REMARKS ON $K$-MIDCONVEX SET-VALUED FUNCTIONS WITH CLOSED EPIGRAPH

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In this note we present some continuity results on $K$-midconvex set-valued functions. In particular, we give conditions under which $K$-midconvex set-valued functions with closed epigraph are $K$-continuous.

Let $X$ and $Y$ be real topological vector spaces (satisfying the $T_0$ separation axiom). Assume that $D$ is a non-empty convex subset of $X$ and $K$ is a convex cone in $Y$. A set-valued function $F : D \to 2^Y$ is called $K$-midconvex if

$$
\frac{1}{2}[F(x) + F(y)] \subset F\left(\frac{x+y}{2}\right) + K \text{ for all } x, y \in D.
$$

Equivalently, $F$ is $K$-midconvex if its epigraph, i.e. the set

$$
epi F := \{(x, y) \in D \times Y : y \in F(x) + K\},$$

is a midpoint convex subset of $X \times Y$. Similarly, $F$ is said to be $K$-convex, if its epigraph is convex. We say that $F$ is $K$-continuous at a point $x_0 \in D$ if for every neighbourhood $W$ of zero in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that

$$
(1) \quad F(x) \subset F(x_0) + W + K
$$

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and

\[(2) \quad F(x_0) \subseteq F(x) + W + K\]

for all \(x \in (x_0 + U) \cap D\). If only condition (1) (condition (2)) is fulfilled, we say that \(F\) is \(K\)-upper semicontinuous (\(K\)-lower semicontinuous) at \(x_0\). For single-valued functions the \(K\)-continuity coincides with continuity whenever \(K\) is a normal cone.

Denote by \(B(Y)\) the family of all non-empty bounded subsets of \(Y\) and by \(\mathbb{N}\) the set of all positive integers.

The main result of this note is the following

**THEOREM 1.** Let \(X\) be a real topological vector Baire space, \(D - a\) non-empty convex open subset of \(X\), \(Y - a\) real topological vector space and \(K - a\) convex cone in \(Y\). Moreover, assume that there exist compact sets \(B_n \subseteq Y, \ n \in \mathbb{N}\), such that

\[(3) \quad \bigcup_{n \in \mathbb{N}} (B_n - K) = Y.\]

If a set-valued function \(F : D \to B(Y)\) is \(K\)-midconvex and its epigraph is closed in \(D \times Y\), then \(F\) is \(K\)-continuous on \(D\).

**Remarks.** The assumption (3) is trivially satisfied if \(Y\) is a locally compact space (and \(K\) is any convex cone). It is also fulfilled if there exists an order unit in the space \(Y\), i.e. such an element \(e \in Y\) that for every \(y \in Y\) we can find an \(n \in \mathbb{N}\) with \(y \in ne - K\) (we put then \(B_n := \{ne\}\)). In particular, if \(int K \neq \emptyset\), then every element of \(int K\) is an order unit in \(Y\). Theorem 1 in the case of single-valued functions extends a closed epigraph theorem proved by R. Ger [4, Th. 1] for midconvex operators. In the case where \(K = \{0\}\) the above theorem gives conditions under which midconvex set-valued functions with closed graph are continuous (in Hausdorff sense). This result crosses with the closed graph theorems due to Borwein [3, Proposition 2.3], Ricceri [8, Th. 2] and Robinson-Ursescu (cf. [1, Th. 1, 3, Chpt. 1]).

**Proof of Theorem 1.** Consider the sets

\[A_n := \{x \in D : F(x) \cap (B_n - K) \neq \emptyset\}, \ n \in \mathbb{N}.\]

We will show that these sets are closed in \(D\). To this end fix an \(n \in \mathbb{N}\) and take a point \(x \in D \setminus A_n\). Then

\[F(x) \cap (B_n - K) = \emptyset,\]
whence

\[(\{x\} \times B_n) \cap epi F = \emptyset.\]

Since the set \(\{x\} \times B_n\) is compact and the epigraph of \(F\) is closed, there exist a neighbourhood \(U\) of zero in \(X\) and a neighbourhood \(V\) of zero in \(Y\) such that

\[(((\{x\} \times B_n) + (U \times V)) \cap epi F = \emptyset.\]

In particular, for every \(u \in U_x := (x + U) \cap D\) we have

\[(\{u\} \times B_n) \cap epi F = \emptyset.\]

Hence

\[F(u) \cap (B_n - K) = \emptyset, \quad u \in U_x.\]

This implies that \(U_x \subset D \setminus A_n\) and proves that \(A_n\) is closed in \(D\).

Now observe that in view of (3) we have

\[\bigcup_{n \in \mathbb{N}} A_n = D.\]

The set \(D\), as a non-empty open subset of the Baire space \(X\), is a Baire space. Therefore there exists an \(n \in \mathbb{N}\) such that

\[\text{int}_D \text{cl}_D A_n \neq \emptyset,\]

where \(\text{int}_D\) and \(\text{cl}_D\) denote the relative interior and closure in \(D\), respectively. Since \(A_n\) is closed in \(D\) and \(D\) is open, we infer that \(\text{int}_n \neq \emptyset\). By the definition of \(A_n\), \(F\) is weakly \(K\)-upper bounded on \(A_n\). Thus \(F\) is a \(K\)-midconvex set-valued function weakly \(K\)-upper bounded on a set with a non-empty interior. This implies that \(F\) is \(K\)-continuous on \(D\) (cf. [6, Corollary 3.3]).

In [8] (cf. also [7]) B. Ricceri proved that if a midconvex set-valued function is closed-valued and lower semicontinuous, then its graph is closed. The following theorem is an analogy of that result for \(K\)-midconvex set-valued functions.

**THEOREM 2.** Let \(X\) and \(Y\) be real topological vector spaces, \(D - a\) non-empty convex open subset of \(X\) and \(K\) - a convex cone in \(Y\). Assume that \(F : D \to B(Y)\) is a \(K\)-midconvex set-valued function and
$F(x) + K$ is closed for every $x \in D$. If $F$ is $K$-lower semicontinuous at a point of $D$, then the epigraph of $F$ is closed in $D \times Y$.

**Proof.** Fix a point $(x_0, y_0) \in (D \times Y) \setminus epi F$. Then $y_0 \notin F(x_0) + K$. Since the set $F(x_0) + K$ is closed, there exists a neighbourhood $W$ of zero in $Y$ such that

$$(y_0 + W) \cap (F(x_0) + K) = \emptyset.$$  

Take a symmetric neighbourhood $V$ of zero in $Y$ satisfying $V + V \subseteq W$. Since $F$ is bounded-valued and $K$-lower semicontinuous at a point of $D$, it is $K$-continuous on $D$ (cf. [6, Th. 3.2]). In particular, there exists a neighbourhood $U$ of zero in $X$ such that $x_0 + U \subseteq D$ and

$$F(x) \subseteq F(x_0) + V + K$$

for all $x \in x_0 + U$. Hence, using (4), we get

$$((x_0 + U) \times (y_0 + V)) \cap epi F = \emptyset,$$

which shows that $epi F$ is closed in $D \times Y$.

**Remark.** J. Gwinner has proved (cf. [5, Lemma 2.2]) that Hausdorff upper semicontinuous set-valued functions with closed values have closed graph (cf. also [9, Remark 3.1]). We can give another proof of Theorem 2 applying this result to the set-valued function $F + K$.

Similarly we may use the result of Ricceri mentioned above.

Finally we shall present an application of Theorem 1. Let $D$ be a set, $Y$ a vector space and $K$ a convex cone in $Y$. We say that a set-valued function $H : D \to 2^Y$ supports a set-valued function $F : D \to 2^Y$ at a point $x_0 \in D$ if $H(x_0) \subseteq F(x_0) + K$ and $F(x) \subseteq H(x) + K$ for all $x \in D$.

The following theorem is a generalization of a standard result for real-valued functions.

**THEOREM 3.** Let $X, D, Y$ and $K$ be as in Theorem 1. If a set-valued function $F : D \to B(Y)$ has at every point of $D$ a $K$-midconvex support with closed epigraph, then $F$ is $K$-convex and $K$-continuous on $D$.

**Proof.** Fix points $x, y \in D$ and take a $K$-midconvex set-valued
function $H$ supporting $F$ at $(x + y)/2$. Then

$$
\frac{1}{2} [F(x) + F(y)] \subset \frac{1}{2} [H(x) + K + H(y) + K] \subset \\
\subset H \left( \frac{x + y}{2} \right) + K \subset F \left( \frac{x + y}{2} \right) + K,
$$

which shows that $F$ is $K$-midconvex. Now observe that

$$
epi F = \bigcap_{z \in D} \text{epi} H_z,
$$

where $H_z$ denotes a $K$-midconvex support of $F$ at $z \in D$ with closed epigraph. Consequently, $\text{epi} F$ is closed in $D \times Y$. Since $F$ is $K$-midconvex, this implies that it is $K$-convex (cf. Borwein [2, Prop. 1.13]). Finally, by Theorem 1, we get that $F$ is $K$-continuous on $D$.

REFERENCE3


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