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STABILIZABILITY OF OSCILLATORY SYSTEMS: A CLASSICAL APPROACH SUPPORTED BY SYMBOLIC COMPUTATION

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In this paper we study the stabilizability of planar single-input nonlinear systems, whose linearization at the origin has purely imaginary eigenvalues. A classical recursive procedure based on polar coordinates transformations is applied to obtain some sufficient conditions. Then we focus on bilinear systems; a complete solution of the problem is given in this case.

Explicit statements of the conditions are possible thanks to the use of symbolic computation packages.

Introduction.

In this paper we deal with nonlinear systems of ordinary differential equations in the plane of the form

$$(1) \qquad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \,. \end{cases}$$

Our study is motivated by the stabilizability problems of certain

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nonlinear control processes. Indeed, as explained in Sect. 3, under suitable assumptions the stabilizability problem reduces to checking local asymptotic stability of an equilibrium position $x = x_0$, $y = y_0$ of a system of the form (1), where the functions $f(x, y)$ and $g(x, y)$ at the right-hand-sides depend on some real parameters. In particular, in this paper we focus on the oscillatory case. This means that the linear approximation of (1), namely the linear system defined by the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

has a pair of purely imaginary conjugate eigenvalues. Oscillatory systems are critical in the sense that stability cannot be decided by looking at the linear approximation.

To analyze stability of oscillatory systems at least three methods have been proposed in the literature. The most recent one is due to [8]. The basic fact is the existence of a Liapunov function $V(x, y)$ such that

$$\frac{\partial V}{\partial x} f + \frac{\partial V}{\partial y} g = \gamma_2(x^2 + y^2) + \ldots + \gamma_{2k}(x^2 + y^2)^k + \ldots \quad .$$

Obviously, the origin is an asymptotically stable equilibrium of (1) if the first nonvanishing term of this expansion is negative. In principle, the coefficients $\gamma_2, \gamma_4, \ldots$ can be determined by the algorithm shown in [8]. Computations are very involved and a heavy use of computer algebra is needed: explicit results are given for particular kinds of systems.

A more classical approach is given by the so-called normal form expansion. By a sequence of coordinate changes, any analytic oscillatory system is transformed into a pair of polar coordinates equations

$$\text{(2)} \qquad \begin{cases} \dot{r} = a_1 r^3 + a_2 r^5 + \ldots + a_k r^{2k+1} + \ldots \\ \dot{\omega} = 1 + b_1 r^2 + b_2 r^4 + \ldots + b_k r^{2k} + \ldots \end{cases} \quad .$$

Now, it is obvious that the asymptotic behaviour near the origin depends on the sign of the first non-zero coefficient of the first equation. As far as the authors know, explicit expressions have been found only for a_1 (see [9]) and a_2 (see [10]). The expression of a_1 is not too difficult to use, but that of a_2 is terribly involved. So, in the highly critical case where $a_1 = 0$, this approach is not very helpful.

A different classical approach to stability analysis of oscillatory systems is reported in [13]. This leads to an infinite system of ordinary differential equations which, in principle, can be solved by hand. However, also in this case computational difficulties have been until now a serious obstruction to practical results. Nevertheless, in the authors' opinion, this third method has an advantage. Namely, computational difficulties are not intrinsic to the theoretical framework, which is on the contrary very simple and clear. The use of symbolic computation packages reduces to computing a sequence of definite integrals involving trigonometric polynomials of higher and higher degree.

The paper is organized as follows. In Sect. 2 the theoretical idea of the method is sketched for sake of completeness. The first step of the procedure for solving the resulting infinite system of ordinary differential equations is carried out in general. At that point, to decide about stability, one has to evaluate the sign of an expression that, except for a positive constant multiple, is the same as the expression of a_1 found in [9]. In Sect. 3 the problem of stabilizability of an affine control system by means of linear feedback is introduced. We present some sufficient conditions, based on the first step of the procedure developed in Sect. 2. In Sect. 4 we consider oscillatory bilinear control processes in highly critical position; stabilizability conditions of this section are obtained by iterating the procedure to higher order steps. The study of the bilinear processes is completed in Sect. 5; we prove that, except for one trivial case, oscillatory bilinear systems can be stabilized by means of constant or linear feedbacks.

Related papers are [5], where planar systems whose linear approximation has a simple zero eigenvalue are treated using a centre manifold approach, and [6], devoted to linear stabilizability of planar bilinear systems. Some results of the present paper are stated in [6] without proof.

Local stabilizability of critical systems is also considered in [1], [2], [3], [4], [7], [12].

2. Sketch of the method.

Let us consider a system of the form (1). Without loss of generality, the equilibrium position is supposed to be the origin. In other words, we assume that $f(0,0) = g(0,0) = 0$. We assume also that f and g are analytic around the origin, with Taylor expansions

$$f(x,y) = -y + f_2(x,y) + f_3(x,y) + \dots \tag{3.1}$$

$$g(x,y) = x + g_2(x,y) + g_3(x,y) + \dots \tag{3.2}$$

where $f_n(x,y)$ and $g_n(x,y)$ are homogeneous polynomials of degree n $(n = 2,3,..)$, namely

$$f_n(x,y) = \sum_{i+j=n} A_{ij} x^i y^j \tag{4.1}$$

$$g_n(x,y) = \sum_{i+j=n} B_{ij} x^i y^j \tag{4.2}$$

$(n = 2,3,..)$. Following the exposition given in [13], the system is rewritten in polar coordinates (ρ, θ). One easily gets

$$\begin{cases} \dot{\rho} = \rho^2 F(\rho,\theta) \\ \dot{\theta} = 1 + \rho G(\rho,\theta) \ , \end{cases} \tag{5}$$

where

$$F(\rho,\theta) = F_3(\theta) + \rho F_4(\theta) + \rho^2 F_5(\theta) + \dots$$

$$G(\rho,\theta) = G_3(\theta) + \rho G_4(\theta) + \rho^2 G_5(\theta) + \dots$$

and

$$F_{n+1}(\theta) = f_n(\cos\theta, \sin\theta)\cos\theta + g_n(\cos\theta, \sin\theta)\sin\theta$$

$$G_{n+1}(\theta) = g_n(\cos\theta, \sin\theta)\cos\theta - f_n(\cos\theta, \sin\theta)\sin\theta$$

....

for each $n \geq 2$. Note that $F_n(\theta)$ and $G_n(\theta)$ are polynomials of degree n in the variables $\cos\theta$ and $\sin\theta$. From (5) we have

$$\begin{aligned} \frac{d\rho}{d\theta} &= \rho^2 F(\rho,\theta)\left[1 - \rho G(\rho,\theta) + \rho^2 G^2(\rho,\theta) + \dots\right] \\ &= \rho^2 F_3(\theta) + \rho^3 \left[F_4(\theta) - F_3(\theta) G_3(\theta)\right] + \dots \quad . \end{aligned} \tag{6}$$

Let us denote by $\rho(\theta, \rho_0)$ the solution of (6) such that $\rho(0, \rho_0) = \rho_0$. We can write

$$\rho(\theta, \rho_0) = c_1(\theta)\rho_0 + c_2(\theta)\rho_0^2 + c_3(\theta)\rho_0^3 + \dots \quad . \tag{7}$$

Substituting in (6) we have

$$\frac{d\rho}{d\theta} = c_1^2(\theta)F_3(\theta)\rho_0^2 + [2c_1(\theta)c_2(\theta)F_3(\theta) + c_1^3(\theta)(F_4(\theta) - F_3(\theta)G_3(\theta))]\rho_0^3 + \dots \quad . \tag{8}$$

On the other hand, taking the derivative of (7), we also have

$$\frac{d\rho}{d\theta} = \frac{dc_1(\theta)}{d\theta}\rho_0 + \frac{dc_2(\theta)}{d\theta}\rho_0^2 + \dots \quad . \tag{9}$$

Equating (8) and (9), an infinite system of differential equations in the unknowns $c_1, c_2, \dots$ is obtained

$$\frac{dc_1(\theta)}{d\theta} = 0 \tag{10.1}$$

$$\frac{dc_2(\theta)}{d\theta} = F_3(\theta)c_1^2(\theta) \tag{10.2}$$

$$\frac{dc_3(\theta)}{d\theta} = 2c_1(\theta)c_2(\theta)F_3(\theta) + c_1^3(\theta)(F_4(\theta) - F_3(\theta)G_3(\theta)) \tag{10.3}$$

....

Furthermore, there are some obvious initial conditions associated with this system. Indeed, since $\rho(0, \rho_0) = \rho_0$, from (7) we obtain

$$c_1(0) = 1 \tag{11.1}$$

$$c_2(0) = 0 \tag{11.2}$$

$$c_3(0) = 0 \tag{11.3}$$

....

Finally, according to (10.1) and (11.1), it is easily seen that $c_1(\theta) \equiv 1$; this allows us to simplify equations (10).

Now, let $M = \{i : i > 1 \text{ and } c_i(2\pi) \neq 0\} \subset \mathrm{N}$ and let $m = \min M$. The following statement is easily deduced.

PROPOSITION 2.1 *Let us consider system (1), with f and g given by (3). Let us assume that $M \neq \emptyset$. If $c_m(2\pi) < 0$, then the origin is (locally) asymptotically stable.* ■

In order to determine the stability properties of the origin of a system defined by (1), (3), we have therefore to compute numbers $c_i(2\pi)$ until a non-zero value is found. From (10.2) and (11.2) we obtain immediately

$$c_2(2\pi) = \int_0^{2\pi} F_3(\theta)\, d\theta$$

and

$$c_3(2\pi) = \left[\int_0^{2\pi} F_3(\theta)\, d\theta\right]^2 + \int_0^{2\pi} F_4(\theta)\, d\theta - \int_0^{2\pi} F_3(\theta) G_3(\theta)\, d\theta.$$

The computation of $c_2(2\pi)$ and $c_3(2\pi)$ has been performed by the authors with the aid of a standard symbolic manipulation package. The results are reported in the following statement.

PROPOSITION 2.2. *For a system defined by (1), (3), (4), we have $c_2(2\pi) = 0$ and*

$$\begin{aligned} c_3(2\pi) = \frac{\pi}{4}[&3A_{30} + A_{12} + B_{21} + 3B_{03} + A_{11}(A_{20} + A_{02}) \\ &- B_{11}(B_{20} + B_{02}) - 2A_{20}B_{20} + 2A_{02}B_{02}] \end{aligned} \tag{12}$$

■

Remark 2.3. As recalled in the Introduction, also the normal form expansion performed in [9] leads to a polar coordinates representation of the system. Thus, it is natural to expect some similarity in the conclusions. Indeed, the expression found for $c_3(2\pi)$ is the same as the expression given in [9] for the coefficient a_1 of (2), except for a positive multiple 2π.

Relationship with results of [9] is briefly discussed in the Appendix, where we shall show that the two approaches can be considered equivalent from a practical point of view. However, the method described in this section looks considerably simpler and clearer.

3. The stabilizability problem for planar affine systems.

In this section we consider single-input planar control systems of the form

$$\begin{cases} \dot{x} = -y + a_{001}u + \phi(x,y,u) \\ \dot{y} = x + b_{001}u + \psi(x,y,u) \end{cases} \tag{13}$$

where u is the control and

$$\phi(x,y,u) = \sum_{n=2}^{\infty} \left[\sum_{i+j+k=n} a_{ijk} x^i y^j u^k \right] \tag{14.1}$$

$$\psi(x,y,u) = \sum_{n=2}^{\infty} \left[\sum_{i+j+k=n} b_{ijk} x^i y^j u^k \right] . \tag{14.2}$$

A smooth function $u = u(x,y)$ is called a stabilizing feedback if the origin turns out to be a (locally) asymptotically stable point for the differential system (13), with $u = u(x,y)$. The stabilizability problem consists in finding sufficient (and/or necessary) conditions for the existence of such a feedback.

We note that if $a_{001}^2 + b_{001}^2 \neq 0$, a well known result implies the existence of a linear stabilizing feedback, that is, a feedback of the form

$$u(x,y) = px + qy. \tag{15}$$

Based on the approach described in Sect. 2, we next state some sufficient conditions for the existence of stabilizing feedbacks of the form (15) also in the critical case where $a_{001} = b_{001} = 0$. We limit ourselves to the so-called affine systems, whose importance in applications is well known. Referring to (14), a system is affine when $a_{ijk} = b_{ijk} = 0$ for each $k > 1$.

Let us think of p and q as real indeterminates. Substituting (15) in (13), a system of the form (1) is obtained. Of course, the coefficients A_{ij} and B_{ij} of (3) are now polynomial expressions of p, q and the coefficients a_{ijk}, b_{ijk} of (14). Stabilizability results can be now deduced from Proposition 2.1 and 2.2, provided that the term $c_3(2\pi)$ has been rewritten as a function of p, q and the a_{ijk}'s and b_{ijk}'s. To this purpose, symbolic computation can be again very useful.

PROPOSITION 3.1. *Let us consider a system of the form (1), obtained by substituting (15) in (13). Let us assume* $a_{001} = b_{001} = 0$ *and* $a_{ijk} = b_{ijk} = 0$ *for each* $k > 1$. *Then,* $c_3(2\pi)$ *is a quadratic expression of the real indeterminates* p *and* q. *More precisely,* $c_3(2\pi) = 2\pi H(p,q)$ *where*

$$H(p,q) = H_{20}p^2 + H_{02}q^2 + H_{11}pq + H_{10}p + H_{01}q + H_{00} \tag{16}$$

and

$$\begin{aligned}
H_{00} &= a_{120} + 3a_{300} + b_{210} + 3b_{030} + a_{200}a_{110} - 2a_{200}b_{200} \\
&\quad + a_{110}a_{020} + 2a_{020}b_{020} - b_{110}b_{200} - b_{110}b_{020} \\
H_{10} &= a_{021} + 3a_{201} + b_{111} + a_{101}a_{110} - 2a_{101}b_{200} + a_{200}a_{011} \\
&\quad - 2a_{200}b_{101} + a_{011}a_{020} - b_{011}b_{200} - b_{011}b_{020} - b_{101}b_{110} \\
H_{01} &= a_{111} + 3b_{021} + b_{201} + a_{101}a_{200} + a_{101}a_{020} + a_{011}a_{110} \\
&\quad + 2a_{011}b_{020} + 2a_{020}b_{011} - b_{011}b_{110} - b_{101}b_{200} - b_{101}b_{020} \\
H_{20} &= a_{101}a_{011} - 2a_{101}b_{101} - b_{011}b_{101} \\
H_{11} &= a_{101}^2 + a_{011}^2 - b_{011}^2 - b_{101}^2 \\
H_{02} &= a_{101}a_{011} + 2a_{011}b_{011} - b_{011}b_{101}
\end{aligned}$$

■

We are now able to state some sufficient conditions of stabilizability.

THEOREM 3.2. *Let us consider a control system of the form (13), with* $a_{001} = b_{001} = 0$ *and* $a_{ijk} = b_{ijk} = 0$ *for* $k > 1$. *There exists a stabilizing feedback of the form (15) if at least one of the following conditions holds:*

(i) $H_{20} < 0$ *or* $H_{02} < 0$;

(ii) $H_{20} = H_{02} = 0$ *but* $H_{11} \neq 0$;

(iii) $H_{20} = H_{11} = H_{02} = 0$, *but* $H_{10} \neq 0$ *(or* $H_{01} \neq 0$*)*;

(iv) $H_{20} = H_{11} = H_{02} = H_{10} = H_{01} = 0$ *but* $H_{00} < 0$;

(v) $H_{20} \geq 0$, $H_{02} \geq 0$ *and* $4H_{20}H_{02} < H_{11}^2$;

(vi) $H_{20} \geq 0$, $H_{02} \geq 0$, $4H_{20}H_{02} > H_{11}^2$ *but*

$$H_{00} < \frac{H_{20}H_{01}^2 + H_{02}H_{10}^2 - H_{11}H_{10}H_{01}}{4H_{20}H_{02} - H_{11}^2}$$

(vii) $H_{20} \geq 0$, $H_{02} \geq 0$, $4H_{20}H_{02} = H_{11}^2$ *and* $H_{11}H_{01} \neq 2H_{02}H_{10}$ *(or, equivalently,* $H_{11}H_{10} \neq 2H_{20}H_{01}$*)*;

(viii) $H_{20} \geq 0$, $H_{02} \geq 0$, $4H_{20}H_{02} = H_{11}^2$, $H_{11}H_{01} = 2H_{02}H_{10}$ *and* $4H_{00}H_{02} < H_{01}^2$.

Proof. The conclusion follows immediately under one of the assumptions (i), (ii), (iii) or (iv), taking into account Propositions 2.1 and 3.1. Let therefore H_{20} and H_{02} be nonnegative, but not both zero. To study the sign of $H(p,q)$ we can now look at its critical points. Note that if $4H_{20}H_{02} \neq H_{11}^2$, then $H(p,q)$ has a unique critical point (p_0, q_0), which may be a minimum or a saddle. In particular, if $4H_{20}H_{02} < H_{11}^2$, then (p_0, q_0) is a saddle. Since the critical point is unique, then $H(p,q)$ must take negative values, for suitable p and q.

If $4H_{20}H_{02} > H_{11}^2$, then (p_0, q_0) is actually a minimum; so, there exists a stabilizing feedback of the form (15) if and only if $H(p,q)$ takes a negative value for $p = p_0$ and $q = q_0$. A little computation shows that

$$H(p_0, q_0) = H_{00} - \frac{H_{02}H_{10}^2 + H_{20}H_{01}^2 - H_{11}H_{10}H_{01}}{4H_{20}H_{02} - H_{11}^2}$$

Hence, the conclusion follows also under the assumption (v) or (vi). In the case (vii), there is no critical points. So, $H(p,q)$ must change sign.

Finally, in the case (viii) there is a straight line of minima. Along this line, we have

$$H(p,q) \equiv H_{00} - \frac{H_{01}^2}{4H_{02}}$$

(it is not restrictive to assume $H_{02} \neq 0$). Hence, there is a linear stabilizing feedback if this value is negative. ■

With the same arguments used in the proof of Theorem 3.2, it is possible to find out the cases where a linear stabilizing feedback does not exist, i.e. the cases where $H(p,q) > 0$ for each pair $(p,q) \in \mathbf{R}^2$. This analysis does not exhaust all the possible situations. It may happen that $H(p,q)$ is nonnegative and vanishes for some pair $(p,q) \neq (0,0)$. When this occurs, it is not possible to decide about linear stabilizability by looking at $c_3(2\pi)$; we need to compute higher order coefficient $c_i(2\pi)$ and study their signs for those values of (p,q) such that $H(p,q) = 0$. In these cases we say that the system is highly critical; an important example is treated in the next section.

4. Stabilizability of some highly critical bilinear systems.

In this section we consider single-input planar oscillatory bilinear systems, that is, systems of the form

$$(17) \qquad \begin{cases} \dot{x} = -y + u(a_{101}x + a_{011}y + a_{001}) \\ \dot{y} = x + u(b_{101}x + b_{011}y + b_{001}) \end{cases} .$$

As before, we assume $a_{001} = b_{001} = 0$, otherwise the problem is trivial. We seek again stabilizing feedbacks of the form (15). The expression of $H(p,q)$ given in Proposition 3.1 now simplifies to

$$H(p,q) = H_{20}p^2 + H_{11}pq + H_{02}q^2.$$

In view of the analysis which will be carried out in Sect. 5, we focus our attention on the case where

$$a_{101} = -b_{011} = a \quad \text{and} \quad a_{011} = b_{101} = b \quad .$$

Putting $u = px + qy$, the system becomes

$$(18) \qquad \begin{cases} \dot{x} = -y + (px + qy)(ax + by) \\ \dot{y} = x + (px + qy)(bx - ay) \end{cases} .$$

It is easily seen that, in this case, $H_{20} = H_{11} = H_{02} = 0$. Hence, the existence of a linear feedback cannot be decided by invoking

Theorem 3.2. Nevertheless, we prove that for a suitable choice of p and q, the origin is an asymptotically stable point of (18). Since (18) is considerably simplified with respect to the general form (1), it is convenient to restart application of our method from the beginning, instead of using formulas of Sect. 2. To this purpose, let

$$\Phi(\theta) = (a\cos 2\theta + b\sin 2\theta)(p\cos\theta + q\sin\theta)$$
$$\Psi(\theta) = (b\cos 2\theta - a\sin 2\theta)(p\cos\theta + q\sin\theta) \quad .$$

With this notation, (6) and (8) can be rewritten as

$$\text{(6')} \qquad \frac{d\rho}{d\theta} = \rho^2\Phi(\theta) - \rho^3\Phi(\theta)\Psi(\theta) + \rho^4\Phi(\theta)\Psi^2(\theta) - \rho^5\Phi(\theta)\Psi^3(\theta) + \ldots$$

$$\text{(8')} \qquad \begin{aligned} \frac{d\rho}{d\theta} &= c_1^2(\theta)\Phi(\theta)\rho_0^2 + [2c_1(\theta)c_2(\theta)\Phi(\theta) - c_1^3(\theta)\Phi(\theta)\Psi(\theta)]\rho_0^3 \\ &+ [c_2^2(\theta)\Phi(\theta) + 2c_1(\theta)c_3(\theta)\Phi(\theta) \\ &- 3c_1(\theta)c_2(\theta)\Phi(\theta)\Psi(\theta) + c_1^4(\theta)\Phi(\theta)\Psi^2(\theta)]\rho_0^4 \\ &+ [2c_1(\theta)c_4(\theta)\Phi(\theta) + 2c_2(\theta)c_3(\theta)\Phi(\theta) \\ &- 3c_1^2(\theta)c_3(\theta)\Phi(\theta)\Psi(\theta) - 3c_1(\theta)c_2^2(\theta)\Phi(\theta)\Psi(\theta) \\ &+ 4c_1^3(\theta)c_2(\theta)\Phi(\theta)\Psi^2(\theta) - c_1^5(\theta)\Phi(\theta)\Psi^3(\theta)]\rho_0^5 + \cdots \end{aligned}$$

where 5^{th} order terms have been included. According to (8') and (9), a simplified system of infinite differential equations is obtained. Using the integration by parts rule, new expression for the $c_i(2\pi)$'s can be found.

$$c_2(2\pi) = \int_0^{2\pi} \Phi(\theta)\, d\theta$$

$$c_3(2\pi) = \left[\int_0^{2\pi} \Phi(\theta)\, d\theta\right]^2 - \int_0^{2\pi} \Phi(\theta)\Psi(\theta)\, d\theta$$

$$\begin{aligned} c_4(2\pi) = \left[\int_0^{2\pi} \Phi(\theta)\, d\theta\right]^3 &- 2\int_0^{2\pi} \Phi(\theta)\, d\theta \int_0^{2\pi} \Phi(\theta)\Psi(\theta)\, d\theta \\ &- \int_0^{2\pi} (\Phi(\theta)\Psi(\theta) \int_0^{\theta} \Phi(s)\, ds)\, d\theta + \int_0^{2\pi} \Phi(\theta)\Psi^2(\theta)\, d\theta \end{aligned}$$

$$
\begin{aligned}
c_5(2\pi) = & \left[\int_0^{2\pi} \Phi(\theta)\, d\theta\right]^4 + \frac{3}{2}\left[\int_0^{2\pi} \Phi(\theta)\Psi(\theta)\, d\theta\right]^2 \\
& - 3\left[\int_0^{2\pi} \Phi(\theta)\, d\theta\right]^2 \int_0^{2\pi} \Phi(\theta)\Psi(\theta)\, d\theta \\
& - 2\int_0^{2\pi} \Phi(\theta)\, d\theta \int_0^{2\pi} (\Phi(\theta)\Psi(\theta) \int_0^{\theta} \Phi(s)\, ds)\, d\theta \\
& + 2\int_0^{2\pi} \Phi(\theta)\, d\theta \int_0^{2\pi} \Phi(\theta)\Psi^2(\theta)\, d\theta \\
& - \int_0^{2\pi} \Phi(\theta)\Psi(\theta)(\int_0^{\theta} \Phi(s)\, ds)^2\, d\theta \\
& + 2\int_0^{2\pi} (\Phi(\theta)\Psi^2(\theta) \int_0^{\theta} \Phi(s)\, ds)\, d\theta - \int_0^{2\pi} \Phi(\theta)\Psi^3(\theta)\, d\theta \quad .
\end{aligned}
$$

We already know that $c_2(2\pi) = c_3(2\pi) = 0$. To compute the remaining integrals, we need once again the aid of a symbolic manipulator package. It is possible to verify that

$$
\begin{aligned}
\int_0^{2\pi} \Phi(\theta)\Psi(\theta)\, d\theta &= \int_0^{2\pi} (\Phi(\theta)\Psi(\theta) \int_0^{\theta} \Phi(s)\, ds)\, d\theta \\
&= \int_0^{2\pi} \Phi(\theta)\Psi^2(\theta)\, d\theta = 0 \quad .
\end{aligned}
$$

Hence, $c_4(2\pi) = 0$ and

$$
\begin{aligned}
c_5(2\pi) = & \, 2\int_0^{2\pi} (\Phi(\theta)\Psi^2(\theta) \int_0^{\theta} \Phi(s)\, ds)\, d\theta \\
& - \int_0^{2\pi} \Phi(\theta)\Psi(\theta)(\int_0^{\theta} \Phi(s)\, ds)^2\, d\theta \\
& - \int_0^{2\pi} \Phi(\theta)\Psi^3(\theta)\, d\theta \qquad (19) \\
& = -\frac{\pi}{6}[(ab^3 + a^3b)(p^4 + q^4) \\
& + 2(a^4 - b^4)(pq^3 - p^3q) - 6(ab^3 + a^3b)p^2q^2] \quad .
\end{aligned}
$$

THEOREM 4.1. *Let a and b be such that $a^2 + b^2 \neq 0$. Then, there exists a pair of real numbers p and q such that the origin is an* *asymptotically stable point for system (18).*

Proof. We distinguish several cases, depending on the sign of the product ab. If $ab > 0$, it can be chosen $p = 0$ and $q = 1$. Then (18) implies

$$c_5(2\pi) = -\frac{\pi}{6}ab(a^2 + b^2) < 0 \quad ,$$

and the conclusion follows by Proposition 2.1, since $c_2(2\pi) = c_3(2\pi) = c_4(2\pi) = 0$. If $ab < 0$, the choice $p = q$ works. Indeed, in this case

$$c_5(2\pi) = \frac{2\pi}{3}p^4 ab(a^2 + b^2) < 0 \quad .$$

If $a = 0$, $b \neq 0$, then according to (18),

$$c_5(2\pi) = \frac{\pi}{3}b^4 pq(q^2 - p^2) \quad .$$

It can be made negative taking any pair (p, q) such that $|q| > |p|$ and $pq < 0$. Finally, if $a \neq 0$, $b = 0$, we have

$$c_5(2\pi) = -\frac{\pi}{3}a^4 pq(q^2 - p^2)$$

which is negative for $|q| > |p|$ and $pq > 0$. ■

5. Bilinear systems with constant or linear controls.

Here we consider again single-input planar bilinear systems of the form (17). In the previous section, we restricted our attention to linear control functions of the form (15). Now, we want to remove this restriction, allowing also constant feedback laws.

THEOREM 5.1. *Any system of the form (17) can be stabilized by means of a linear or constant feedback, except for the case where*

$$a_{001} = b_{001} = a_{101} = b_{011} = 0, \quad a_{011} = -b_{101} \tag{20}$$

When (20) holds, the system is not stabilizable at all.

Proof. We already noted that if $a_{001}^2 + b_{001}^2 \neq 0$ there exists a linear stabilizing feedback. So, we can assume that $a_{001} = b_{001} = 0$.

First we check the existence of constant stabilizing feedbacks. Note that if u is constant, then (17) is a linear process. Thus, eigenvalues can be directly computed by solving the equation

$$\lambda^2 - \lambda u(a_{101} + b_{011}) + u^2(a_{101}b_{011} - a_{011}b_{101}) + u(b_{101} - a_{011}) + 1 = 0 \quad .$$

We conclude that a constant feedback exists provided that $a_{101} + b_{011} \neq 0$. Indeed, we can choose any sufficiently small value u such that $u(a_{101} + b_{011}) < 0$.

Let therefore be $a_{101} = -b_{011} = a$. The expressions of H_{20}, H_{11} and H_{02} given in Sect. 3 simplify now to

$$H_{20} = a(a_{011} - b_{101})$$

$$H_{11} = a_{011}^2 - b_{101}^2$$

$$H_{02} = a(b_{101} - a_{011}) \quad .$$

If $H_{20} \neq 0$, then also H_{02} is not zero and has opposite sign. Thus, a linear feedback exists by Theorem 3.1(i).

The last case to be considered is when $H_{20} = H_{02} = 0$. If $a \neq 0$, then $a_{011} = b_{101}$ and Theorem 4.1 applies.

Let therefore assume $a = 0$; when the condition $|a_{011}| \neq |b_{101}|$ holds, we have $H_{11} \neq 0$ and a linear feedback exists by virtue of Theorem 3.1(ii). Using again Theorem 4.1, the existence of a linear feedback can be proved also when $a_{011} = b_{101} \neq 0$.

Only two cases are still missing: the trivial one, when all the coefficients vanish, and when condition (20) holds. Here it is easy to see that the system is not stabilizable at all; indeed, in polar coordinates the system takes the form $\dot{\rho} = 0$, $\dot{\theta} = 1 + ub_{101}$. The statement is so proved. ■

Remark 5.2. The previous proof requires the application of Theorem 4.1. This is a motivation for the analysis carried out in Sect. 4.

Remark 5.3. According to the Jurdjevic and Quinn theory ([11]), stabilizability of system (17) can be achieved also by means of the quadratic control law

$$u(x, y) = a_{101}x^2 + (a_{011} + b_{101})xy + a_{011}y^2$$

provided that (20) is not satisfied.

6. Appendix.

In this appendix we briefly indicate the relationships between the approach of [9] and the method described in Sect. 2. First of all we recall the following basic result about normal form expansions.

Let us consider a system of the form (1), with f and g given by (3.1), (3.2). Then, for each positive N there exists an analytic change of coordinates

$$\begin{cases} x = \xi + \dots \\ y = \eta + \dots \end{cases} \tag{21}$$

transforming (1) to

$$\begin{cases} \dot{\xi} = -\eta + \sum_{i=1}^{N} (\xi^2 + \eta^2)^i (a_i \xi - b_i \eta) + o(2N+1) \\ \dot{\eta} = \xi + \sum_{i=1}^{N} (\xi^2 + \eta^2)^i (a_i \eta + b_i \xi) + o(2N+1) \end{cases}$$

where $o(2N+1)$ denotes analytic functions of ξ and η of order strictly greater than $2N+1$.

Note that the Jacobian matrix at the origin of the change of coordinates coincides with the identity matrix. Using polar coordinates (r, ω), the system takes now the form (2). Starting from it, and repeating the procedure of Sect. 2, we obtain the equation

$$\frac{dr}{d\omega} = a_1 r^3 + (a_2 - b_1 a_1) r^5 + \dots \quad . \tag{22}$$

that represents the same differential equation as (6) in the new system of polar coordinates.

Let $r = r(\omega, r_0) = d_1(\omega) r_0 + d_2(\omega) r_0^2 + \dots$ be the solution of (22) such that $r(0, r_0) = r_0$. Then, it is easily seen that

$$d_2(2\pi) = 0$$

$$d_3(2\pi) = 2\pi a_1$$

$$d_4(2\pi) = 0$$

$$d_5(2\pi) = 6\pi^2 a_1^2 + 2\pi a_2 - 2\pi b_1 a_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The expression of $r(2\pi, r_0)$ has to be compared with the expression of $\rho(2\pi, \rho_0)$ found in Sect. 2. To this purpose, it should be noted that the two systems of polar coordinates are related by means of (21). In particular, it can be verified that

$$\rho = r + \alpha_2(\theta)r^2 + \alpha_3(\theta)r^3 + \ldots$$

for some functions $\alpha_2(\theta)$, $\alpha_3(\theta)$,... . This last expression with $\theta = 2\pi$ can be used to find the relation between the coefficients $c_i(2\pi)$'s and $d_i(2\pi)$'s. In particular, it is an exercise from elementary calculus to verify that if $d_2(2\pi) = \ldots = d_{m-1}(2\pi) = 0$, and $d_m(2\pi) \neq 0$, then $c_2(2\pi) = \ldots = c_{m-1}(2\pi) = 0$ and $c_m(2\pi) = d_m(2\pi)$. Taking into account the expression of $d_3(2\pi)$ found above, the numerical outcome pointed out in Remark 2.3 is now explained. Moreover, in the highly critical case where $a_1 = 0$, we see that $c_5(2\pi) = d_5(2\pi) = 2\pi a_2$. In general, it can be seen that if a_m is the first non zero term in (2), then $c_{2m+1}(2\pi) = d_{2m+1}(2\pi) = 2\pi a_m$, while $c_i(2\pi) = d_i(2\pi) = 0$ for each $i = 2, 3, \ldots, 2m$. This shows the practical equivalence of the method of Sect. 2 and the normal form expansion approach.

In a similar manner, relationships with the Liapunov function method of [8] can be explored. Indeed, consider the system in the normal form (2), and take $V(x,y) = \frac{1}{2}(x^2 + y^2)$. It is easy to see that if a_m is, as above, the first nonvanishing coefficient in (2), then $\gamma_2 = \gamma_4 = \ldots = \gamma_{2m} = 0$ and $\gamma_{2m+2} = a_m$. This is confirmed by the expression of γ_4 found in [8] in the cubic case, which agrees with the expression of a_1 given in [9].

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