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CONTROLLABILITY OF PARTIAL DIFFERENTIAL INCLUSIONS DEPENDING ON A PARAMETER AND DISTRIBUTED PARAMETER CONTROL PROCESSES

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Let Λ be a topological space and let F be a multifunction from $[0, a] \times [0, b] \times (\mathbb{R}^n)^4 \times \Lambda$ into $\mathbb{R}^n$. In this paper we prove that, under suitable assumptions, the set of all $\lambda \in \Lambda$ such that the partial differential inclusion

$$z_{xy} \in F(x, y, z, z_x, z_y, z_{xy}, \lambda)$$

is locally controllable around the origin of $\mathbb{R}^n$ at the point (a, b) (resp. $\mathbb{R}^n$-completely controllable) is open in Λ. Next, we present an application to the study of two kinds of controllability for the distributed parameter control process

$$z_{xy} = A(x, y)z + B(x, y)z_x + C(x, y)z_y + G(x, y, u(x, y)).$$

Introduction.

Throughout this paper a, b are two positive real numbers; Q is the rectangle $[0, a] \times [0, b]$ with the Lebesgue σ-algebra; Λ is a first countable topological space; m, n are two positive integers; $\mathbb{R}^n$ is the

(*) Entrato in redazione il 2 ottobre 1990 real Euclidean n-space, whose null element is denoted by $\vartheta_{\mathbb{R}^n}$; F is a multifunction from $Q \times (\mathbb{R}^n)^4 \times \Lambda$ into $\mathbb{R}^n$, with non-empty closed values; $p \in [1, +\infty[$.

If I is a compact real interval, we denote by $AC_p(I, \mathbb{R}^n)$ the space of all absolutely continuous functions $\varphi : I \to \mathbb{R}^n$ such that $\frac{d\varphi}{dt} \in L^p(I, \mathbb{R}^n)$. Given $f \in AC_p([0,a], \mathbb{R}^n)$, $g \in AC_p([0,b], \mathbb{R}^n)$, with $f(0) = g(0)$, and $\lambda \in \Lambda$, denote by $\Gamma_p(f, g, \lambda)$ the set of all generalized solutions (in the sense of [9], p. 282) of the problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial x \partial y} \in F\left(x, y, z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x \partial y}, \lambda\right) \\ z(x, 0) = f(x) \\ z(0, y) = g(y). \end{cases}$$

Moreover, put:

$$\mathcal{A}((a,b); (f,g); \lambda) = \{z(a,b) : z \in \Gamma_p(f, g, \lambda)\}.$$

If, for fixed $\lambda \in \Lambda$,

$$\vartheta_{\mathbb{R}^n} \in \operatorname{int}(\mathcal{A}((a,b); (\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n}); \lambda))$$

then, like the case of ordinary differential inclusions (see for instance [1] and [5]), we say that the partial differential inclusion

$$(I_\lambda) \qquad \frac{\partial^2 z}{\partial x \partial y} \in F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}, \lambda\right)$$

is *locally controllable around* $\vartheta_{\mathbb{R}^n}$ *at the point* (a, b).

If

$$\mathcal{A}((a,b); (f,g); \lambda) = \mathbb{R}^n$$

for every $(f, g) \in AC_p([0,a], \mathbb{R}^n) \times AC_p([0,b], \mathbb{R}^n)$ such that $f(0) = g(0)$, then we say that (I_λ) is $\mathbb{R}^n$*-completely controllable.*

In this paper we prove that, under suitable assumptions, the set of all $\lambda \in \Lambda$ such that (I_λ) is locally controllable around $\vartheta_{\mathbb{R}^n}$ at the point (a, b) (resp., $\mathbb{R}^n$-completely controllable) is open in Λ (see Theorems 2.1 and 2.2). To do this, we need some preliminary results (Propositions 1.1 and 1.2 and Theorems 1.1 and 1.2). The proof of Proposition 1.1 is implicitly contained in that of Theorem 3.1 of [10];

Theorems 1.1 and 1.2 improve, respectively, Theorems 2.1 and 2.3 of [9].

Next, we consider the following distributed parameter control process:

$$(E) \qquad \frac{\partial^2 z}{\partial x \partial y} = A(x,y)z + B(x,y)\frac{\partial z}{\partial x} + C(x,y)\frac{\partial z}{\partial y} + G(x,y,u(x,y)),$$

where (x, y) ranges over Q, the vector function z is n-dimensional, the control vector function u is constrained within a non-empty closed subset Ω of $\mathbb{R}^m$, the matrix-valued functions A, B, C and the vector function G satisfy some rather general assumptions (see $(a_1) - (a_4)$). As an application of the previous results we prove that, if (E) is locally controllable around $\vartheta_{\mathbb{R}^n}$ at the point (a, b) (resp., $\mathbb{R}^n$-completely controllable; see Definition 2.1) and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{G}$ are *close* to A, B, C, G in a sense to be specified then, the perturbed control process

$$(\tilde{E}) \qquad \frac{\partial^2 z}{\partial x \partial y} = \tilde{A}(x,y)z + \tilde{B}(x,y)\frac{\partial z}{\partial x} + \tilde{C}(x,y)\frac{\partial z}{\partial y} + \tilde{G}(x,y,u(x,y))$$

is locally controllable around $\vartheta_{\mathbb{R}^n}$ at the point (a, b) (resp., $\mathbb{R}^n$-completely controllable) too (see Theorems 2.3 and 2.4).

For other papers related, in some way, to the subject of the present one, we refer to [2], [4], [10], [11], [14], [15]. Here, we point out only that Theorem 3.2 of [2] is a particular case of Theorem 2.4 and that Theorem 3 of [15] (which holds only for $p > 1$) and Theorem 2.4 are independent. Indeed, the hypotheses of [15] on A, B, C are more general than $(a_1) - (a_3)$ (see, for instance, [4] Remark 4.1) but, in the setting of coefficients A, B, C which satisfy assumptions $(a_1) - (a_3)$, the *closeness* required in Theorem 2.4 is not so restrictive as that required in Theorem 3 of [15].

1. Preliminaries.

Let X, Y be two non-empty sets. A multifunction $\Phi : X \to 2^Y$ is a function from X into the family of all subsets of Y. The graph of Φ, denoted by $\mathrm{gr}(\Phi)$, is the set $\{(x, y) \in X \times Y : y \in \Phi(x)\}$. If $W \subseteq Y$, we put $\Phi^-(W) = \{x \in X : \Phi(x) \cap W \neq \emptyset\}$. If X, Y are two topological spaces, we say that Φ is lower semicontinuous if, for every open set $W \subseteq Y$, the set $\Phi^-(W)$ is open in X. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, we say that Φ is measurable if $\Phi^-(W) \in \mathcal{F}$ for all open set $W \subseteq Y$. If X, Y are two real vector spaces, we say that Φ is a convex process if gr(Φ) is a convex cone of $X \times Y$ containing the origin. This is equivalent to say that $\alpha\Phi(x) + \beta\Phi(z) \subseteq \Phi(\alpha x + \beta z)$ for every $\alpha, \beta \geq 0$, $x, z \in X$. If (Σ, δ) is a metric space, for every $x \in \Sigma$ and every pair of non-empty sets $V, W \subseteq \Sigma$, we put:

$$\delta(x, V) = \inf_{z \in V} \delta(x, z); \quad \delta^*(V, W) = \sup_{z \in V} \delta(z, W);$$
$$\delta_H(V, W) = \max\{\delta^*(V, W), \delta^*(W, V)\}.$$

Let (X, δ), (Y, ρ) be two metric spaces. A multifunction $\Phi : X \to 2^Y$, with non-empty values, is said to be Lipschitzian if there exists a real number $L \geq 0$ (Lipschitz constant) such that

$$\rho_H(\Phi(x), \Phi(z)) \leq Ld(x, z)$$

for all $x, z \in X$.

Finally, if S is a topological space and $V \subseteq S$, we denote by int(V) the interior of V and by cl(V) the closure of V.

In the sequel, we will apply the following two propositions. The first of them is implicitly contained in the proof of Theorem 3.1 of [10].

PROPOSITION 1.1. *Let S be a topological space and let Φ be a lower semicontinuous multifunction from S into $\mathbb{R}^n$, with non-empty convex values. If there exists $s_0 \in S$ such that $\vartheta_{\mathbb{R}^n} \in int(\Phi(s_0))$, then there exists a neighbourhood U_0 of s_0 such that $\vartheta_{\mathbb{R}^n} \in int(\Phi(s))$ for all $s \in U_0$.*

Proof. Let $\Omega_1, \Omega_2, \ldots, \Omega_{2^n}$ be the open orthants of $\mathbb{R}^n$. Since Φ is lower semicontinuous and $\vartheta_{\mathbb{R}^n} \in$int$(\Phi(s_0))$, there exists a neighbourhood U_0 of s_0 such that $\Phi(s) \cap \Omega_i \neq \emptyset$ for every $s \in U_0$ and every $i = 1, 2, \ldots, 2^n$. Fix $s \in U_0$ and, for each $i = 1, 2, \ldots, 2^n$, take $v_i \in \Phi(s) \cap \Omega_i$. Moreover, denote by V the convex hull of the set $\{v_1, v_2, \ldots, v_{2^n}\}$. Thanks to Theorem 6.2 of [12] one has ri$(V) \neq \emptyset$, where ri(V) is the relative interior of V. We claim that $\vartheta_{\mathbb{R}^n} \in$ ri(V). To verify this, observe first that if W is any proper linear subspace of $\mathbb{R}^n$, then $W \cap \Omega_i = \emptyset$ for some $i \in \{1, 2, \ldots, 2^n\}$. Indeed, if $(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \setminus \{\vartheta_{\mathbb{R}^n}\}$ is such that $\sum_{i=1}^{n} w_i z_i = 0$ for all $(w_1, w_2, \ldots, w_n) \in W$ then, denoted by $\Omega_{i'}$ the orthant associated with the arrangement (sign z_1, sign $z_2, \ldots,$ sign z_n), one has $\sum_{i=1}^{n} w_i z_i > 0$ for all $(w_1, w_2, \ldots, w_n) \in \Omega_{i'}$; therefore, $W \cap \Omega_{i'} = \emptyset$. Now, assume that $\vartheta_{\mathbb{R}^n} \notin \mathrm{ri}(V)$. By Theorem 11.3 of [12], there exists a non-null linear functional φ on $\mathbb{R}^n$ such that $\varphi(v) \geq 0$ for every $v \in V$. Let Ω_{i_0} be such that $\varphi^{-1}(0) \cap \Omega_{i_0} = \emptyset$ and let $\Omega_{j_0} = -\Omega_{i_0}$. Since φ is continuous, Ω_{j_0} is a convex set not meeting $\varphi^{-1}(0)$ and $\varphi(-v_{i_0}) < 0$, one has $\varphi(v) < 0$ for every $v \in \Omega_{j_0}$, against the fact that $v_{j_0} \in V \cap \Omega_{j_0}$. Hence, $\vartheta_{\mathbb{R}^n} \in \mathrm{ri}(V)$. This implies that the linear hull and the affine hull of V coincide. Taking into account that, by a previous remark, the liner hull of V is $\mathbb{R}^n$ and that $V \subseteq \Phi(s)$, we get $\vartheta_{\mathbb{R}^n} \in \mathrm{int}(\Phi(s))$. This completes the proof. ∎

Remark 1.1. Proposition 1.1 does not hold if Φ takes its values in an infinite dimensional Hilbert space. Indeed, let $\mathbb{N}$ be the set of all positive integers, H a separable Hilbert space such that $\dim(H) = +\infty$, $\{e_r\}$ an orthonormal basis of H, H_r the linear hull of the set $\{e_1, e_2, \ldots, e_r\}$ $(r \in \mathbb{N})$ and S the one-point compactification of $\mathbb{N}$ with the usual topology. For every $s \in S$, put:

$$\Phi(s) = \begin{cases} H_r & \text{if } s < +\infty,\ s = r \\ H & \text{if } s = +\infty. \end{cases}$$

Then, the multifunction $\Phi : S \to 2^H$ so defined is non-empty convex-valued and lower semicontinuous, $\vartheta_H \in \mathrm{int}(\Phi(+\infty))$, but $\vartheta_H \notin \mathrm{int}(\Phi(s))$ for every $s \in \mathbb{N}$ (of course, ϑ_H is the null element of H).

PROPOSITION 1.2. *Let (X, δ) be a metric space; $(Y, \|\cdot\|_Y)$ a real normed space; Φ a multifunction from X into Y, with non-empty convex values. If Φ is Lipschitzian and there exists $x_0 \in X$ such that $cl(\Phi(x_0)) = Y$, then $cl(\Phi(x)) = Y$ for every $x \in X$.*

Proof. Assume that $\mathrm{cl}(\Phi(x_1)) \neq Y$ for some $x_1 \in X$. Then, there exist a real number c and a non-null continuous linear functional φ on Y such that $\Phi(x_1) \subseteq \varphi^{-1}([c, +\infty[)$. Let ρ be the metric on Y induced by $\|\cdot\|_Y$, d the Euclidean metric of $\mathbb{R}$ and $n(\varphi)$ the norm of φ. Since

$\mathrm{cl}(\Phi(x_0)) = Y$ and φ is onto, one has:

$$\rho^*(\Phi(x_0), \Phi(x_1)) \geq \rho^*(\Phi(x_0), \varphi^{-1}([c, +\infty[)) \geq$$
$$\geq \rho^*(\varphi^{-1}(]-\infty, c]), \varphi^{-1}([c, +\infty[)) \geq$$

$$\geq \frac{1}{n(\varphi)} d^*(\varphi(\varphi^{-1}(]-\infty, c])), \varphi(\varphi^{-1}([c, +\infty[))) =$$
$$= \frac{1}{n(\varphi)} d^*(]-\infty, c], [c, +\infty[) = +\infty.$$

This implies that $\rho_H(\Phi(x_0), \Phi(x_1)) = +\infty$, against the fact that Φ is Lipschitzian. ∎

Remark 1.2. Let Φ be as in Proposition 1.2. The assumption $\Phi(x_0) = Y$ does not imply that $\Phi(x) = Y$ for every $x \in X$. Indeed, let Y be infinite dimensional and complete and let Y_0 be a dense convex proper subset of Y (see, for instance, [3], Exercise 27 p. 437). For every $x \in X$, put:

$$\Phi(x) = \begin{cases} Y & \text{if } x = x_0 \\ Y_0 & \text{if } x \neq x_0. \end{cases}$$

Then, the multifunction $\Phi : X \to 2^Y$ so defined is non-empty convex valued and Lipschitzian, $\Phi(x_0) = Y$, but $\Phi(x) \neq Y$ for every $x \in X \backslash \{x_0\}$.

We denote by $C_p^*(Q, \mathrm{I\!R}^n)$ the space of all continuous functions $z : Q \to \mathrm{I\!R}^n$ for which there exist $h \in L^p(Q, \mathrm{I\!R}^n)$, $h_1 \in L^p([0, a], \mathrm{I\!R}^n)$, $h_2 \in L^p([0, b], \mathrm{I\!R}^n)$, $z_0 \in \mathrm{I\!R}^n$ such that

$$z(x, y) = \int_0^x \int_0^y h(s, t) ds dt + \int_0^x h_1(s) ds + \int_0^y h_2(t) dt + z_0$$

for all $(x, y) \in Q$. It is possible to show that if $z \in C_p^*(Q, \mathrm{I\!R}^n)$, then there exist $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$ and, a.e. in Q, one has

$$\frac{\partial z(x,y)}{\partial x} = \int_0^y h(x,t)dt + h_1(x);$$
$$\frac{\partial z(x,y)}{\partial y} = \int_0^y h(s,y)ds + h_2(y);$$
$$\frac{\partial^2 z(x,y)}{\partial x \partial y} = h(x,y).$$

For every $z \in C_p^*(Q, \mathbb{R}^n)$ put:

$$\|z\|_{C_p^*(Q,\mathbb{R}^n)} = \max_{(x,y)\in Q} \|z(x,y)\| + \left\| \frac{\partial z}{\partial x} \right\|_{L^p(Q,\mathbb{R}^n)} + \left\| \frac{\partial z}{\partial y} \right\|_{L^p(Q,\mathbb{R}^n)} + + \left\| \frac{\partial^2 z}{\partial x \partial y} \right\|_{L^p(Q,\mathbb{R}^n)},$$

where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$. It is clear that $\|\cdot\|_{C_p^*(Q,\mathbb{R}^n)}$ is a norm on $C_p^*(Q, \mathbb{R}^n)$. Moreover, $C_p^*(Q, \mathbb{R}^n)$ endowed with this norm is complete (see, for instance, [9] Proposition 1.1).

If $AC_p(I, \mathbb{R}^n)$ is as in Introduction and $\varphi \in AC_p(I, \mathbb{R}^n)$, put:

$$\|\varphi\|_{AC_p(I,\mathbb{R}^n)} = \max_{t\in I} \|\varphi(t)\| + \left\| \frac{d\varphi}{dt} \right\|_{L^p(I,\mathbb{R}^n)}.$$

Of course, $(AC_p(I, \mathbb{R}^n), \|\cdot\|_{AC_p(I,\mathbb{R}^n)})$ is complete.

Now, put:

$$\Xi_p = \{(f,g) \in AC_p([0,a], \mathbb{R}^n) \times AC_p([0,b], \mathbb{R}^n) : f(0) = g(0)\}.$$

On Ξ_p we consider the norm of the graph $\|\cdot\|_{\Xi_p}$. The space Ξ_p, endowed with this norm, is a closed linear subspace of $AC_p([0,a], \mathbb{R}^n) \times AC_p([0,b], \mathbb{R}^n)$.

In the sequel, we will use the following theorems, which improve (when $X = \mathbb{R}^n$), respectively, Theorem 2.1 and Theorem 2.3 of [9].

THEOREM 1.1. *Assume that:*

(i) the multifunction $(x,y) \to F(x,y,z_1,z_2,z_3,z_4,\lambda)$ *is measurable for every* $z_1, z_2, z_3, z_4 \in \mathbb{R}^n$, $\lambda \in \Lambda$;

(ii) *there exist* $L_1 \in L^p(Q)$, $L_2 \in L^p([0,b])$, $L_3 \in L^p([0,a])$, $M \in [0,1[$ *such that, for every* $z_i', z_i'' \in \mathbb{R}^n$, $i = 1,2,3,4$, $\lambda \in \Lambda$ *and for almost every* $(x,y) \in Q$ *one has*

$$d_H(F(x,y,z_1',z_2',z_3',z_4',\lambda), F(x,y,z_1'',z_2'',z_3'',z_4'',\lambda)) \leq$$

$$\leq L_1(x,y)\|z_1' - z_1''\| + L_2(y)\|z_2' - z_2''\| + L_3(x)\|z_3' - z_3''\| + M\|z_4' - z_4''\|,$$

where d *is the metric induced by* $\|\cdot\|$;

(iii) *the multifunction* $\lambda \to F(x,y,z_1,z_2,z_3,z_4,\lambda)$ *is lower semicontinuous for every* $z_1,z_2,z_3,z_4 \in \mathbb{R}^n$ *and for almost every* $(x,y) \in Q$.

Then, the following assertions are equivalent:

(i_1) *for each convergent sequence* $\{\lambda_r\} \subseteq \Lambda$ *the set functions* $A \to \iint_A [d(\vartheta_{\mathbb{R}^n}, F(x,y,\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\lambda_r))]^p\,dxdy$ *are equi-absolutely continuous;*

(i_2) *for every* $(f,g) \in \Xi_p$ *the multifunction* $\lambda \to \Gamma_p(f,g,\lambda)$ *is nonempty closed-valued and lower semicontinuous with respect to the norm topology on* $C_p^*(Q,\mathbb{R}^n)$.

Proof. Let us prove that $(i_1) \Rightarrow (i_2)$. Fix $k > 0$ such that

$$M + [(ab)^{1-1/p} + a^{1-1/p} + b^{1-1/p}](pk)^{-1/p} < 1$$

and, for every $(x,y) \in Q$, put

$$L(x,y) = \int_0^x\int_0^y L_1(s,t)^p\,dsdt + \int_0^y L_2(t)^p\,dt + \int_0^x L_3(s)^p\,ds,$$

where $\int_0^x\int_0^y L_1(s,t)^p\,dsdt$ stands for $\int_0^x \left(\int_0^y L_1(s,t)^p\,dt\right) ds$. Next, consider on $L^p(Q,\mathbb{R}^n)$ the norm (equivalent to the usual one)

$$\|\varphi\|_0 = \max_{(x,y)\in Q} e^{-kL(x,y)} \left(\int_0^x\int_0^y \|\varphi(s,t)\|^p\,dsdt\right)^{1/p},$$

and go on exactly as in the proof of Theorem 2.1 of [9]. Conversely, let us prove that $(i_2) \Rightarrow (i_1)$. Let $\{\lambda_r\}$ be a sequence in Λ converging to $\lambda \in \Lambda$, $(f,g) \in \Xi_p$ and $z \in \Gamma_p(f,g,\lambda)$. Arguing as in the proof of Theorem 2.1 of [9], we get a sequence $\{z_r\} \subseteq C_p^*(Q, \mathbb{R}^n)$ such that $z_r \in \Gamma_p(f,g,\lambda_r)$ for all $r \in \mathbb{N}$,

(1) $$\lim_{r\to\infty} \|z_r - z\|_{C_p^*(Q,\mathbb{R}^n)} = 0$$

and

(2) $$d(\vartheta_{\mathbb{R}^n}, F(x,y,\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\lambda_r)) \leq$$

$$\leq L_1(x,y)\|z_r(x,y)\| + L_2(y)\left\|\frac{\partial z_r(x,y)}{\partial x}\right\| + L_3(x)\left\|\frac{\partial z_r(x,y)}{\partial y}\right\| +$$

$$+(M+1)\left\|\frac{\partial^2 z_r(x,y)}{\partial x\partial y}\right\| + 1$$

for every $r \in \mathbb{N}$ and almost every $(x,y) \in Q$. Let $\varphi, \varphi_1, \varphi_2, \ldots$ be a sequence in $L^p(Q, \mathbb{R}^n)$ such that $z(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y \varphi(s,t)dsdt$, $z_r(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y \varphi_r(s,t)dsdt$ $(r \in \mathbb{N})$ for all $(x,y) \in Q$. Since, thanks to (1), one has $\lim_{r\to\infty} \varphi_r = \varphi$ in $L^p(Q,\mathbb{R}^n)$, the set functions

$$A \to \int\int_A \left(L_2(y)\left\|\frac{\partial z_r(x,y)}{\partial x}\right\|\right)^p dxdy,$$

$$A \to \int\int_A \left(L_3(x)\left\|\frac{\partial z_r(x,y)}{\partial y}\right\|\right)^p dxdy,$$

$$A \to \int\int_A \left\|\frac{\partial^2 z_r(x,y)}{\partial x\partial y}\right\|^p dxdy$$

are equi-absolutely continuous. At this point, assertion (i_1) is a simple consequence of (1) and (2). ■

THEOREM 1.2. *Let F satisfy assumptions (i) and (ii) of Theorem 1.1. Further, assume that the real function*

$$(x,y) \to d(\vartheta_{\mathbb{R}^n}, F(x,y,\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\vartheta_{\mathbb{R}^n},\lambda))$$

belongs to $L^p(Q)$ for all $\lambda \in \Lambda$.

Then, there exists a constant c, depending only on a, b, p, $\|L_1\|_{L^p(Q)}$, $\|L_2\|_{L^p([0,b])}$, $\|L_3\|_{L^p([0,a])}$ *and M, such that, if D is the metric induced by* $\|\cdot\|_{C_p^*(Q,\mathbb{R}^n)}$, *one has*

$$D_H(\Gamma_p(f_1, g_1, \lambda), \Gamma_p(f_2, g_2, \mu)) \leq c(\|(f_1, g_1) - (f_2, g_2)\|_{\Xi_p} + \|f_1(0) - f_2(0)\| +$$

$$+ \operatorname{ess\,sup}_{(x,y)\in Q} \sup_{\xi\in(\mathbb{R}^n)^4} d_H(F(x, y, \xi, \lambda), F(x, y, \xi, \mu)))$$

for every (f_1, g_1), $(f_2, g_2) \in \Xi_p$, $\lambda, \mu \in \Lambda$.

The proof of this result is similar at all to that of Theorem 2.3 of [9]; hence we omit it.

Remark 1.3. Theorems 1.1 and 1.2 remain true (with the same proof) if $\mathbb{R}^n$ is replaced by a separable real Banach space X.

We denote by $\mathbb{R}^{n,m}$ the space of all real $n \times m$-matrices. For every $D \in \mathbb{R}^{n,m}$ put

$$|D| = \sup\{\|Du\| : u \in \mathbb{R}^m, \|u\|_{\mathbb{R}^m} \leq 1\}$$

($\|\cdot\|_{\mathbb{R}^m}$ is the Euclidean norm of $\mathbb{R}^m$). If $\mathcal{M}(Q, \mathbb{R}^{n,m})$ is the space of all (equivalence classes of) measurable functions from Q into $\mathbb{R}^{n,m}$, for every $D_1, D_2 \in \mathcal{M}(Q, \mathbb{R}^{n,m})$ put:

$$\delta(D_1, D_2) = \int\int_Q \frac{|D_1(x, y) - D_2(x, y)|}{1 + |D_1(x, y) - D_2(x, y)|} dxdy.$$

It is clear that δ is a metric on $\mathcal{M}(Q, \mathbb{R}^{n,m})$. Moreover, for every sequence $D, D_1, D_2, \ldots$ in $\mathcal{M}(Q, \mathbb{R}^{n,m})$ one has $\lim_{k\to\infty} \delta(D_k, D) = 0$ if and only if $\lim_{k\to\infty} D_k = D$ in measure in Q (see, for instance, [7] pp. 5-6).

The definition of the space $\mathcal{M}(Q, \mathbb{R}^n)$ is similar at all.

Given a non-empty closed subset Ω of $\mathbb{R}^m$, we denote by $\mathcal{Z}(Q \times \Omega, \mathbb{R}^n)$ the space of all functions $G: Q \times \Omega \to \mathbb{R}^n$ such that $G(\cdot, u) \in \mathcal{M}(Q, \mathbb{R}^n)$ for every $u \in \Omega$ and the function $u \to G(x, y, u)$ is continuous for every $(x, y) \in Q$. Moreover, for every $G \in \mathcal{Z}(Q \times \Omega, \mathbb{R}^n)$ we put $\mathcal{M}_G(Q, \Omega) = \{u : Q \to \Omega$ such that u is measurable and $G(\cdot, u(\cdot)) \in L^p(Q, \mathbb{R}^n)\}$.

If Ω is compact, we denote by $\mathcal{Z}_c$ the class of all functions $G \in \mathcal{Z}(Q \times \Omega, \mathbb{R}^n)$ such that for almost every $(x, y) \in Q$ the set

$\{G(x, y, u) : u \in \Omega\}$ is convex and contains $\vartheta_{\mathbb{R}^n}$. If $G_1, G_2 \in Z_c$, put:

$$\rho(G_1, G_2) = \int\int_Q \frac{\sup_{u\in\Omega} ||G_1(x, y, u) - G_2(x, y, u)||}{1 + \sup_{u\in\Omega} ||G_1(x, y, u) - G_2(x, y, u)||} dxdy.$$

Obviously, ρ is a metric on Z_c and, for every sequence $G, G_1, G_2, \dots$ in Z_c one has $\lim_{k\to\infty} \rho(G_k, G) = 0$ if and only if $\lim_{k\to\infty} \sup_{u\in\Omega} ||G_k(\cdot, u) - G(\cdot, u)|| = 0$ in measure in Q.

2. Results.

Our first theorem is a stability result for the local controllability of (I_λ) around $\vartheta_{\mathbb{R}^n}$ at the point (a, b).

THEOREM 2.1. *Let F satisfy assumptions (i), (iii) and (i_1) of Theorem 1.1. Moreover, assume that:*

(j) for every $\lambda \in \Lambda$ there exist a neighbourhood U_λ of λ and $L_{1,\lambda} \in L^p(Q)$, $L_{2,\lambda} \in L^p([0, b])$, $L_{3,\lambda} \in L^p([0, a])$, $M_\lambda \in [0, 1[$ such that

$$d_H(F(x, y, z_1', z_2', z_3', z_4', \mu),\ F(x, y, z_1'', z_2'', z_3'', z_4'', \mu)) \leq$$

$$\leq L_{1,\lambda}(x, y)||z_1' - z_1''|| + L_{2,\lambda}(y)||z_2' - z_2''|| + L_{3,\lambda}(x)||z_3' - z_3''|| + M_\lambda||z_4' - z_4''||$$

for every $z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, $\mu \in U_\lambda$ and for almost every $(x, y) \in Q$;

(jj) for every $\lambda \in \Lambda$ the set $\mathcal{A}((a, b); (\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n}); \lambda)$ is convex.

Then, the set

$$V_1 = \{\lambda \in \Lambda : (I_\lambda) \text{ is locally controllable around } \vartheta_{\mathbb{R}^n} \text{ at the point } (a, b)\}$$

is open in Λ.

Proof. Fix $\lambda_0 \in V_1$. Thanks to (j), we can apply Theorem 1.1 to the multifunction $F|_{Q\times(\mathbb{R}^n)^4\times U_{\lambda_0}}$ and we obtain that, for every $(f, g) \in \Xi_p$, the multifunction $\lambda \to \Gamma_p\ (f, g, \lambda)$, $\lambda \in U_{\lambda_0}$, is non-empty closed-valued and lower semicontinuous with respect to the norm topology on $C_p^*(Q, \mathbb{R}^n)$. Now, for every $z \in C_p^*(Q, \mathbb{R}^n)$, put

$$T(z) = z(a, b).$$

Of course, T is a continuous linear operator from $C_p^*(Q, \mathbb{R}^n)$ onto $\mathbb{R}^n$ and one has

$$\mathcal{A}((a,b);(f,g);\lambda) = T(\Gamma_p(f,g,\lambda)) \tag{3}$$

for all $(f,g) \in \Xi_p$, $\lambda \in U_{\lambda_0}$. Taking into account (jj), this implies that the multifunction $\lambda \to \mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda)$, $\lambda \in U_{\lambda_0}$, is non-empty convex closed-valued and lower semicontinuous. Since $\vartheta_{\mathbb{R}^n} \in \mathrm{int}(\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda_0))$, the conclusion follows at once from Proposition 1.1. ∎

Remark 2.1. A simple sufficient condition in order that (i_1) of Theorem 1.1 and (jj) of Theorem 2.1 hold is the following:

(jj)' *for every $\lambda \in \Lambda$ and for almost every $(x,y) \in Q$ the graph of the multifunction $(z_1, z_2, z_3, z_4) \to F(x,y,z_1,z_2,z_3,z_4,\lambda)$ is convex and contains the origin.*

The following theorem deals with the $\mathbb{R}^n$-complete controllability of (I_λ).

THEOREM 2.2. *Let F satisfy assumptions (i), (iii) of Theorem 1.1 and (j) of Theorem 2.1. Further, assume that for almost every $(x,y) \in Q$ and every $\lambda \in \Lambda$, the multifunction $(z_1, z_2, z_3, z_4) \to F(x,y,z_1,z_2,z_3,z_4,\lambda)$ is a convex process. Then the set*

$$V_c = \{\lambda \in \Lambda : (I_\lambda) \text{ is } \mathbb{R}^n\text{-completely controllable}\}$$

is open in Λ.

Proof. Fix $\lambda_0 \in V_c$. Since $\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda_0) = \mathbb{R}^n$ and (jj)' holds, by Theorem 2.1 there exists a neighbourhood W_{λ_0} of λ_0 such that $W_{\lambda_0} \subseteq V_1$. Let us prove that $W_{\lambda_0} \subseteq V_c$. To this, end, fix $\lambda \in W_{\lambda_0}$ and observe that from our assumptions it follows that $\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda)$ is a convex cone of $\mathbb{R}^n$ and $\vartheta_{\mathbb{R}^n} \in \mathrm{int}(\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda))$. Hence,

$$\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda) = \mathbb{R}^n. \tag{4}$$

If we apply Theorem 1.2 to the multifunction $F|_{Q \times (\mathbb{R}^n)^4 \times U_{\lambda_0}}$, we obtain that the multifunction $(f,g) \to \Gamma_p(f,g,\lambda)$ is Lipschitzian, with Lipschitz constant $2c$, and non-empty closed-valued. Moreover, for every $(f,g) \in \Xi_p$, the set $\Gamma_p(f,g,\lambda)$ is convex.

Now, observe that, thanks to (3) and Theorem 6 of [13], for every (f_1, g_1), $(f_2, g_2) \in \Xi_p$ one has

$$d_H(\mathcal{A}((a,b);(f_1,g_1);\lambda), \mathcal{A}((a,b);(f_2,g_2);\lambda)) \leq$$
$$\leq n(T) D_H(\Gamma_p(f_1,g_1,\lambda), \Gamma_p(f_2,g_2,\lambda)) \leq$$
$$\leq D_H(\Gamma_p(f_1,g_1,\lambda), \Gamma_p(f_2,g_2,\lambda)) \leq 2c\|(f_1,g_1) - (f_2,g_2)\|_{\Xi_p},$$

where $n(T)$ is the norm of the operator T introduced in the proof of Theorem 2.1. This shows that the multifunction $(f,g) \to \mathcal{A}((a,b);(f,g);\lambda)$ is Lipschitzian and non-empty convex closed-valued. Taking into account (4), from Proposition 1.2 it follows that $\mathcal{A}((a,b);(f,g);\lambda) = \mathbb{R}^n$ for every $(f,g) \in \Xi_p$, that is $\lambda \in V_c$. So, our claim is proved. ■

Now, let us apply the previous results to the study of two kinds of controllability for the distributed parameter control process (E). Suppose that the following assumptions hold (cf. [14] p. 566).

(a_1) $A \in L^p(Q, \mathbb{R}^{n,n})$;

(a_2) $B \in \mathcal{M}(Q, \mathbb{R}^{n,n})$ and there exists $\beta \in L^p([0,b])$ such that $|B(x,y)| \leq \beta(y)$ a.e. in Q;

(a_3) $C \in \mathcal{M}(Q, \mathbb{R}^{n,n})$ and there exists $\gamma \in L^p([0,a])$ such that $|C(x,y)| \leq \gamma(x)$ a.e. in Q;

(a_4) $G \in \mathcal{Z}(Q \times \Omega, \mathbb{R}^n)$.

For every $(x,y) \in Q$, $z_1, z_2, z_3 \in \mathbb{R}^n$ put:

$$\Phi(x,y,z_1,z_2,z_3) = \{A(x,y)z_1 + B(x,y)z_2 + C(x,y)z_3 + G(x,y,u) : u \in \Omega\}.$$

The following proposition allows us to study the control process (E) by means of a suitable partial differential inclusion.

PROPOSITION 2.1. *A function* $z \in C_p^*(Q, \mathbb{R}^n)$ *is a solution of the differential inclusion*

$$(I) \qquad \frac{\partial^2 z}{\partial x \partial y} \in \Phi\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

if and only if it is a solution of (E) *for some* $u \in \mathcal{M}_G(Q, \Omega)$.

Proof. Let z be a solution of (I). For almost every $(x, y) \in Q$, put:

$$\Psi_z(x, y) = \left\{ \left(z(x, y), \frac{\partial z(x, y)}{\partial x}, \frac{\partial z(x, y)}{\partial y}, u \right) : u \in \Omega \right\}.$$

Of course, the multifunction $\Psi_z : Q \to 2^{(\mathbb{R}^n)^3 \times \mathbb{R}^m}$ so defined is measurable and non-empty closed-valued. Now, for every $(x, y) \in Q$, $z_1, z_2, z_3 \in \mathbb{R}^n$, $u \in \Omega$, put

$$\psi(x, y, z_1, z_2, z_3, u) = A(x, y)z_1 + B(x, y)z_2 + C(x, y)z_3 + G(x, y, u).$$

Observe that for all $(z_1, z_2, z_3, u) \in (\mathbb{R}^n)^3 \times \Omega$ the function $(x, y) \to \psi(x, y, z_1, z_2, z_3, u)$ is measurable and for all $(x, y) \in Q$ the function $(z_1, z_2, z_3, u) \to \psi(x, y, z_1, z_2, z_3, u)$ is continuous. Since one has

$$\psi(x, y, \Psi_z(x, y)) = \Phi\left(x, y, z(x, y), \frac{\partial z(x, y)}{\partial x}, \frac{\partial z(x, y)}{\partial y}\right),$$

$$\frac{\partial^2 z(x, y)}{\partial x \partial y} \in \psi(x, y, \Psi_z(x, y))$$

a.e. in Q, we are allowed to apply Theorem 7.1 of [6]. Then, there exists a measurable function $u : Q \to \Omega$ such that

$$\frac{\partial^2 z(x, y)}{\partial x \partial y} = \psi\left(x, y, z(x, y), \frac{\partial z(x, y)}{\partial x}, \frac{\partial z(x, y)}{\partial y}, u(x, y)\right)$$

a.e. in Q. It follows that $u \in \mathcal{M}_G(Q, \Omega)$ and z is a solution of (E). The converse is trivial. ■

For every $u \in \mathcal{M}_G(Q, \Omega)$, $(f, g) \in \Xi_p$, denote by $R_u((a, b); (f, g); (A, B, C, G))$ the set $\{z(a, b) : z$ is a solution of (E) such that $z(x, 0) = f(x)$ for all $x \in [0, a]$, $z(0, y) = g(y)$ for all $y \in [0, b]\}$.

Moreover, put:

$$R((a, b); (f, g); (A, B, C, G)) = \bigcup_{u \in \mathcal{M}_G(Q, \Omega)} R_u((a, b); (f, g); (A, B, C, G)).$$

We give the following

DEFINITION 2.1. *The control process* (E) *is said to be locally* Ω*-controllable around* $\vartheta_{\mathbb{R}^n}$ *at the point* (a, b) *if*

$$\vartheta_{\mathbb{R}^n} \in \operatorname{int}(R((a, b); (\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n}); (A, B, C, G))).$$

If, for every $(f,g) \in \Xi_p$, *one has*

$$R((a,b);(f,g);(A,B,C,G)) = \mathbb{R}^n,$$

then we say that (E) *is* $(\mathbb{R}^n,\Omega)$*-completely controllable.*

The next theorem is a stability result for the local Ω-controllability of (E) around $\vartheta_{\mathbb{R}^n}$ at the point (a,b).

THEOREM 2.3. *Assume that* $G \in Z_c$ *and that the control process* (E) *is locally* Ω*-controllable around* $\vartheta_{\mathbb{R}^n}$ *at the point* (a,b).

Then, for every $\tilde{\beta} \in L^p([0,b])$, $\tilde{\gamma} \in L^p([0,a])$ *such that* $\tilde{\beta}(y) \geq \beta(y)$ *a.e. in* $[0,b]$ *and* $\tilde{\gamma}(x) \geq \gamma(x)$ *a.e. in* $[0,a]$, *there exists* $\sigma > 0$ *such that if* $\tilde{A} \in L^p(Q,\mathbb{R}^{n,n})$, $\tilde{B},\tilde{C} \in \mathcal{M}(Q,\mathbb{R}^{n,n})$, $\tilde{G} \in Z_c$ *and*

(b_1) $|\tilde{B}(x,y)| \leq \tilde{\beta}(y)$, $|\tilde{C}(x,y)| \leq \tilde{\gamma}(x)$ *a.e. in* Q,

(b_2) $\|A-\tilde{A}\|_{L^p(Q,\mathbb{R}^{n,n})} + \delta(B,\tilde{B}) + \delta(C,\tilde{C}) + \rho(G,\tilde{G}) < \sigma$

then, the control process $(\tilde{E})$ *is locally* Ω*-controllable around* $\vartheta_{\mathbb{R}^n}$ *at the point* (a,b).

Proof. Suppose that the conclusion of the theorem does not hold. Then, there exist $\tilde{\beta} \in L^p([0,b])$, $\tilde{\gamma} \in L^p([0,a])$, a sequence $\{A_k\}$ in $L^p(Q,\mathbb{R}^{n,n})$, two sequences $\{B_k\}$, $\{C_k\}$ in $\mathcal{M}(Q,\mathbb{R}^{n,n})$ and a sequence $\{G_k\}$ in Z_c such that:

(5) $\tilde{\beta}(y) \geq \beta(y)$ a.e. in $[0,b]$, $\tilde{\gamma}(x) \geq \gamma(x)$ a.e. in $[0,a]$;

(6) $|B_k(x,y)| \leq \tilde{\beta}(y)$, $|C_k(x,y)| \leq \tilde{\gamma}(x)$ a.e. in Q and for all $k \in \mathbb{N}$;

(7) $\|A-A_k\|_{L^p(Q,\mathbb{R}^{n,n})} + \delta(B,B_k) + \delta(C,C_k) + \rho(G,G_k) < \dfrac{1}{k}$ for all $k \in \mathbb{N}$;

(8) for every $k \in \mathbb{N}$ the system

$$\frac{\partial^2 z}{\partial x \partial y} = A_k(x,y)z + B_k(x,y)\frac{\partial z}{\partial x} + C_k(x,y)\frac{\partial z}{\partial y} + G_k(x,y,u(x,y))$$

is not locally Ω-controllable around $\vartheta_{\mathbb{R}^n}$ at the point (a,b). Since, by (7), the sequence $\{|A(\cdot)-A_k(\cdot)|\}$ converges to zero in $L^p(Q)$ and the sequence $\{|B(\cdot)-B_k(\cdot)| + |C(\cdot)-C_k(\cdot)| + \sup_{u\in\Omega}\|G(\cdot,u)-G_k(\cdot,u)\|\}$ converges to zero in measure in Q, from Theorem 2.8.1 and Remark (e) p. 88 of [8] it follows that there exist $\tilde{\alpha} \in L^p(Q)$ and an increasing sequence $\{k_r\}$ of integers such that

$$|A_{k_r}(x,y)| \leq \tilde{\alpha}(x,y), \tag{9}$$

$$\lim_{r\to\infty}[|A(x,y)-A_{k_r}(x,y)|+|B(x,y)-B_{k_r}(x,y)|+ \tag{10}$$
$$+|C(x,y)-C_{k_r}(x,y)|+$$
$$+\sup_{u\in\Omega}\|G(x,y,u)-G_{k_r}(x,y,u)\|]=0$$

a.e. in Q and for all $r \in \mathbb{N}$.

Let Λ be the one-point compactification of $\{k_r\}$, with the usual topology. For every $(x,y) \in Q$, $z_1, z_2, z_3 \in \mathbb{R}^n$, $\lambda \in \Lambda$ put:

$$F(x,y,z_1,z_2,z_3,\lambda) =$$

$$= \begin{cases} \{A_{k_r}(x,y)z_1 + B_{k_r}(x,y)z_2 + C_{k_r}(x,y)z_3 + G_{k_r}(x,y,u) : u \in \Omega\} \\ \qquad \text{if } \lambda < +\infty,\ \lambda = k_r \\ \{A(x,y)z_1 + B(x,y)z_2 + C(x,y)z_3 + G(x,y,u) : u \in \Omega\} \\ \qquad \text{if } \lambda = +\infty. \end{cases}$$

The multifunction $F : Q\times(\mathbb{R}^n)^3\times\Lambda \to 2^{\mathbb{R}^n}$ so defined is non-empty closed-valued and, thanks to Theorem 6.5 of [6], measurable with respect to $(x,y) \in Q$. Moreover, by (5), (6), (9) and (10), for almost every $(x,y) \in Q$ and every $z_i', z_i'' \in \mathbb{R}^n$, $i = 1,2,3, \lambda \in \Lambda$, one has

$$d_H(F(x,y,z_1',z_2',z_3',\lambda),\ F(x,y,z_1'',z_2'',z_3'',\lambda)) \le$$
$$\le |A_\lambda(x,y)|\,\|z_1'-z_1''\| + |B_\lambda(x,y)|\,\|z_2'-z_2''\| + |C_\lambda(x,y)|\,\|z_3'-z_3''\| \le$$
$$\le \tilde\alpha(x,y)\|z_1'-z_1''\| + \tilde\beta(y)\|z_2'-z_2''\| + \tilde\gamma(x)\|z_3'-z_3''\|,$$

where $(A_\lambda, B_\lambda, C_\lambda) = (A_{k_r}, B_{k_r}, C_{k_r})$ if $\lambda = k_r$ and $(A_\lambda, B_\lambda, C_\lambda) = (A,B,C)$ if $\lambda = +\infty$. Now fix (x,y) a.e. in Q and $z_1,z_2,z_3 \in \mathbb{R}^n$. If W is a non-empty open subset of $\mathbb{R}^n$ such that $F(x,y,z_1,z_2,z_3,+\infty)\cap W \neq \emptyset$, then, for some $\bar u \in \Omega$, one has

$$A(x,y)z_1 + B(x,y)z_2 + C(x,y)z_3 + G(x,y,\bar u) \in W.$$

Since (10) holds, there exists $\bar r \in \mathbb{N}$ such that

$$A_{k_r}(x,y)z_1 + B_{k_r}(x,y)z_2 + C_{k_r}(x,y)z_3 + G_{k_r}(x,y,\bar u) \in W$$

for all $r \ge \bar r$, that is $F(x,y,z_1,z_2,z_3,k_r)\cap W \neq \emptyset$ for all $r \ge \bar r$. This implies that the multifunction $\lambda \to F(x,y,z_1,z_2,z_3,\lambda)$ is lower semicontinuous.

Finally, observe that, since $G, G_{k_1}, G_{k_2}, \dots \in Z_c$, for every $\lambda \in \Lambda$ and for almost every $(x,y) \in Q$ the graph of multifunction $(z_1, z_2, z_3) \to F(x,y,z_1,z_2,z_3,\lambda)$ is convex and contains the origin. At this point, we can apply Theorem 2.1. Then, taking into account that, thanks to Proposition 2.1, one has

$$\mathcal{A}((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});\lambda) = R((a,b);(\vartheta_{\mathbb{R}^n}, \vartheta_{\mathbb{R}^n});(A_\lambda, B_\lambda, C_\lambda, G_\lambda))$$

for every $\lambda \in \Lambda$, from $+\infty \in V_1$ it follows that there exists $r^* \in \mathbb{N}$ such that for every $r \geq r^*$ the control process

$$\frac{\partial^2 z}{\partial x \partial y} = A_{k_r}(x,y)z + B_{k_r}(x,y)\frac{\partial z}{\partial x} + C_{k_r}(x,y)\frac{\partial z}{\partial y} + G_{k_r}(x,y,u(x,y))$$

is locally Ω-controllable around $\vartheta_{\mathbb{R}^n}$ at the point (a,b), against (8). ■

If Ω is a closed convex cone of $\mathbb{R}^m$ and $G(x,y,u) = G_1(x,y)u$ ($(x,y) \in Q$, $u \in \Omega$), with $G_1 \in \mathcal{M}(Q, \mathbb{R}^{n,m})$, then, arguing as in the proof of the previous theorem, but using Theorem 2.2 instead of Theorem 2.1, it is possible to verify the following

THEOREM 2.4. *Assume that: Ω is a closed convex cone of $\mathbb{R}^m$; $G(x,y,u) = G_1(x,y)u$ ($(x,y) \in Q$, $u \in \Omega$), with $G_1 \in \mathcal{M}(Q, \mathbb{R}^{n,m})$; the control process (E) is $(\mathbb{R}^n, \Omega)$-completely controllable. Then, for every $\tilde{\beta} \in L^p([0,b])$, $\tilde{\gamma} \in L^p([0,a])$ such that $\tilde{\beta}(y) \geq \beta(y)$ a.e. in $[0,b]$ and $\tilde{\gamma}(x) \geq \gamma(x)$ a.e. in $[0,a]$, there exists $\sigma > 0$ such that if $\tilde{A} \in L^p(Q, \mathbb{R}^{n,n})$, $\tilde{B}, \tilde{C} \in \mathcal{M}(Q, \mathbb{R}^{n,n})$, $\tilde{G}_1 \in \mathcal{M}(Q, \mathbb{R}^{n,m})$ and*

(b_1') $|\tilde{B}(x,y)| \leq \tilde{\beta}(y)$, $|\tilde{C}(x,y)| \leq \tilde{\gamma}(x)$ *a.e. in* Q,

(b_2') $\|A - \tilde{A}\|_{L^p(Q,\mathbb{R}^{n,n})} + \delta(B,\tilde{B}) + \delta(C,\tilde{C}) + \delta(G_1,\tilde{G}_1) < \sigma$

then, the control process ($\tilde{E}$), where $\tilde{G}(x,y,u) = \tilde{G}_1(x,y)u$ ($(x,y) \in Q$, $u \in \Omega$), is $(\mathbb{R}^n, \Omega)$-completely controllable.

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