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THE REPRESENTATION OF PREMODAL ALGEBRAS

GIULIA SIMI (Siena) (*)

"Premodal structures" and "premodal algebras", introduced in [5], are the result of an act of abstraction, just as monadic algebras, taken as starting point in [5], are abstracted from functional monadic algebras introduced by P.R. Halmos [1].

Subsequently, Halmos shows that the relevant abstraction is, so to speak, faithful in the sense that every monadic algebra is isomorphic to the abstract of a suitable functional monadic algebra. Here is proved, in strict comparison, that every premodal algebra is isomorphic to the abstract of a suitable concrete premodal algebra.

1. Some previous results.

Let us recall the following definitions introduced in [5].

DEFINITION 1. *A premodal structure is a system* $M = (A, C, f, a)$ *such that:*

(1,0) (A, C) *is a monadic algebra in the sense of Halmos*

(1,1) f *is a Boolean homomorphism from* $C(A)$ *to* A

(1,2) a *is a constant in the sense of Halmos*

(1,3) $afp = p$ $\qquad (p \in C(A))$

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It is often useful to write, instead of f, $g = fC$.
Then we have:

DEFINITION 2. *A premodal algebra is a system* $A = (A, C, g, a)$ *such that:*

(2,0) (A, C) *is a monadic algebra in the sense of Halmos*

(2,1) g *is a hemimorphism from the Boolean algebra into itself*

(2,2) $gvCp = vgp$ *(v is the Boolean complementation)*

(2,3) a *is a constant in the sense of Halmos*

(2,4) $agp = Cp$ $\qquad (p \in A)$

In [5] we have shown what follows:

THEOREM 3. *The positions* $g = fC$, $f = g_{|C(A)}$ *give two bijective functions inverse one to the other, between the two classes of structures.*

2. Representation.

The following definition is a consequence of the ideas exposed in the introduction and in section 1.

DEFINITION 4. *A concrete premodal algebra (B-valued with domain X) is a triple* (A, f, a) *such that:*

(4,0) A *is subalgebra of* B^X *(B is a Boolean algebra and X is a set) such that for every element* p *of* A *the mapping* $\exists p$*, defined by* $(\exists p)x = V\mathbf{a}\,p$,

$$(\mathbf{a}\,p = \{px : x \in X\})$$

belongs to A.

(4,1) f *is a homomorphism from* B *to* A

(4,2) $a \in X$

(4,3) $(fq)a = q$ $\qquad (q \in B)$

DEFINITION 5. *The abstract of a concrete premodal algebra* (A, f, a) *is the system* $(\bar{A}, C, g, \bar{a})$ *such that:*

(5,0) $\bar{A} = A$

(5,1) $(\bar{a}p)x = pa$

(5,2) $(Cp)x = V\mathbf{a}\,p$ $\qquad (p \in A, x \in X)$

(5,3) $gp = f(V\mathbf{a}\,p)$

Then we have:

LEMMA 6. *The abstract of a concrete premodal algebra is a premodal algebra.*

Proof. Let $(\bar{A}, C, g, \bar{a})$ be the abstract of a concrete premodal algebra (A, f, a).

It is to be proved that $(\bar{A}, C, g, \bar{a})$ is a premodal algebra.

(2,0) is trivial.

For $p, q \in A$ we have:

$$g0 = f(V\mathbf{a}\,0) = f0 = 0$$

$$g(p+q) = f(V\mathbf{a}\,(p+q)) = f(V\mathbf{a}\,p + V\mathbf{a}\,q) = fV\mathbf{a}\,p + fV\mathbf{a}\,q = gp + gq$$

so that g is a hemimorphism.

Then we have for $p \in \bar{A}$:

$$gvCp = f(V\mathbf{a}\,vCp) = f(vV\mathbf{a}\,p) = vf(V\mathbf{a}\,p) = vgp.$$

i.e. (2,2) holds.

For p, $q \in \bar{A}$, we have

$$\bar{a}(p+q)x = (p+q)a = pa + qa = (\bar{a}p)x + (\bar{a}q)x$$

$$\bar{a}(vp)x = (vp)a = v(pa) = v(\bar{a}p)x.$$

So that $\bar{a}$ is an endomorphism of A.

For $p \in \bar{A}$:

$$(\bar{a}CP)x = (Cp)a = V\mathbf{a}\,p = (Cp)x$$

$$(C\bar{a}p)x = V\mathbf{a}\,\bar{a}p = pa = (\bar{a}p)x$$

therefore $\bar{a}$ is a constant in the sense of Halmos.

Finally, for $p \in \bar{A}$

$$(\bar{a}gp)x = (\bar{a}(fV\mathbf{a}\,p)x = (f(V\mathbf{a}\,p))a = (\text{for } (4,3)) = V\mathbf{a}\,p = (Cp)x$$

i.e. (2,4) holds.

Now we prove that every premodal algebra is isomorphic to the abstract of a suitable concrete premodal algebra.

Let us recall the proof of the representation theorem for monadic algebra (see Halmos [1]).

It is useful to give the followings:

DEFINITION 6. *A rich algebra is a monadic algebra* (A, C) *where for each element* p *of* A *there exists at least one costant a (in the sense of Halmos) such that*

$$ap = Cp.$$

The following holds:

THEOREM 7. *Every monadic algebra is a subalgebra of a rich algebra.*

We have:

THEOREM 8. (Halmos Representation Theorem).

If (A, C) is a monadic algebra, then there exist a set X and a Boolean algebra B such that:

(8,1) (A, C) is isomorphic to a B-valued functional algebra $\bar{A}$ with domain X.

(8,2) for every element $\bar{p}$ of $\bar{A}$ there exists a point x in X with $\bar{p}x = \exists\bar{p}x$.

Proof. The conclusions of the theorem are such that if they are valid for an algebra, then they are automatically valid for all its subalgebras. It follows from this comment and from theorem 7 that there is no loss of generality in assuming that A is rich.

Let X be a set of constants of A such that for every $p \in A$ there exists at least one $x_p \in X$ with $x_p p = Cp$.

Let the Boolean algebra B be the range $C(A)$ of the quantifier C on A; Define a mapping h from A into B^X such that:

$$(hp)x = \bar{p}x = xp.$$

Since $Cxp = xp$, there follows $hp = \bar{p} \in B^X$ for every $p \in A$.

Since each x in X is a Boolean endomorphism on A and the Boolean operations in B^X are defined pointwise, a routine verification shows that h is a Boolean homomorphism.

If $hp = 0$, i.e., if $\bar{p}x = 0$ for all x in X, then, in particular, $Cp = x_p p = \bar{p}x_p = 0$, and therefore $p = 0$; this proves that the homomorphism h is one-one.

Let $\bar{A}$ be the range of h, so that $\bar{A}$ is Boolean subalgebra of B^X; it is to be proved that $\bar{A}$ is a functional monadic algebra and h is a monadic isomorphism between A and $\bar{A}$.

If $\bar{p} = hp \in \bar{A}$, the range $\mathfrak{a}\bar{p} = \{\bar{p}x : x \in X\}$ contains, in particular, the element $Cp = x_p p = \bar{p}x_p$ of B; since $xp \leq Cp(p \leq Cp \Rightarrow xp \leq xCp = Cp)$ for every constant x, it follows that $\mathfrak{a}\bar{p}$ has a largest element, namely Cp. This proves that $\exists\bar{p}$ exists and has the value Cp at each x in X.

On the other hand, $(hCp)x = xCp = Cp$ for all $x \in X$ so that $\exists hp = \exists\bar{p} = hCp$.

Finally, we can prove our main result.

THEOREM 9. *Every premodal algebra is isomorphic to the abstract of a suitable concrete premodal algebra.*

Proof. Let (A, C, g, a) be a premodal algebra.

Let the Boolean algebra B be the range $C(A)$ of the quantifier C on A; let X be a set of constants of A in the sense of Halmos.

By the representation theorem for monadic algebras it follows that the mapping h from A into B^X defined by $\bar{p} = hp$, where $\bar{p}x = xp$ $(x \in X)$ is a monadic monomorphism.

Let $\bar{A} = h(A)$;

Let $\bar{a}$ be the mapping from the Boolean algebra $\bar{A}$ into B^X defined by

$$(\bar{a}\bar{p})x = \bar{p}a \qquad (\bar{p} \in \bar{A},\ x \in X)$$

We show that $\bar{a}(hp) = h(ap)$: for each $p \in A$ and $x \in X$ we have

$$(\bar{a}(hp))x = (hp)a = ap = Cap = xCap = xap = (h(ap))x;$$

therefore $\bar{a}$ is a constant of $\bar{A}$ in the sense of Halmos.

Let $f = g_{|C(A)}$ and let $\bar{f}$ be the mapping from B to $\bar{A}$ defined by: $\bar{f}p = hfp$. $\bar{f}$ is clearly a homomorphism and $(\bar{f}p)a = (hfp)a = afp = p$ for all $p \in B$.

Now let us consider the following mapping $\bar{g}$ from $\bar{A}$ into itself defined by: $\bar{g}\bar{p} = \bar{f}(V\mathfrak{a}\bar{p})$.

It still remains to be shown that $\bar{g}hp = hgp$.

We have

$$\bar{g}hp = \bar{f}(V\mathbf{a}\,\bar{p}) = \bar{f}Cp = hfCp = hgp$$

for every $p \in A$.

We proved that premodal algebra (A, C, g, a) is isomorphic to $(A, \exists, \bar{g}, \bar{a})$, where $(\bar{A}, \exists, \bar{g}, \bar{a})$ is the abstract of the concrete premodal algebra $(A, \bar{f}, a)$.

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Dipartimento di Matematica
Via del Capitano, 15
53100 Siena