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REMARKS ON K-MIDCONVEX SET-VALUED FUNCTIONS WITH CLOSED EPIGRAPH

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In this note we present some continuity results on K-midconvex set-valued functions. In particular, we give conditions under which K-midconvex set-valued functions with closed epigraph are K-continuous.

Let X and Y be real topological vector spaces (satisfying the T_0 separation axiom). Assume that D is a non-empty convex subset of X and K is a convex cone in Y. A set-valued function $F : D \to 2^Y$ is called K-midconvex if

$$\frac{1}{2}[F(x) + F(y)] \subset F\left(\frac{x+y}{2}\right) + K \text{ for all } x, y \in D.$$

Equivalently, F is K-midconvex if its epigraph, i.e. the set

$$epiF := \{(x,y) \in D \times Y : y \in F(x) + K\},$$

is a midpoint convex subset of $X \times Y$. Similarly, F is said to be K-convex, if its epigraph is convex. We say that F is K-continuous at a point $x_0 \in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

(1) $$F(x) \subset F(x_0) + W + K$$

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and

$$F(x_0) \subset F(x) + W + K \tag{2}$$

for all $x \in (x_0 + U) \cap D$. If only condition (1) (condition (2)) is fulfilled, we say that F is K-upper semicontinuous (K-lower semicontinuous) at x_0. For single-valued functions the K-continuity coincides with continuity whenever K is a normal cone.

Denote by $B(Y)$ the family of all non-empty bounded subsets of Y and by $\mathbb{N}$ the set of all positive integers.

The main result of this note is the following

THEOREM 1. *Let X be a real topological vector Baire space, D – a non-empty convex open subset of X, Y – a real topological vector space and K – a convex cone in Y. Moreover, assume that there exist compact sets $B_n \subset Y$, $n \in \mathbb{N}$, such that*

$$\bigcup_{n\in\mathbb{N}} (B_n - K) = Y. \tag{3}$$

If a set-valued function $F : D \to B(Y)$ is K-midconvex and its epigraph is closed in $D \times Y$, then F is K-continuous on D.

Remarks. The assumption (3) is trivially satisfied if Y is a locally compact space (and K is any convex cone). It is also fulfilled if there exists an order unit in the space Y, i.e. such an element $e \in Y$ that for every $y \in Y$ we can find an $n \in \mathbb{N}$ with $y \in ne - K$ (we put then $B_n := \{ne\}$). In particular, if $intK \neq \emptyset$, then every element of $intK$ is an order unit in Y. Theorem 1 in the case of single-valued functions extends a closed epigraph theorem proved by R. Ger [4, Th. 1] for midconvex operators. In the case where $K = \{0\}$ the above theorem gives conditions under which midconvex set-valued functions with closed graph are continuous (in Hausdorff sense). This result crosses with the closed graph theorems due to Borwein [3, Proposition 2.3], Ricceri [8, Th. 2] and Robinson-Ursescu (cf. [1, Th. 1, 3, Chpt. 1]).

Proof of Theorem 1. Consider the sets

$$A_n := \{x \in D : F(x) \cap (B_n - K) \neq \emptyset\},\ n \in \mathbb{N}.$$

We will show that these sets are closed in D. To this end fix an $n \in \mathbb{N}$ and take a point $x \in D \setminus A_n$. Then

$$F(x) \cap (B_n - K) = \emptyset,$$

whence

$$(\{x\} \times B_n) \cap epiF = \emptyset.$$

Since the set $\{x\} \times B_n$ is compact and the epigraph of F is closed, there exist a neighbourhood U of zero in X and a neighbourhood V of zero in Y such that

$$((\{x\} \times B_n) + (U \times V)) \cap epiF = \emptyset.$$

In particular, for every $u \in U_x := (x + U) \cap D$ we have

$$(\{u\} \times B_n) \cap epiF = \emptyset.$$

Hence

$$F(u) \cap (B_n - K) = \emptyset, \; u \in U_x.$$

This implies that $U_x \subset D \backslash A_n$ and proves that A_n is closed in D. Now observe that in view of (3) we have

$$\bigcup_{n \in \mathbb{N}} A_n = D.$$

The set D, as a non-empty open subset of the Baire space X, is a Baire space. Therefore there exists an $n \in \mathbb{N}$ such that

$$int_D cl_D A_n \neq \emptyset,$$

where int_D and cl_D denote the relative interior an closure in D, respectively. Since A_n is closed in D and D is open, we infer that $intA_n \neq \emptyset$. By the definition of A_n, F is weakly K-upper bounded on A_n. Thus F is a K-midconvex set-valued function weakly K-upper bounded on a set with a non-empty interior. This implies that F is K-continuous on D (cf. [6, Corollary 3.3]).

In [8] (cf. also [7]) B. Ricceri proved that if a midconvex set-valued function is closed-valued and lower semicontinuous, then its graph is closed. The following theorem is an analogy of that result for K-midconvex set-valued functions.

THEOREM 2. *Let X and Y be real topological vector spaces, D – a non-empty convex open subset of X and K – a convex cone in Y. Assume that $F : D \rightarrow B(Y)$ is a K-midconvex set-valued function and*

$F(x)+K$ is closed for every $x \in D$. If F is K-lower semicontinuous at a point of D, then the epigraph of F is closed in $D \times Y$.

Proof. Fix a point $(x_0, y_0) \in (D \times Y) \backslash epiF$. Then $y_0 \notin F(x_0)+K$. Since the set $F(x_0)+K$ is closed, there exists a neighbourhood W of zero in Y such that

$$(y_0+W) \cap (F(x_0)+K) = \emptyset. \tag{4}$$

Take a symmetric neighbourhood V of zero in Y satisfying $V+V \subset W$. Since F is bounded-valued and K-lower semicontinuous at a point of D, it is K-continuous on D- (cf. [6, Th. 3.2]). In particular, there exists a neighbourhood U of zero in X such that $x_0+U \subset D$ and

$$F(x) \subset F(x_0)+V+K$$

for all $x \in x_0+U$. Hence, using (4), we get

$$((x_0+U) \times (y_0+V)) \cap epiF = \emptyset,$$

which shows that $epiF$ is closed in $D \times Y$.

Remark. J. Gwinner has proved (cf. [5, Lemma 2.2]) that Hausdorff upper semicontinuous set-valued functions with closed values have closed graph (cf. also [9, Remark 3.1]). We can give another proof of Theorem 2 applying this result to the set-valued function $F+K$.

Similarly we may use the result of Ricceri mentioned above.

Finally we shall present an application of Theorem 1. Let D be a set, Y – a vector space and K – a convex cone in Y. We say that a set-valued function $H: D \to 2^Y$ supports a set-valued function $F: D \to 2^Y$ at a point $x_0 \in D$ if $H(x_0) \subset F(x_0)+K$ and $F(x) \subset H(x)+K$ for all $x \in D$.

The following theorem is a generalization of a standard result for real-valued functions.

THEOREM 3. *Let X, D, Y and K be as in Theorem 1. If a set-valued function $F: D \to B(Y)$ has at every point of D a K-midconvex support with closed epigraph, then F is K-convex and K-continuous on D.*

Proof. Fix points $x, y \in D$ and take a K-midconvex set-valued function H supporting F at $(x+y)/2$. Then

$$\frac{1}{2}[F(x)+F(y)] \subset \frac{1}{2}[H(x)+K+H(y)+K] \subset$$
$$\subset H\left(\frac{x+y}{2}\right)+K \subset F\left(\frac{x+y}{2}\right)+K,$$

which shows that F is K-midconvex. Now observe that

$$epiF = \bigcap_{z\in D} epiH_z,$$

where H_z denotes a K-midconvex support of F at $z \in D$ with closed epigraph. Consequently, $epiF$ is closed in $D \times Y$. Since F is K-midconvex, this implies that it is K-convex (cf. Borwein [2, Prop. 1.13]). Finally, by Theorem 1, we get that F is K-continuous on D.

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