Linear upper bounds for local Ramsey numbers.* Graphs and Combinatorics **3** (1987) 67-73.

[29] Tuza Zs., *Critical hypergraphs and intersecting set-pair systems.* J. Combin. Theory, Ser. B, **39** (1985) 134-145.

[30] Tuza Zs., *Minimum number of elements representing a set system of given rank.* J. Combin. Theory, Ser. A, **52** (1989) 84-89.

[31] Tuza Zs., *Intersection properties and extremal problems for set systems.* in: Irregularities of Partitions (G. Halász and V.T. Sós, Eds.), Proc. Colloq. Math. Soc. János Bolyai, Fertöd (Hungary) 1986, Springer-Verlag, 1989, pp. 141-151.

[32] Tuza Zs., *Representations of relation algebras and patterns of colored triplets.* in: Algebraic Logic (H. Andréka, J.D. Monk, and I. Németi, Eds.), Proc. Colloq. Math. Soc. János Bolyai, Budapest (Hungary) 1988, North-Holland, in print.

[33] Tuza Zs., *Contractions and minimal k-colorability.* Graphs and Combinatorics, **6** (1990) 51-59.

[34] Tuza Zs., *Graph coloring in linear time.* J. Combin. Theory, Ser. B, to appear.

[35] Tuza Zs., *Perfect graph decompositions.* Graphs and Combinatorics **7** (1991) 89-93.

[36] Tuza Zs., *Inequalities for minimal covering sets in set systems of given rank.* in preparation.

[37] Tuza Zs., *Perfect-graph recognition.* in preparation.

[38] Tuza Zs., *Helly property in finite set systems.* J. Combin. Theory, Ser. A, to appear.

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