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A FURTHER RESTRICTION IN RELATIVISTIC EXTENDED THERMODYNAMICS

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The best result on relativistic thermodynamics of these last years is the partial differential equations system (henceforth called PDE) that has been found to describe it by Liu, Müller and Ruggeri. This system is determined except for a single variable function A. here it is imposed that this system is hyperbolic and that the shock speeds do not exceed the speed of light c, finding some necessary and sufficient conditions to this end (Liu, Müller and Ruggeri have imposed only some necessary conditions to this regard); the compatibility of these conditions is then studied obtaining as result that the fucntion A must be non positive.

Introduction.

Relatavistic extended thermodynamics is a theory where the 14 independent components of the particle number – particle flux vector V^α and the stress – energy momentum symmetric tensor $T^{\alpha\beta}$ may be determined from the balance equations

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$$(1.1)\qquad \begin{cases} V^{\alpha}_{,\alpha} = 0 & \text{- conservation of particle number} \\ T^{\alpha\beta}_{,\alpha} = 0 & \text{- conservation of energy - momentum} \\ A^{\beta\gamma\alpha}_{,\alpha} - I^{\beta\gamma} = 0 & \text{- balance equation for the fluxes} \end{cases}$$

where $A^{\beta\gamma\alpha}$, $I^{\beta\gamma}$ are the following explicitly expressed functions of V^{α}, $T^{\alpha\beta}$:

$$(1.2)\qquad A^{\alpha\beta\gamma} = (C_1^0 + C_1^{\pi}\pi)u^{\alpha}u^{\beta}u^{\gamma} + (c^2/2)(nm^2 - C_1^0 - C_1^{\pi}\pi)g^{(\alpha\beta}u^{\gamma)} + \\ +3C_3(g^{(\alpha\beta} - 6c^{-2}u^{(\alpha}u^{\beta})q^{\gamma)} + 3C_5 t^{(\langle\alpha\beta\rangle}u^{\gamma)},$$

$$(1.3)\qquad I^{\alpha\beta} = B_1^{\pi}\pi(g^{\alpha\beta} - 4c^{-2}u^{\alpha}u^{\beta}) + B_3 t^{\langle\alpha\beta\rangle} + 2B_4 q^{(\alpha}u^{\beta)}$$

with

$$g_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$$

the metric tensor,

$$(1.4)\qquad \begin{aligned} n &= C^{-1}(V_{\beta}V^{\beta})^{1/2}; \\ u^{\alpha} &= c(V_{\beta}V^{\beta})^{-1/2}V^{\alpha} \\ h^{\alpha\beta} &= c^{-2}u^{\alpha}u^{\beta} - g^{\alpha\beta}; \\ t^{\langle\alpha\beta\rangle} &= \left(h^{\alpha}_{\mu}h^{\beta}_{\nu} - \frac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right)T^{\mu\nu}; \\ q^{\alpha} &= -h^{\alpha}_{\mu}u_{\nu}T^{\mu\nu}; \\ e &= c^{-2}u_{\mu}u_{\nu}T^{\mu\nu}; \\ \pi &= \frac{1}{3}h_{\mu\nu}T^{\mu\nu} - p(n,e), \end{aligned}$$

m is the particle mass, $p(n,e)$ is the value of $\frac{1}{3}\ h_{\mu\nu}T^{\mu\nu}$ in thermodynamical equilibrium as function of n, e and all coefficients $C_1^0, C_1^{\pi}, C_3, C_5, B_1^{\pi}, B_3, B_4$ are functions of n, e and the first four of them are determined except for a single variable function A.

The expressions (1.2), (1.3) are found by Liu, Müller and Ruggeri (henceforth referred as LMR) in ref [9] by imposing the relativity and entropy principles; to this end they use another set of independent variables; the so called Lagrange multipliers that compels to impose also the symmetry of $T^{\alpha\beta}$ and $A^{\alpha\beta}$ (henceforth referred as "the symmetry conditions") and the condition $A^{\alpha\beta}_{\beta} = m^2C^2V^\alpha$ (the trace condition) that come from kinetic considerations.

LMR impose exactly the entropy principle but only approximatively the symmetry and the trace conditions, i.e. in a linear departure with respect to thermodynamical equilibrium; after that they return to the suggestive variables V^α, $T^{\alpha\beta}$ and obtain the expression of $A^{\alpha\beta\gamma}$ at first order with respect to thermodynamical equilibrium.

A different procedure that avoids all these twists may be that of always remaining with the old variables $(V^\alpha, T^{\alpha\beta})$, keeping up the symmetry and the trace conditions exactly verified and imposing the entropy principle in a linear departure from equilibrium. This procedure is followed in appendix A for the sake of completeness; from the result it can be seen that the two procedures are not equivalent; in fact in appendix A all coefficients of (1.2) are determined except for the two variables function $C_3(n,e)$ while in LMR's approach this same function is determined except for a single – variable function A. Obviously this restriction on C_3 may be obtained with the approach of this paper by imposing the entropy principle also in orders greater than 1 with respect to equilibrium.

This fact shows that coefficients appearing in a certain order ν may be influenced by the equations studied in order greater than ν; therefore the suspicion arises that, if one studies the equations in a sufficiently great order, restrictions may be found for the terms linear with respect to equilibrium such to deprive them of physical significance. The attempt to eliminate this suspicion may be the subject for future works.

The entropy principle states that a vector – valued function $h^\alpha(V^\beta, T^{\gamma\delta})$ exists, such that

(1.5) $h^\alpha_{,\alpha} \geq 0$ holds for all solutions of eqs. (1.1).

As shown by Liu in ref [8] and already used by many authors [6,7,9], condition (1.5) is equivalent to assume the existence of

Lagrange multipliers ξ, Ω_β, $\Sigma_{\beta\gamma}$ such that

$$h^\alpha_{,\alpha} + \xi V^\alpha_{,\alpha} + \Omega_\beta T^{\beta\alpha}_{,\alpha} + \Sigma_{\beta\gamma}(A^{\beta\gamma\alpha}_{,\alpha} - l^{\beta\gamma}) \geq 0 \tag{1.6}$$

holds $\forall V^\alpha$, $T^{\alpha\beta}$, or, equivalently,

$$h'^\alpha_{,\alpha} - V^\alpha \xi_{,\alpha} - T^{\beta\alpha}\Omega_{\beta,\alpha} - A^{\beta\gamma\alpha}\Sigma_{\beta\gamma,\alpha} - \Sigma_{\beta\gamma}I^{\beta\gamma} \leq 0 \tag{1.7}$$

where

$$h'^\alpha = h^\alpha + \xi V^\alpha + \Omega_\beta T^{\beta\alpha} + \Sigma_{\beta\gamma}A^{\beta\gamma\alpha}. \tag{1.8}$$

Now the expressions of ξ, Ω_β, $\Sigma_{\beta\gamma}$ found here, or in [9], are invertible fucntions of V^α, $T^{\alpha\beta}$; therefore it can be adopted an idea conceived by other outhors [3, 13, 15] to take them as independent variables so that (1.7) becomes

$$\begin{aligned} V^\alpha &= \frac{\partial h'^\alpha}{\partial \xi}, \quad T^{\beta\alpha} = \frac{\partial h'^\alpha}{\partial \Omega_\beta}, \\ A^{\beta\gamma\alpha} &= \frac{\partial h'^\alpha}{\partial \Sigma_{\beta\gamma}}, \ \Sigma_{\beta\gamma}I^{\beta\gamma} \leq 0 \end{aligned} \tag{1.9}$$

and consequently the PDE (1.1) assumes the symmetric conservative form. Then hyperbolicity holds in the time – like direction ξ_α iff

$$\left\| \frac{\partial^2 h'^\alpha}{\partial X_A \partial X_B}\xi_\alpha \right\| \text{ is positive definite.} \tag{1.10}$$

Where X_A are the independent components of ξ, Ω_μ, $\Sigma_{\beta\gamma}$. LMR have imposed only some necessary conditions that come from the convexity condition (1.10) in a neighbourhood of equilibrium and with $\xi_\alpha = u_\alpha$.

In this paper, sect. II, I obtain necessary and sufficient conditions assuring the validity of property (1.10) for every four – vector ξ_α such that $\xi_\alpha \xi^\alpha = c^2$, in order that (see ref [16]) the shock speeds do not exceed the speed of light.

In sect. III it is proved that these conditions imply $A \leq 0$, where A is the single variable fucntion appearing in the expressions of C_1^0, C_1^π, C_3, C_5; moreover they imply that relativistic extended thermodynamics can be applied only for values of temperature satisfying certain relations; for example these ones surely hold for sufficiently little values of temperature as in the application of ref [1].

In sect. IV are drawn conclusions.

2. convexity of the function $\xi_\alpha h'^\alpha$.

Let us call y_A the independent components of V^α, $T^{\alpha\beta}$. From relations (1.8) (1.9) we obtain

$$h^\alpha = h'^\alpha - X_A \frac{\partial h'^\alpha}{\partial X_A}$$

and

$$\frac{\partial h^\alpha}{\partial Y_p \partial Y_Q} \delta Y_p \delta Y_Q = -\frac{\partial^2 h'^\alpha}{\partial X_A \partial X_B} \delta X_A \delta X_B - X_A \frac{\partial^3 h'^\alpha}{\partial X_A \partial Y_p \partial_Q} \delta Y_p \delta Y_Q;$$

but for $X_A = \xi$, we have

$$-X_A \frac{\partial^3 h'^\alpha}{\partial X_A \partial Y_p \partial Y_Q} = -\xi \frac{\partial^2 V^\alpha}{\partial Y_p \partial Y_Q} = 0;$$

for $X_A = \Omega_\mu$, we have

$$-X_A \frac{\partial^3 h'^\alpha}{\partial X_A \partial Y_p \partial Y_Q} = -\Omega_\mu \frac{\partial^2 T^{\alpha\mu}}{\partial Y_p \partial Y_Q} = 0;$$

for $X_A = \Sigma_{\mu\nu}$, we have

$$-X_A \frac{\partial^3 h'^\alpha}{\partial X_A \partial Y_p \partial Y_Q} = -\Sigma_{\mu\nu} \frac{\partial^2 A^{\mu\nu\alpha}}{\partial Y_p \partial Y_Q}$$

that is null at equilibrium; then

$$Q = \xi_\alpha \frac{\partial^2 h'^\alpha}{\partial X_A \partial X_B} \delta X_A \delta X_B = -\xi_\alpha \frac{\partial^2 h^\alpha}{\partial Y_p \partial Y_Q} \delta Y_p \delta Y_Q \tag{2.1}$$

at equilibrium. As consequence, the statement (1.10) holds iff the quadratic form Q is positive definite. This condition is satisfied for $\xi_\alpha = u_\alpha$ iff

$$A_1^t < 0; \; 2c^2(e+p)TA_1^q > 1; \; A_1^{\pi^2} < 0 \tag{2.2}$$

hold.

It is satisfied for every timelike ξ_α iff also

$$-2c^2 A_1^q - 2A_3^0 - (e+p)(A_3^0)^2 T - 4A_1^t[2c^2(e+p)TA_1^q - 1] \geq 0 \tag{2.3}$$

$$(3A_1^t + 2A_1^{\pi^2})e_T - 6TA_1^t A_1^{\pi^2}[T(p_T)^2 + ne_T p_n] + 6TA_1^t A_1^{\pi^2} e_T(e+p) \geq 0 \tag{2.4}$$

$$A_7 = \alpha\eta^2 + \beta\eta + \delta > 0 \; A\eta \in [0, c^2[\tag{2.5}$$

hod. (See appendix B for the mathematical details). The expressions of A_1^t, A_1^q, $A_1^{\pi^2}$, A_3^0, α, β, δ are presented below in eqs. (B.3)-(B.8), (B.13)-(B.15).

LMR have considered the condition (1.10) only for $\xi_\alpha = u_\alpha$ and then they would have found the conditions (2.2); but they impose only a necessary condition to obtain (1.10) with $\xi_\alpha = u_\alpha$, i.e. that the elements of the principal diagonal of the matrix associated to Q are positive; thus they obtain $A_1^t < 0$; $A_1^q > 0$; $A_1^{\pi^2} < 0$ (see ref [10] that corrects ref [9] on this regard). Here these ones are substituted by the more restrictive eqs. (2.2) that assure the principal minors of the above matrix to be positive; then they are necessary and sufficient conditions for (1.10) with $\xi_\alpha = u_\alpha$.

Now in ref [16] and [14] has been proved that (1.10) $\forall$ timelike ξ_α holds iff it is verified for $\xi_\alpha = u_\alpha$ and moreover the characteristic velocities are less than c; in ref [4] Boillat has found some algebraic equations for these characteristic velocities.

Consequently by imposing that their solutions are less than c one obtains the conditions (2.3)-(2.5) with an equivalent procedure.

In the next section the conditions (2.2)-(2.5) are investigated for little values of temperature.

3. Compatibility of conditions (2.2)-(2.5).

Let us firstly prove that as consequence of these conditions one has $A \leq 0$ which allows the more restrictive theories where $A = 0$ (see ref [5] as example) to remain compatible with LMR's one.

Now in the non-degenerate case we have

$$e = nmc^2(G - 1/\gamma);\ p = nmc^2/\gamma \tag{3.1}$$

and for little values of $\frac{1}{\gamma}$ the following development holds

$$\begin{aligned} G(\gamma) = 1 + \frac{5}{2}\frac{1}{\gamma} + \frac{15}{8}\frac{1}{\gamma^2} - \frac{15}{8}\frac{1}{\gamma^3} + \frac{135}{2^7}\frac{1}{\gamma^4} + \\ + \frac{45}{2^5}\frac{1}{\gamma^5} + \frac{1}{\gamma^6}g_1(\gamma) \end{aligned} \tag{3.2}$$

with

$$\lim_{1/\gamma \to 0} g_1(\gamma) = \bar{g}_1;$$

(henceforth other functions $g_i(\gamma)$ will be introduced and it will be understood that they are such that $\lim_{1/\gamma \to 0} g_i(\gamma) = \bar{g}_i$).

Consequently we have

$$\begin{aligned} e = nmc^2\Bigg[1 + \frac{3}{2}\frac{1}{\gamma} + \frac{15}{8}\frac{1}{\gamma^2} - \frac{15}{8}\frac{1}{\gamma^3} + \frac{135}{2^7}\frac{1}{\gamma^4} + \\ \frac{45}{2^5}\frac{1}{\gamma^5} + \frac{1}{\gamma^6}g_1(\gamma)\Bigg] \end{aligned} \tag{3.3}$$

$$e_T = nk\left[\frac{3}{2} + \frac{15}{4}\frac{1}{\gamma} - \frac{45}{8}\frac{1}{\gamma^2} + \frac{135}{2^5}\frac{1}{\gamma^3} + \frac{1}{\gamma^4}g_2(\gamma)\right] \tag{3.4}$$

$$p_n = \frac{mc^2}{\gamma} > 0;\ p_T = nk.$$

Now the condition $(2.2)_1$ can be written as

$$\frac{AK_2(\gamma)}{n\gamma} < \frac{15}{m^3c^3K}\gamma^4 K_2(\gamma)\left(1 + \frac{6}{\gamma}G\right);$$

if we consider $\frac{K_2(\gamma)}{n\gamma}$ and γ as independeet variables and take the limit of this relation for $\frac{1}{\gamma} \to 0$, the first member remains unchanged (because it does not depend on $\frac{1}{\gamma}$), while the second member becomes zero; therefore $A \leq 0$ holds.

Another possible consideration is that LMR's theory may be not applicable for all values of T; in fact $e_T > 0$ is a necessary condition for the convexity and from the above expression it can be seen that it may be not verified for all values of γ. However the field of applicability of the theory is not the empty set; in fact

$$\lim_{1/\gamma \to 0} e_T = \frac{3}{2} n_k > 0$$

and then $e_T > 0$ for little values of T.

We can see now that all the conditions (2.2)-(2.5) are satisfied for sufficiently little values of T.

In fact from $A \leq 0$ we can see that $(2.2)_1$ is verified for all values of n, T. Moreover all the other first members of (2.2)-(2.5) may be developed in growing powers of $\frac{1}{\gamma}$; the corresponding expressions are presented in appendix C, for simplicity. By taking their limits for $\frac{1}{\gamma} \to 0$, we can see that (2.2)-(2.5) hold at least for little values of temperature.

The property $A \leq 0$ and the verification of the conditions (2.2) (2.5) have been proved in the non-degenerate case; however this case is obtained from the degenerate one letting $-\frac{\mu}{T}$ go to infinity, where μ is the specific Gibbs free energy or chemical potential; consequently, for the permanence sign theorem we obtain that the results of this section are verified also in the degenerate case but with sufficiently great values of $-\frac{\mu}{T}$.

4. Conclusions.

Relativistic extended thermodynamics is a theory of great beauty and significance. Even if some questions may arise, they find satisfactory answers when put under sufficient consideration as in this paper. I think that the present work contributes to improving the understanding of this theory. In fact it completely exploit the convexity requirement showing that it is verified at least for little values of temperature.

Moreover, by showing that $A \leq 0$, it draws this theory nearer to those where $A = 0$, at last it shows (in the appendix) that it is not strange the difference of this theory with that in ref [2] where the unknown fucntions are determined except for a two – variables function (and not for a single – variable fucntion as in LMR's work).

I retain that other aspects may be checked up in the future.

Appendix A. Exploitation of the entropy principle.

The entropy principle in the form (1.6) can be expressed as

$$\sum_A X_A F^{A\alpha}_{,\alpha} = 0 \ \Sigma_{\beta\gamma} I^{\beta\gamma} \leq 0 \qquad \forall V_\mu, T_{\mu\nu} \tag{A.1}$$

where

$$X_1 = 1;\ F^{1\alpha} = h^\alpha;\ X_2 = \xi,\ F^{2\alpha} = V^\alpha;\ X_{3+\beta} = \Omega_\beta;\ F^{3+\beta} = T^{\beta\alpha};$$

and for $A \geq 7$, $X_A = \Sigma_{\beta\gamma}$; $F^\alpha = A^{\beta\gamma\alpha}$.

The relation $(A.1)_1$ gives

$$\sum_A X_A \frac{\partial F^{A\alpha}}{\partial V_\mu} = 0;\ \sum_A X_A \frac{\partial F^{A\alpha}}{\partial T_{\nu\mu}} = 0. \tag{A.2}$$

If we consider the functions $F^{A\alpha}$ of V_μ, $T_{\mu\nu}$ as composite functions by means of n, e, π, u_μ, q_μ, $t_{\langle\mu\nu\rangle}$ and use the relations (B.1), that are presented below, we obtain that $(A.2)_2$ multiplied times $h_{\mu\nu}$, $u_\mu u_\nu$, u_μ gives the following eq. (A.7) and

$$\sum_A X_A \frac{\partial F^{A\alpha}}{\partial e} = 0;\ \sum_A X_A \frac{\partial F^{A\alpha}}{\partial q_\gamma} h_{\gamma\nu} = 0 \tag{A.3}$$

of which the second one can be written as the following eq. (A.9) with f_1^α given by (A.11). By using these relations $(A.2)_2$ becomes

$$\sum_A X_A \frac{\partial F^{A\alpha}}{\partial t_{\langle\gamma\delta\rangle}} \left(h_\gamma^{(\mu} h_\delta^{\nu)} - \frac{1}{3} h^{\mu\nu} h_{\gamma\delta} \right) = 0$$

that is equivalent to (A.10) with $f_1^{\alpha\beta} h_\beta^\mu$, f_2^α, f_3^α respectively given by (A.15), (A.12), (A.13).

By using all these relations, $(A.2)_1$ multiplied times u_μ gives

$$(A.4) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial n} = 0$$

and of $(A.2)_1$ it remains the relation (A.8) with f_4^α given by (A.14).

Moreover if we consider the function $\gamma = \gamma(e, n)$ and invert it to obtain $e = e(\gamma, n)$, we can consider γ and n as independent variables instead of e, n; if $F^{A\alpha}(e, n) = G^{A\alpha}[\gamma(e, n), n]$ we have

$$\begin{aligned} X_A \frac{\partial F^{A\alpha}}{\partial n} &= X_A \left(\frac{\partial G^{A\alpha}}{\partial \gamma} \frac{\partial \gamma}{\partial n} + \frac{\partial G^{A\alpha}}{\partial n} \right) \\ X_A \frac{\partial F^{A\alpha}}{\partial e} &= X_A \frac{\partial G^{A\alpha}}{\partial \gamma} \frac{\partial \gamma}{\partial e} \end{aligned}$$

and then $(A.3)_1$ and (A.4) are equivalent to (A.5) and (A.6) where the notation "$F^{A\alpha}$" instead of "$G^{A\alpha}$" has been again used.

In this way we have written the conditions (A.2) in the form (A.5)-(A.15) where the presence of f_1^α, $f_1^{\alpha\beta}$ h_β^μ, f_2^α, f_3^α, f_4^α takes into account of the fact that u^α, q^α, $t^{\langle\alpha\beta\rangle}$ are restricted by $u^\alpha q^\alpha = 0$; $u^\alpha t_{\langle\alpha\beta\rangle} = 0$; $h_{\alpha\beta} t^{\langle\alpha\beta\rangle} = 0$.

Then in conclusion of this first step we can say that the entropy principle is equivalent to the following relations (A.5)-(A.15) and to the residual inequality $(A.1)_2$.

$$(A.5) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial n} = 0$$

$$(A.6) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial \gamma} = 0$$

$$(A.7) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial \pi} = 0$$

$$(A.8) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial u^\mu} + f_4^\alpha u_\mu + t_{\langle\mu\beta\rangle} h_\delta^\beta f_1^{\delta\alpha} + q_\mu f_1^\alpha = 0$$

$$(A.9) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial q_\nu} + f_1^\alpha u^\nu = 0$$

$$(A.10) \qquad \sum_A X_A \frac{\partial F^{A\alpha}}{\partial t_{\langle \nu\mu \rangle}} + f_1^{\alpha\beta} h_\beta^{(\mu} u^{\nu)} + f_2^\alpha u^\mu u^\nu + f_3^\alpha h^{\mu\nu} = 0$$

where

$$(A.11) \qquad f_1^\alpha = -\sum_A X_A \frac{\partial F^{A\alpha}}{\partial q_\gamma} \frac{1}{c^2} u_\gamma$$

$$(A.12) \qquad f_2^\alpha = -\frac{1}{c^4} u_\gamma u_\delta \sum_A X_A \frac{\partial F^{A\alpha}}{\partial t_{\langle \gamma\delta \rangle}}$$

$$(A.13) \qquad f_3^\alpha = -\frac{1}{3} h_{\gamma\delta} \sum_A X_A \frac{\partial F^{A\alpha}}{\partial t_{\langle \gamma\delta \rangle}}$$

$$(A.14) \qquad f_4^\alpha = -\frac{1}{c^2} u^\gamma \sum_A X_A \frac{\partial F^{A\alpha}}{\partial u^\gamma}$$

$$(A.15) \qquad f_1^{\alpha\mu} h_\mu^\beta = \frac{2}{c^2} u_\delta h_\gamma^\beta \sum_A X_A \frac{\partial F^{A\alpha}}{\partial t_{\langle \gamma\delta \rangle}}$$

The second step is to consider the expressions of V^α, $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$, $I^{\beta\gamma}$, ξ, Ω^α, $\sum^{\alpha\beta}$ at first order with respect to equilibrium and that of h^α at second order; By imposing the relativity principle (see refs. [11], [12]) we obtain $A^{\alpha\beta\gamma}$, $I^{\beta\gamma}$ given by (1.2), (1.3), h^α given by (B.2),

$$\xi = \xi_0 + \chi\pi;$$

$$\Omega^\alpha = \left(l_0 - \frac{\lambda_0}{c^2}\pi \right) u^\alpha - \lambda_1 q^\alpha$$

$$\Sigma_{\alpha\beta} = \sigma_2 t_{\langle \alpha\beta \rangle} + \frac{2}{c^2}\sigma_1 u_{(\alpha} q_{\beta)} + \sigma_0 \pi \left(h_{\alpha\beta} + \frac{3}{c^2} u_\alpha u_\beta \right)$$

and obviously $V^\alpha = nu^\alpha$,

$$T^{\alpha\beta} = t^{\langle \alpha\beta \rangle} + [p(n,\gamma) + \pi] h^{\alpha\beta} + \frac{2}{c^2} u^{(\alpha} q^{\beta)} + \frac{e}{c^2} u^\alpha u^\beta .$$

By using these expressions (A.11)-(A.15) become

$$f_1^\alpha = \left[-\frac{1}{c^2}A_2^0 - \frac{2}{c^2}l_0 + \pi\left(-\frac{1}{c^2}A_2^\pi - \frac{2}{c^4}\lambda_0 + \frac{48}{c^2}C_3\sigma_0\right)\right]u^\alpha - \frac{2}{c^2}C_3\sigma_1 q^\alpha$$

$$f_2^\alpha = -\left(l_0 + \lambda_0\frac{\pi}{c^2}\right)\frac{u^\alpha}{c^2} - 9C_5\sigma_0\frac{\pi}{c^2}u^\alpha$$

$$f_3^\alpha = \frac{1}{3}(A_3^0 - \lambda_1 + 2\sigma_1 C_5)q^\alpha - \sigma_0\pi C_5 u^\alpha$$

$$f_4^\alpha = \frac{-u^\alpha}{c^2}\Big\{ns + \xi_0 n + 2(e+p)l_0 + \pi\Big[A_1^\pi + n\chi +$$
$$+2(e+p)\frac{\lambda_0}{c^2} + 2l_0 + \sigma_0 c^2(nm^2 + 8C_1^0)\Big] -$$
$$-\frac{1}{3}(nm^2 - c_1^0)\sigma_1 q^\alpha\Big\}$$

$$f_1^{\alpha\mu}h_\mu^\beta = u^\alpha q^\beta\left(-\frac{A_3^0}{c^2} + \frac{\lambda_1}{c^2} - 4\frac{c_5}{c^2}\sigma_1\right) +$$
$$+h^{\alpha\beta}\left(l_0 + \frac{\lambda_0}{c^2}\pi + 6C_5\sigma_0\pi\right).$$

After that we can impose (A.5)-(A.10) at first order with respect to equilibrium. The following relations are respectively the coefficients of u^α, πu^α, q^α in (A.5), those of u^α, πu^α, q^α in (A,6), those of u^α, πu^α, q^α in (A.7), those of $g^{\alpha\nu}$, $\pi g^{\alpha\nu}$, $t^{\langle\alpha\nu\rangle}$, $q^\alpha u^\nu$, $q^\nu u^\alpha$ in (A.9) those of $t^{\langle\gamma\delta\rangle}u^\alpha$, $q^\alpha g^{\gamma\delta}$ in (A.10) and those of $t^{\langle\alpha\mu\rangle}$, $q^\alpha u^\mu$, $u^\alpha q^\mu$, $g^{\alpha u}$ $\pi g^{\alpha\mu}$ in (A,8). They are

$$\frac{\partial(ns)}{\partial n} + \xi_0 + e_n l_0 = 0 \tag{A.16}$$

$$\frac{\partial A_1^\pi}{\partial n} + \chi + \sigma_0 c^2\left(m^2 + 2\frac{\partial C_1^0}{\partial n}\right) + e_n\frac{\lambda_0}{c^2} = 0 \tag{A.17}$$

$$\frac{\partial A_2^0}{\partial n} + \frac{c^2}{3}\left(m^2 - \frac{\partial C_1^0}{\partial n}\right)\sigma_1 + p_n\lambda_1 = 0 \tag{A.18}$$

$$\frac{\partial(ns)}{\partial n} + e_\gamma l_0 = 0 \tag{A.19}$$

$$\frac{\partial A_1^\pi}{\partial \gamma} + e_\gamma \frac{\lambda_0}{c^2} + 2\sigma_0 c^2 \frac{\partial C_1^0}{\partial \gamma} = 0 \tag{A.20}$$

$$\frac{\partial A_2^0}{\partial \gamma} + p_\gamma \lambda_1 - \frac{c^2}{3}\sigma_1 \frac{\partial C_1^0}{\partial \gamma} = 0 \tag{A.21}$$

$$A_1^\pi = 0 \tag{A.22}$$

$$2A_1^{\pi^2} + 2C_1^\pi \sigma_0 c^2 = 0 \tag{A.23}$$

$$A_2^\pi + \lambda_1 - \frac{c^2}{3} C_1^\pi \sigma_1 = 0 \tag{A.24}$$

$$A_2^0 + 1_0 = 0 \tag{A.25}$$

$$A_2^\pi + \frac{\lambda_0}{c^2} - 20\sigma_0 C_3 = 0 \tag{A.26}$$

$$A_3^0 + 2\sigma_2 C_3 = 0 \tag{A.27}$$

$$2A_1^q - \frac{\lambda_1}{c^2} - 10C_3 \frac{\sigma_1}{c^2} = 0 \tag{A.28}$$

$$2A_1^t + \sigma_2 C_5 = 0 \tag{A.29}$$

$$A_3^0 - \lambda_1 + 2\sigma_1 C_5 = 0 \tag{A.30}$$

$$\frac{c^2}{3}(nm^2 - C_1^0)\sigma_2 - 1_0 = 0 \tag{A.31}$$

$$-\frac{\lambda_1}{c^2}(e+p) + 2\sigma_1 C_1^0 + \frac{1}{3}(nm^2 - C_1^0)\sigma_1 - \frac{1}{c^2}A_2^0 - \frac{2}{c^2} l_0 = 0 \tag{A.32}$$

$$ns + \xi_0 n + (e+p) l_0 = 0 \tag{A.33}$$

$$(A.34) \qquad A_1^{\pi} + n\chi + (e+p)\frac{\lambda_0}{c^2} + l_0 - \frac{c^2}{3}(nm^2 - 10C_1^0)\sigma_0 = 0$$

Now the next step is to evaluate these conditions (A.16)-(A.34). From (2.4) in the variables n, γ we obtain

$$\frac{\partial s}{\partial \gamma} = \frac{1}{nT}e_\gamma,$$

$$\frac{\partial s}{\partial n} = \frac{1}{nT}\left(e_n - \frac{e+p}{n}\right)$$

and then (A.19), (A.16) give

$$1_0 = -\frac{1}{nT};$$

$$\xi_0 = -s + \frac{e+p}{nT}$$

while (A.33) is identically verified. (A.25) gives

$$A_2^0 = \frac{1}{T};$$

(A.18) and (A.21) form a linear algebraic system in the unknowns λ_1, σ_1, whose solution is

$$\sigma_1 = \frac{3p_n}{p_\gamma\left(m^2 - \frac{\partial C_1^0}{\partial n}\right) + p_n\frac{\partial C_1^0}{\partial \gamma}}\frac{k}{mc^4}$$

$$\lambda_1 = \frac{-k}{mc^2}\frac{m^2 - \frac{\partial C_1^0}{\partial n}}{p_\gamma\left(m^2 - \frac{\partial C_1^0}{\partial n}\right) + p_n\frac{\partial C_1^0}{\partial \gamma}}$$

From (A.31) we obtain

$$\sigma_2 = -\frac{3}{c^2T}\frac{1}{nm^2 - C_1^0}.$$

(A.17), (A.20) (A.34) form a linear algebraic system in the unknowns χ, $\sigma_0 c^2$, λ_0/c^2 whose solution is

$$\chi = \frac{3}{T}D^{-1}\left[-2e_n\frac{\partial C_1^0}{\partial \gamma} + e_\gamma\left(m^2 + 2\frac{\partial C_1^0}{\partial n}\right)\right]$$

$$\sigma_0 = -\frac{3}{Tc^2} e_\gamma D^{-1}$$

$$\lambda_0 = \frac{6c^2}{T} \frac{\partial C_1^0}{\partial \gamma} D^{-1}$$

where

$$D = 6\frac{\partial C_1^0}{\partial \gamma}(e + p - ne_n) + e_\gamma \left(6n\frac{\partial C_1^0}{\partial \gamma} + 4nm^2 - 10C_1^0\right).$$

The relation (A.32) gives a condition on C_1^0

$$(A.35) \qquad \left(m^2 - \frac{\partial C_1^0}{\partial n}\right)(e + p + \gamma p_\gamma) + p_n \left(nm^2 + 5C_1^0 + \gamma\frac{\partial C_1^0}{\partial \gamma}\right) = 0$$

From (A.26), (A.24) we obtain

$$A_2^\pi = -\left(\frac{6}{T}\frac{\partial C_1^0}{\partial \gamma} + \frac{60}{c^2 T} e_\gamma C_3\right) D^{-1};$$

$$C_1^\pi = \frac{-m^2 + \dfrac{\partial C_1^0}{\partial n}}{p_n} - \frac{6\gamma}{p_n} D^{-1} \left[p_\gamma \left(m^2 - \frac{\partial C_1^0}{\partial n}\right) + \right.$$
$$\left. + P_n \frac{\partial C_1^0}{\partial \gamma}\right] \cdot \left(\frac{\partial C_1^0}{\partial \gamma} + 10 e_\gamma \frac{C_3}{c^2}\right).$$

(A.27) and (A.30) give

$$A_3^0 = \frac{6}{c^2 T} \frac{1}{nm^2 - C_1^0} C_3$$

$$C_5 = -\frac{c^2}{6} \frac{m^2 - \dfrac{\partial C_1^0}{\partial n}}{p_n} -$$
$$-\gamma \frac{p_\gamma \left(m^2 - \dfrac{\partial C_1^0}{\partial n}\right) + p_n \dfrac{\partial C_1^0}{\partial \gamma}}{(nm^2 - C_1^0) p_n} C_3.$$

The relations (A.23), (A.28), (A.29) respectively give

$$A_1^{\pi^2} = -\frac{3e_\gamma}{Tp_n} \left(m^2 - \frac{\partial C_1^0}{\partial n}\right) D^{-1} -$$

$$-18\frac{\gamma e_\gamma}{Tp_n}D^{-2}\left[p_\gamma\left(m^2-\frac{\partial C_1^0}{\partial n}\right)+\right.$$

$$\left.+p_n\frac{\partial C_1^0}{\partial \gamma}\right]\cdot\left(\frac{\partial C_1^0}{\partial \gamma}+10e_\gamma\frac{C_3}{c^2}\right);$$

$$A_1^q=\frac{K}{mc^4}\left[p_\gamma\left(m^2-\frac{\partial C_1^0}{\partial n}\right)+\right.$$

$$\left.+p_n\frac{\partial C_1^0}{\partial \gamma}\right]^{-1}\left[15\frac{p_n}{c^2}C_3-\frac{1}{2}\left(m^2-\frac{\partial C_1^0}{\partial n}\right)\right]$$

$$A_1^t=-\frac{1}{4T}\frac{m^2-\frac{\partial C_1^0}{\partial n}}{p_n(nm^2-C_1^0)}-$$

$$-\frac{3}{2}\frac{m}{KT^2}\frac{p_\gamma\left(m^2-\frac{\partial C_1^0}{\partial n}\right)+p_n\frac{\partial C_1^0}{\partial \gamma}}{(nm^2-C_1^0)^2p_n}C_3.$$

The functions e, p, C_1^0 are determined in thermodynamical equilibrium with the statistical mechanics; in the non-degenerate case they are given by (3.1) and (see ref. [9])

$$C_1^0=nm^2\left(1+\frac{6}{\gamma}G\right);$$

this function satisfies condition (A.35).

Then all the unknown functions are determined except for $C_3(n,\gamma)$.

In ref. [9] the entropy principle is partially imposed also in order greater than 1 with respect to equilibrium; as consequence the further restriction is found

$$C_3=-\frac{m}{\gamma}\left(1+\frac{5}{\gamma}G-G^2\right)^{-1}\left(1+\frac{6}{\gamma}G-G^2-\frac{m^3c^3KA}{15n\gamma^5}\right)$$

where A is a single variable function A $(K_2(\gamma/n))$.

If this expression for C_3 is used in the above relations then the corresponding expressions in ref [9] are found; in particular the expressions of $A_1^{\pi^2}$, A_1^q, A_1^t, A_1^π, A_2^0, A_2^π, A_3^0 are those listed in appendix B , eqs. (B.3)-(B.8) .

In remains to impose the inequality $(A.1)_2$; by using (1.3) and the expression of $\Sigma_{\beta\gamma}$ one finds at second order

$$\sigma_2 B_3 < 0;$$

$$\sigma_1 B_4 > 0;$$

$$\sigma_0 B_1^{\pi} > 0.$$

Appendix B. Conditions assuring that the quadratic form Q is positive definite.

To exploit this condition on Q it is useful to notice that from (1.4) it follows,

$$(B.1)\qquad \frac{\partial n}{\partial V^{\mu}} = \frac{u_{\mu}}{c^2};\ \frac{\partial u^{\alpha}}{\partial V_{\mu}} = -\frac{1}{n}h^{\alpha\mu};\ \frac{\partial h^{\alpha\beta}}{\partial V_{\mu}} = -\frac{2}{nc^2}h^{\mu(\alpha}u^{\beta)};$$

$$\frac{\partial t^{\langle\alpha\beta\rangle}}{\partial V_{\mu}} = \frac{2}{nc^2}\left(q^{(\alpha}h^{\beta)\mu} - \frac{1}{3}h^{\alpha\beta}q^{\mu} - t^{\langle\mu(\alpha\rangle}u^{\beta)}\right);$$

$$\frac{\partial q^{\alpha}}{\partial V_{\mu}} = \frac{1}{n}t^{\langle\alpha\mu\rangle} - \frac{u^{\alpha}q^{\mu}}{nc^2} + \frac{e+p+\pi}{n}h^{\alpha\mu};$$

$$\frac{\partial e}{\partial V^{\mu}} = \frac{2}{nc^2}q^{\mu};$$

$$\frac{\partial \pi}{\partial V^{\mu}} = \frac{2}{3nc^2}(1-3p_e)q^{\mu} - p_n\frac{u^{\mu}}{c^2};$$

$$\frac{\partial t^{\langle\alpha\beta\rangle}}{\partial T_{\mu\nu}} = h^{\beta(\mu}h^{\nu)\alpha} - \frac{1}{3}h^{\alpha\beta}h^{\mu\nu};$$

$$\frac{\partial q^{\alpha}}{\partial T_{\mu\nu}} = -h^{\alpha(\mu}u^{\nu)};$$

$$\frac{\partial e}{\partial T_{\mu\nu}} = \frac{u^{\mu}u^{\nu}}{c^2};$$

$$\frac{\partial \pi}{\partial T_{\mu\nu}} = \frac{1}{3}h^{\mu\nu} - P_e\frac{u^{\mu}u^{\nu}}{c^2}.$$

Moreover we need the expression of h^{α}, at second order with respect to equilibrium; from ref. [9] we read

$$(B.2) \qquad h^\alpha = (ns + A_1^\pi \pi + A_1^{\pi^2} \pi^2 + A_1^q q^\beta q_\beta + A_1^t t^{\langle\beta\gamma\rangle} t_{\langle\gamma\beta\rangle}) u^\alpha + \\ + (A_2^0 + A_2^\pi \pi) q^\alpha + A_3^0 t^{\langle\alpha\beta\rangle} q_\beta$$

where (in the non - degenerate case),

$$(B.3) \qquad A_1^{\pi^2} = -\frac{3}{2}\frac{K\gamma^2}{m^2c^4n}\left(1 - \frac{1}{\gamma^2} + \frac{5}{\gamma}G - G^2\right)\left[\frac{3}{\gamma} - \right. \\ \left. - \left(2 - \frac{20}{\gamma^2}\right)G - \frac{13}{\gamma}G^2 + 2G^3\right]^{-2}\left[2 - \frac{5}{\gamma^2} + \left(\frac{19}{\gamma} - \right.\right. \\ \left. - \frac{30}{\gamma^3}\right)G - \left(2 - \frac{45}{\gamma^2}\right)G^2 - \frac{9}{\gamma}G^3 - \frac{m^3c^3KA}{3n\gamma^5}\left(1 - \frac{1}{\gamma^2} + \right. \\ \left.\left. + \frac{5}{\gamma}G - G^2\right)\right];$$

$$(B.4) \qquad A_1^q = \frac{K\gamma}{2m^2c^6n}\left(1 + \frac{5}{\gamma}G - G^2\right)^{-2}\left[\frac{5}{\gamma} + \right. \\ \left. + \left(\frac{30}{\gamma^2} - 1\right)G - \frac{10}{\gamma}G^2 + G^3 - \frac{m^3c^3KA}{3n\gamma^6}\right]$$

$$(B.5) \qquad A_1^t = \frac{-k\gamma^2}{4m^2c^4nG^2}\left(1 + \frac{6}{\gamma}G - \frac{m^3c^3KA}{15n\gamma^5}\right)$$

$$(B.6) \qquad A_2^0 = \frac{1}{T}; \; A_1^\pi = 0$$

$$(B.7) \qquad A_2^\pi = \frac{k\gamma}{m^2c^4n}\left(1 + \frac{5}{\gamma}G - G^2\right)^{-1}\left[-\frac{3}{\gamma} + \right. \\ \left. + \left(2 - \frac{20}{\gamma^2}\right)G + \frac{13}{\gamma}G^2 - 2G^3\right]^{-1}\left[-2 + \frac{5}{\gamma^2} - \right. \\ - \left(\frac{22}{\gamma} - \frac{30}{\gamma^3}\right)G + \left(4 - \frac{65}{\gamma^2}\right)G^2 + \frac{22}{\gamma}G^3 - 2G^4 + \\ \left. + \left(1 - \frac{1}{\gamma^2} + \frac{5}{\gamma}G - G^2\right)\frac{m^3c^3KA}{3n\gamma^5}\right];$$

$$(B.8) \qquad A_3^0 = \frac{k\gamma}{m^2c^4nG}\left(1+\frac{5}{\gamma}G-G^2\right)^{-1}\Big(1+ + \frac{6}{\gamma}G - G^2 - \frac{m^3c^3KA}{15n\gamma^5}\Big)$$

where $\gamma = \dfrac{mc^2}{KT}$, T is the temperature, K the Boltzmann's constant, $K_\nu(\gamma)$ is the Bessel function of order ν

$$K_\nu(\gamma) = \int_0^\infty \cosh \nu\rho e^{-\gamma\cos\rho} d\rho$$

satisfying the relations

$$K_{\nu+1}(\gamma) - K_{\nu-1}(\gamma) = \frac{2\nu}{\gamma}K_\nu(\gamma);$$

$$\frac{d}{d\gamma}K_\nu(\gamma) = -K_{\nu-1}(\gamma) - \frac{\nu}{\gamma}K_\nu(\gamma),$$

$$G = K_3/K_2$$

and then

$$\frac{dG(\gamma)}{d\gamma} = -1 - \frac{5}{\gamma}G + G^2;$$

moreover A is an arbitrary function of the variable $\dfrac{K_2}{n\gamma}$. We can use these expressions and those for the derivatives of n, u^α, $h^{\alpha\beta}$, $t^{\langle\alpha\beta\rangle}$, q^α, e, π with respect to V_μ, $T_{\mu\nu}$;

then, if we define $\xi = \xi_\alpha u^\alpha c^{-2}$, $\bar{\xi}^\alpha = -h^{\alpha\mu}\xi_\mu$ (and therefore $\xi^\alpha = \bar{\xi}^\alpha + \xi u^\alpha$), the quadratic form Q at equilibrium is

$$Q = -2c^2\xi[A_1^{\pi^2}(\delta\pi)^2 + A_1^q\delta q_\beta\delta q^\beta + A_1^t\delta t^{\langle\beta\gamma\rangle}\delta t_{\langle\beta\gamma\rangle}] -$$

$$-2A_2^\pi\delta\pi\delta q^\alpha\bar{\xi}_\alpha - 2A_3^0\bar{\xi}_\alpha\delta t^{\langle\alpha\beta\rangle}\delta q_\beta - \xi(dn)^2c^2\frac{\partial^2(ns)}{\partial n^2} -$$

$$-2\xi\delta n\delta ec^2\frac{\partial^2(ns)}{\partial e\partial n} - \xi c^2(\delta e)^2\frac{\partial^2(ns)}{\partial e^2} + \xi\delta u_\mu\delta u^\mu\frac{e+p}{T} -$$

$$-2\xi\frac{1}{T}[\delta q_\mu\delta u^\mu + (e+p)\delta u_\mu\delta u^\mu] + 2\bar{\xi}^\mu\delta u_\mu\delta T\frac{e+p}{T^2} +$$

$$+2\delta A_2^0[(e+p)\bar{\xi}^\alpha \delta u_\alpha + \bar{\xi}^\alpha \delta q_\alpha]+$$

$$+2A_2^0[\bar{\xi}_\mu \delta t^{\langle\mu\nu\rangle}\delta u_\nu - (\delta p + \delta\pi)\bar{\xi}_\mu \delta u^\mu].$$

Let us now define

$$\bar{T}^{\alpha\beta} = t^{\langle\alpha\beta\rangle} + \pi h^{\alpha\beta}$$

from which it follows

$$\pi = \frac{1}{3}\bar{T}^{\alpha\beta} h_{\alpha\beta}$$

$$t^{\langle\alpha\beta\rangle} = \bar{T}^{\alpha\beta} - \pi h^{\alpha\beta}$$

and consequently, at equilibrium,

$$\delta\pi = \frac{1}{3} h_{\alpha\beta}\delta\bar{T}^{\alpha\beta}$$

$$\delta t^{\langle\alpha\beta\rangle} = \delta\bar{T}^{\alpha\beta} - h^{\alpha\beta}\delta\pi$$

$$\delta t^{\langle\alpha\beta\rangle}\delta t_{\langle\alpha\beta\rangle} = \delta\bar{T}^{\alpha\beta}\delta\bar{T}_{\alpha\beta} - 3(\delta\pi)^2$$

$$\bar{\xi}^\alpha \delta t_{\langle\alpha\beta\rangle}\delta q^\beta = \bar{\xi}^\alpha \delta\bar{T}_{\alpha\beta}\delta q^\beta + \bar{\xi}_\alpha \delta q^\alpha \delta\pi$$

$$\bar{\xi}_\alpha \delta t^{\langle\alpha\beta\rangle}\delta u_\beta = \bar{\xi}^\alpha \delta\bar{T}_{\alpha\beta}\delta u^\beta + \bar{\xi}^\alpha \delta u_\alpha \delta\pi;$$

Let us also remember the Gibbs relation

$$Tds = d\left(\frac{e}{n}\right) + pd\left(\frac{1}{n}\right) \tag{B.9}$$

that, if e and n are used as independent variables, gives

$$\frac{\partial s}{\partial e} = \frac{1}{nT}; \quad \frac{\partial s}{\partial n} = -\frac{e+p}{n^2 T}, \tag{B.10}$$

while if n, T are the independent variables, gives

$$\frac{\partial s}{\partial T} = \frac{1}{nT} e_T; \quad \frac{\partial s}{\partial n} = -\frac{e+p}{n^2 T} + \frac{1}{nT} e_n; \tag{B.11}$$

moreover the symmetry condition on (B.11) is

$$n e_n + T p_T - (e+p) = 0. \tag{B.12}$$

By using these equations and $A_2^0 = \frac{1}{T}$ one obtains

$$Q = -\xi c^2(\delta n \ \delta e) \begin{pmatrix} \frac{\partial^2(ns)}{\partial n^2} & \frac{\partial^2(ns)}{\partial e \partial n} \\ \frac{\partial^2(ns)}{\partial e \partial n} & \frac{\partial^2(ns)}{\partial e^2} \end{pmatrix} \begin{pmatrix} \delta n \\ \delta e \end{pmatrix} -$$

$$-2\xi \frac{1}{T} \delta q_\mu \delta u^\mu - \frac{2}{T^2} \bar{\xi}^\alpha \delta q_\alpha \delta T - \frac{2}{T} \bar{\xi}^\alpha \delta u_\alpha \delta P -$$

$$-2c^2\xi(A_1^{\pi^2} - 3A_1^t)(\delta\pi)^2 - 2\xi c^2 A_1^q \delta q_\mu \delta q^\mu - 2c^2 \xi A_1^t \delta \bar{T}^{\beta\gamma} \delta \bar{T}_{\beta\gamma} -$$

$$-\xi \frac{e+p}{T} \delta u_\mu \delta u^\mu - 2(A_1^\pi + A_3^0) \delta\pi \delta q^\alpha \bar{\xi}_\alpha -$$

$$-2A_3^0 \bar{\xi}_\alpha \delta \bar{T}^{\alpha\beta} \delta q_\beta + \frac{2}{T} \bar{\xi}^\mu \delta \bar{T}_{\mu\nu} \delta u^\nu;$$

But

$$(\delta n \ \delta e) \begin{pmatrix} \frac{\partial^2(ns)}{\partial n^2} & \frac{\partial^2(ns)}{\partial e \partial n} \\ \frac{\partial^2(ns)}{\partial e \partial n} & \frac{\partial^2(ns)}{\partial e^2} \end{pmatrix} \begin{pmatrix} \delta n \\ \delta e \end{pmatrix} =$$

$$= \delta \left[\frac{\partial(ns)}{\partial n} \right] \delta n + \delta \left[\frac{\partial(ns)}{\partial e} \right] \delta e$$

that by using (B.10) becomes equal to

$$\delta \left(s - \frac{e+p}{nT} \right) \delta n + \delta \left(\frac{1}{T} \right) \delta e$$

that by using (B.11) becomes

$$-\frac{p_n}{nT}(\delta n)^2 - \frac{e_T}{T^2}(\delta T)^2 - \frac{1}{n^2 T}[n e_n + T p_T - (e+p)] \delta n \delta T$$

that is equal to

$$-\frac{1}{nT^2}[p_n T (\delta n)^2 + n e_T (\delta T)^2]$$

thanks to (B.12).

Consequently, in the reference frame Σ where $u^\alpha \equiv (c, 0, 0, 0)$; $\bar{\xi}^\alpha \equiv (0, \xi^1, 0, 0)$ and by using

$$\delta\pi = \frac{1}{3}(\delta \bar{T}^{11} + \delta \bar{T}^{22} + \delta \bar{T}^{33})$$

the quadratic form Q becomes

$$Q = \xi \left[A(\delta \bar{T}^{23})^2 + \sum_{i,j=1}^{7} A_{ij} X^i X^j + \sum_{i,j=1}^{3} B_{ij}(Y^i Y^j + Z^i Z^j) \right]$$

where

$$A = -4c^2 A_1^T$$

$$X^1 = \delta n;\ X^2 = \delta T;\ X^3 = \delta \bar{T}^{11};\ X^4 = \delta \bar{T}^{22};$$

$$X^5 = \delta \bar{T}^{33};\ X^6 = \delta u_1;\ X^7 = \delta q_1$$

$$Y^1 = \delta u^2;\ Y^2 = \delta q^2;\ y^3 = \delta \bar{T}_{12};$$

$$Z^1 = \delta u^3;\ Z^2 = \delta q^3;\ Z^3 = \delta \bar{T}_{13};$$

$$A_{11} = \frac{c^2}{nT} p_n;\ A_{12} = A_{13} = A_{14} = A_{15} = 0;$$

$$A_{16} = -\xi^{-1} \xi^1 \frac{p_n}{T};\ A_{17} = 0$$

$$A_{22} = \frac{c^2}{T^2} e_T;\ A_{23} = A_{24} = A_{25} = 0;$$

$$A_{26} = -\xi^{-1} \xi^1 \frac{p_T}{T};\ A_{27} = -\xi^{-1} \xi^1 \frac{1}{T^2}$$

$$A_{33} = -\frac{2}{9} c^2 (A_1^{\pi^2} + 6A_1^t);$$

$$A_{34} = A_{35} = -\frac{2}{9} c^2 (A_1^{\pi^2} - 3A_1^t);$$

$$A_{36} = -\frac{1}{T} \xi^{-1} \xi^1;$$

$$A_{37} = \frac{1}{3} \xi^{-1} \xi^1 \left(\frac{2}{3} A_3^0 - A_2^\pi \right);$$

$$A_{44} = A_{33};\ A_{45} = A_{34};\ A_{46} = 0;$$

$$A_{47} = -\frac{1}{3} (A_2^\pi + A_3^0) \xi^{-1} \xi^1;$$

$$A_{55} = A_{33};\ A_{56} = 0;\ A_{57} = A_{47};$$

$$A_{66} = \frac{e+p}{T};\ A_{67} = \frac{1}{T};\ A_{77} = 2c^2 A_1^q;$$

$$B_{11} = \frac{e+p}{T};\ B_{12} = \frac{1}{T};\ B_{13} = \frac{1}{T}\xi^{-1}\xi^{1};$$

$$B_{22} = 2c^2 A_1^q;\ B_{23} = \xi^{-1}\xi^1 A_3^0;$$

$$B_{33} = -4c^2 A_1^t;\ A_{ij} = A_{ji};\ B_{ij} = B_{ji}.$$

Now the classical stability conditions $p_n > 0;\ e_T > 0$ show that $A_{11} > 0$; consequently Q is definite iff $\xi^{-1}Q$ is positive definite, i.e. iff $A > 0;\ A_i > 0;\ B_i > 0$ where A_i, B_i are the determinants with i rows, obtained by deleting the last $7-i$, $3-i$ rows and columns of the matrices A_{ij}, B_{ij} respectively.

Now, after some calculations, we obtain

$$A = -4c^2 A_1^t;\ B_1 = \frac{e+p}{T};$$

$$B_2 = \frac{1}{T^2}[2c^2(e+p)TA_1^q - 1]$$

$$B_3 = \frac{1}{T^2}[2c^2 A_1^q + 2A_3^0 + (e+p)T(A_3^0)^2]\eta -$$

$$-4\frac{c^2}{T^2}A_1^t[2c^2(e+p)TA_1^q - 1]$$

where

$$\eta = (\xi^{-1}\xi^1)^2;$$

$$A_1 = \frac{c^2}{nT}p_n;\ A_2 = \frac{c^2}{nT^3}p_n e_T > 0;$$

$$A_3 = -\frac{2}{9}\frac{c^6}{nT^3}p_n e_T(A_1^{\pi^2} + 6A_1^t);$$

$$A_4 = \frac{4}{9}c^8\frac{1}{nT^3}p_n e_T A_1^t(2A_1^{\pi^2} + 3A_1^t);$$

$$A_5 = -\frac{8}{3}c^{10}\frac{1}{nT^3}p_n e_T (A_1^t)^2 A_1^{\pi^2};$$

$$A_6 = -\frac{4}{9}\frac{c^8}{nT^5}p_n A_1^t \left\{\left[\left(3A_1^t + 2A_1^{\pi^2}\right)e_T - \right.\right.$$

$$\left.\left. -6TA_1^t A_1^{\pi^2}(Tp_T^2 + ne_T p_n)\right]\eta + 6TA_1^t A_1^{\pi^2} e_T(e+p)c^2\right\}$$

$$A_7 = \alpha\eta^2 + \beta\eta + \delta$$

with

$$\alpha = \frac{1}{A_5}\left\{\left[M^{*11}\left(\frac{p_n}{T}\right)^2 + M^{*22}\left(\frac{p_T}{T}\right)^2 + M^{*33}\frac{1}{T^2}\right]\cdot\right. \tag{B.13}$$

$$\cdot\left[M^{*22}\frac{1}{T^4} + M^{*33}(\bar{A}_{37})^2 + 2M^{*33}(\bar{A}_{47})^2 + 4M^{*34}\bar{A}_{37}\bar{A}_{47} + \right.$$

$$\left.\left. + 2M^{*34}(\bar{A}_{47})^2\right] - \frac{1}{T^2}\left(\frac{p_T}{T^2}M^{*22} - M^{*33}\bar{A}_{37} - 2M^{*34}\bar{A}_{47}\right)^2\right\}$$

$$\beta = -\frac{e+p}{T}\left[M^{*22}\frac{1}{T^4} + M^{*33}(\bar{A}_{37})^2 + 2M^{*33}(\bar{A}_{47})^2 + \right. \tag{B.14}$$

$$\left. + 4M^{*34}\bar{A}_{37}\bar{A}_{47} + 2M^{*34}(\bar{A}_{47})^2\right] + \frac{2}{T^2}\left(\frac{p_T}{T^2}M^{*22} - \right.$$

$$\left. - M^{*33}\bar{A}_{37} - 2M^{*34}\bar{A}_{47}\right) - 2c^2 A_1^q\left[M^{*11}\left(\frac{p_n}{T}\right)^2 + \right.$$

$$\left. + M^{*22}\left(\frac{p_T}{T}\right)^2 + M^{*33}\frac{1}{T^2}\right]$$

$$\delta = \frac{1}{T^2}[2c^2(e+p)TA_1^q - 1]A_5 \tag{B.15}$$

$$M^{*11} = A_5\frac{nT}{c^2 p_n}; \; M^{*22} = A_5\frac{T^2}{c^2 e_T};$$

$$M^{*33} = \frac{4}{9}\frac{c^8}{nT^3}p_n e_T A_1^t(2A_1^{\pi^2} + 3A_1^t);$$

$$M^{*34} = -\frac{4}{9}\frac{c^8}{nT^3}p_n e_T A_1^t(A_1^{\pi^2} - 3A_1^t)$$

$$\bar{A}_{37} = \frac{2}{9}A_3^0 - \frac{1}{3}A_2^\pi;$$

$$\bar{A}_{47} = -\frac{1}{3}(A_3^0 + A_2^\pi);$$

Consequently $A > 0$, $A_5 > 0$ and $B_2 > 0$ give the conditions (2.2) from which $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, $A_4 > 0$, $B_1 > 0$ are verified as consequence. The conditions $B_3 > 0$, $A_6 > 0$ must be verified for all $\eta \in [0, c^2[$.

Now they already hold for $\eta = 0$ and are linear in η; therefore they are verified for all $\eta \in [0, c^2[$ iff $B_3 \geq 0$; $A_6 \geq 0$ for $\eta = c^2$. Consequently the requested convexity holds if (2.2) and (2.3)-(2.5) are satisfied .

For the evaluations of B_3, A_6, A_7, the following properties of matrices have been used :

Let M be a matrix with $\det M \neq 0$ and M^* its adjoined matrix; then

$$\det \begin{pmatrix} M & \mathbf{v} \\ \mathbf{u}^T & 0 \end{pmatrix} = -\mathbf{v} \cdot M^* \mathbf{u}$$

$$\det \begin{pmatrix} M & \mathbf{u}_1 & \mathbf{v}_1 \\ \mathbf{u}^T & 0 & 0 \\ \mathbf{v}^T & 0 & 0 \end{pmatrix} =$$

$$= (\det M)^{-1} \begin{vmatrix} \mathbf{u}_1 \cdot M^* \mathbf{u} & \mathbf{v}_1 \cdot M^* \mathbf{u} \\ \mathbf{u}_1 \cdot M^* \mathbf{v} & \mathbf{v}_1 \cdot M^* \mathbf{v} \end{vmatrix};$$

$$\det \begin{pmatrix} M & x\mathbf{v} & x\mathbf{u} \\ x\mathbf{v}^T & a & b \\ x\mathbf{u}^T & b & c \end{pmatrix} = \det \begin{pmatrix} M & 0 & 0 \\ 0^T & a & b \\ \mathbf{v}^T & b & c \end{pmatrix} +$$

$$+ \left\{ a \det \begin{pmatrix} M & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix} - b \left[\det \begin{pmatrix} M & \mathbf{v} \\ \mathbf{u}^T & 0 \end{pmatrix} + \right. \right.$$

$$\left. \left. + \det \begin{pmatrix} M & \mathbf{u} \\ \mathbf{v}^T & 0 \end{pmatrix} \right] + c \det \begin{pmatrix} M & \mathbf{v} \\ \mathbf{v}^T & 0 \end{pmatrix} + \right\} x^2 +$$

$$+ \left[\det \begin{pmatrix} M & \mathbf{v} & \mathbf{u} \\ \mathbf{v}^T & 0 & 0 \\ \mathbf{u}^T & 0 & 0 \end{pmatrix} \right] x^4.$$

Appendix C. Orderings in growing powers of $1/\gamma$.

In section 3 , the following orderings of the first members of (2.2)-(2.5) in growing powers of $1/\gamma$, have been used. They are obtained from (B.3)-(B.8), (B.13)-(B.15) and by using the expansions (3.2)-(3.4). We have

$$2A_1^q(e+p)c^2T - 1 = f_1\left(\frac{1}{\gamma}\right) - G\left(1+\frac{5}{\gamma}G - G^2\right)^{-2}\frac{m^3c^3KA}{3n\gamma^6}$$

where

$$f_1\left(\frac{1}{\gamma}\right) = \left(1+\frac{5}{\gamma}G - G^2\right)^{-2} G\left[\frac{5}{\gamma}\left(\frac{30}{\gamma^2}-1\right)G - \right.$$
$$\left. -\frac{10}{\gamma}G^2 + G^3\right] - 1,$$

from which

$$\lim_{1/\gamma\to 0}\gamma^{-2}f_1\left(\frac{1}{\gamma}\right) = \frac{2}{5} > 0 \Rightarrow f_1 > 0$$

and consequently $(2.2)_2$ is verified for little values of T.

Similarly we have

$$A_1^{\pi^2} = \frac{K}{m^2c^4n}\left\{-\frac{3}{5}\gamma^4[1+g_3(\gamma)]+\right.$$
$$\left.+\frac{4}{25}\gamma^4[1+g_4(\gamma)]A^*\right\}$$

with

$$A^* = \frac{m^3c^3KA}{3n\gamma}$$

and therefore $(2.2)_3$ is verified for litte values of temperature.

In the same way we obtain

$$A_1^t = -\frac{K}{4m^2c^4n}\gamma^2\left[1+\frac{1}{\gamma}g_5(\gamma) - \right.$$
$$\left. -\frac{1}{5}\frac{1}{\gamma^4}A^*\left(1+\frac{1}{\gamma}g_6(\gamma)\right)\right];$$

$$A_1^q = \frac{K}{nm^2c^6}\left\{\frac{1}{5}\gamma^3\left[1+\frac{1}{\gamma}g_7(\gamma)\right] - \right.$$

$$-\frac{2}{25}\left[1+\frac{1}{\gamma}g_8(\gamma)\right]A^*\Bigg\};$$

$$e+p=nmc^2\left[1+\frac{1}{\gamma}g_9(\gamma)\right];$$

$$A_3^0=\frac{K}{nm^2c^4}\left\{\frac{2}{5}\gamma^2\left[1+\frac{1}{\gamma}g_{10}(\gamma)\right]-\right.$$

$$\left.-\frac{2}{25}\frac{1}{\gamma}A^*\left[1+\frac{1}{\gamma}g_{11}(\gamma)\right]\right\};$$

$$A_2^\pi=\frac{K}{nm^2c^4}\left\{-\frac{4}{5}\gamma^3\left[1+\frac{1}{\gamma}g_{12}(\gamma)\right]+\right.$$

$$\left.+\frac{4}{25}\gamma A^*\left[1+\frac{1}{\gamma}g_{13}(\gamma)\right]\right\};$$

$$A_5=\frac{K^7}{n^3m^8c^3}\gamma^6\left\{\frac{3}{20}\gamma^4\left[1+\frac{1}{\gamma}g_{14}(\gamma)\right]-\right.$$

$$-\frac{1}{25}\gamma^4A^*\left[1+\frac{1}{\gamma}g_{15}(\gamma)\right]+\frac{2}{125}(A^*)^2\left[1+\frac{1}{\gamma}g_{16}(\gamma)\right]-$$

$$\left.-\left(\frac{1}{25}\right)^2\frac{1}{\gamma^4}(A^*)^3\left[1+\frac{1}{\gamma}g_{17}(\gamma)\right]\right\}>0,$$

$$M^{*11}=\frac{n}{c^2K}A_5;$$

$$M^{*22}=\frac{2}{3}\frac{m^2c^2}{nk^3}A_5\frac{1}{\gamma^2}\left[1+\frac{1}{\gamma}g_{18}(\gamma)\right];$$

$$M^{*34}=\frac{K^6}{n^2c^4m^6}\left\{\frac{1}{10}\gamma^8\left[1+\frac{1}{\gamma}g_{19}(\gamma)\right]-\right.$$

$$-\frac{2}{75}\gamma^8A^*\left[1+\frac{1}{\gamma}g_{20}(\gamma)\right]+$$

$$\left.+\frac{2}{375}\gamma^4(A^*)^2\left[1+\frac{1}{\gamma}g_{21}(\gamma)\right]\right\}=$$

$$=\frac{1}{6}\frac{A_5}{c^2A_1^t}\left[1+\frac{1}{\gamma}g_{22}(\gamma)\right]$$

$$M^{*33} = -2M^{*34}\left[1+\frac{1}{\gamma}g_{23}(\gamma)\right];$$

$$\bar{A}_{37} = -\frac{1}{3}A_2^\pi\left[1+\frac{1}{\gamma}g_{24}(\gamma)\right];$$

$$\bar{A}_{47} = -\frac{1}{3}A_2^\pi\left[1+\frac{1}{\gamma}g_{25}(\gamma)\right];$$

$$\alpha = \frac{1}{18\cdot 5^4}\frac{K^9}{n^3c^{14}m^{10}}\gamma^5\left\{5\left[1+\frac{1}{\gamma}g_{26}(\gamma)\right]-\right.$$
$$\left.-\frac{1}{\gamma^4}A^*\left[1+\frac{1}{\gamma}g_{27}(\gamma)\right]\right\}\cdot\left\{15\left[1+\frac{1}{\gamma}g_{28}(\gamma)\right]-\right.$$
$$\left.-4A^*\left[1+\frac{1}{\gamma}g_{29}(\gamma)\right]\right\}\left\{35\left[1+\frac{1}{\gamma}g_{30}(\gamma)\right]-\right.$$
$$\left.-3\frac{1}{\gamma^4}A^*\left[1+\frac{1}{\gamma}g_{31}(\gamma)\right]\right\};$$

$$\beta = \frac{1}{2^4\cdot 3\cdot 5^5}\frac{K^9}{n^3c^{12}m^{10}}\gamma^8\left\{5\left[1+\frac{1}{\gamma}g_{32}(\gamma)\right]-\right.$$
$$\left.-\frac{1}{\gamma^4}A^*\left[1+\frac{1}{\gamma}g_{33}(\gamma)\right]\right\}\left\{15\left[1+\frac{1}{\gamma}g_{34}(\gamma)\right]-\right.$$
$$\left.-4A^*\left[1+\frac{1}{\gamma}g_{35}(\gamma)\right]\right\}\left\{-5\cdot 77\gamma^5\left[1+\frac{1}{\gamma}g_{36}(\gamma)\right]+\right.$$
$$\left.+4\cdot 36\gamma^2A^*\left[1+\frac{1}{\gamma}g_{37}(\gamma)\right]-\frac{16}{\gamma^2}(A^*)^2\left[1+\frac{1}{\gamma}g_{38}(\gamma)\right]\right\}<0$$

$$\delta = \frac{1}{2\cdot 5^5}\frac{K^9}{n^3m^{10}c^{10}}\gamma^{11}\left\{5\left[1+\frac{1}{\gamma}g_{39}(\gamma)\right]-\right.$$
$$\left.-\frac{1}{\gamma^4}A^*\left[1+\frac{1}{\gamma}g_{40}(\gamma)\right]\right\}\left\{15\left[1+\frac{1}{\gamma}g_{41}(\gamma)\right]-\right.$$
$$\left.-4A^*\left[1+\frac{1}{\gamma}g_{38}(\gamma)\right]\right\}\cdot\left\{25\gamma^3\left[1+\frac{1}{\gamma}g_{43}(\gamma)\right]-\right.$$
$$\left.-10A^*\left[1+\frac{1}{\gamma}g_{44}(\gamma)\right]+\frac{2}{\gamma^4}(A^*)^2\left[1+\frac{1}{\gamma}g_{45}(\gamma)\right]\right\}>0.$$

After that we see that the first member of (2.3) becomes

$$\frac{K}{nm^2c^4}\left\{\frac{2}{5}\gamma^4\left[1+\frac{1}{\gamma}g_{46}(\gamma)\right]-\frac{4}{25}A^*\gamma\left[1+\frac{1}{\gamma}g_{47}(\gamma)\right]+\right.$$
$$\left.+\left(\frac{4}{25}\right)^2\frac{1}{\gamma^3}(A^*)^2\left[1+\frac{1}{\gamma}g_{48}(\gamma)\right]\right\}$$

and therefore (2.3) is verified for little values of T.

The first member of (2.4) is

$$\frac{K^2}{m^2c^4}\left\{\frac{27}{20}\gamma^5\left[1+\frac{1}{\gamma}g_{49}(\gamma)\right]-\right.$$
$$-\frac{9}{25}\gamma^5A^*\left[1+\frac{1}{\gamma}g_{50}(\gamma)\right]+$$
$$\left.+\frac{9}{125}\gamma(A^*)^2\left[1+\frac{1}{\gamma}g_{51}(\gamma)\right]\right\}$$

and therefore (2.4) is verified for little values of temperature.

Moreover

$$2\alpha c^2+\beta=\beta\left[1+\frac{1}{\gamma}g_{52}(\gamma)\right]<0$$

and consequently

$$\frac{dA_7}{d\eta}=2\alpha\eta+\beta$$

is negative for $\eta=0$ and for $\eta=c^2$; therefore from its linearity it follows

$$\frac{dA_7}{d\eta}<0\quad\forall\eta\in[0,c^2].$$

From this fact we see that

$$A_7(\eta)>A_7(c^2)=\alpha c^4+\beta c^2+\delta=\gamma\left[1+\frac{1}{\gamma}g_{53}(\gamma)\right]>0\qquad\forall\,\eta\in[0,c^2]$$

This shows that also the condition (2.5) is verified for little values of temperature.

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REFERENCES

[1] Anile A.M., Pennisi S., *Fluid Models for Relativistic Electron Beams*, Continuum Mech. Thermodyn. **1** (1989) 267 .

[2] Anile A.M.,Pennisi S., *Fluid Models for Relativistic Electron Beams. An independent derivation*, submitted to Physics of Fluids.

[3] Boillat G., *Sur l'existence et la Recherche d'Equation de Conservation Supplèmentaires pour les Systemes Hyperboliques*, C.R. Acad. Sci. Paris **278** - A, (1974) 909 .

[4] Boillat G., *Wave velocities in relativistic Extended Thermodynamics*, In ISIMM Symposium on Kinetic Theory and Extended thermodynamics (Bologna 1987). Ed. by I.Müller and T.Ruggeri, Pitagora (1987) 55 .

[5] Dreyer W., *Maximisation of Entropy for relativistic and degenerate gases in non equilibrium.* In ISIMM Symposium on Kinetic Theory and Extended Thermodynamics (Bologna 1987). Ed. by I.Müller and T.Ruggeri, Pitagora (1987) 107 .

[6] Friedrichs K.O., *Symmetric Hyperbolic Linear Differential Equations*, Comm. Pure Appl. Math. 7, (1954) 345 .

[7] Friedrichs K.O., Lax P.D., *Systems of conservation Equations with a Convex Extension*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971) 1686 .

[8] Liu I - Shih, *Method of Lagrange Multipliers for Exploitation of the Entropy Principle*, Arch. Rational Mech. Anal. **46**, (1972) 131 .

[9] Liu I - Shih, Müller I., Ruggeri T. *Relativistic thermodynamics of Gases*, Ann. of Phys. **169**, (1986) 191 .

[10] Müller I., *Relativistic Extended Thermodynamics II*, Lecture Notes in Mathematics, Relativistic Fluid Dynamics, Noto 1987, Springer Verlag, (1987) 257 .

[11] Pennisi S., *Some Representation Theorems in a 4-dimensional inner product space*, Suppl. B.U.M.I. Fisica Matematica, 5, (1986) 191 .

[12] Pennisi S., Trovato M., *Mathematical Characterization of Functions Underlying the Principle of Relativity*, accepted for pubblication in Le Matematiche.

[13] Ruggeri T., *Struttura dei Sistemi alle Derivate Parziali compatibili con un Principio di Entropia*, Suppl. B.U.M.I. del G.N.F.M., Fisica

Matematica 4, (1985) 5 .

[14] Ruggeri T., *Convexity and symmetrization in relativistic theories*, Continuum Mech. Thermodyn. 2 (1990) 163 .

[15] Ruggeri T., Strumia A., *Main Field and Convex Covariant Density for Quasi - Linear Hyperbolic Systems; Relativistic Fluid Dynamics*, Ann. Inst. H. Poincarè 34 (1), (1981) 165 .

[16] Strumia A., *Wave Propagation and Symmetric Hyperbolic Systems of Conservation Laws with constrained Field Variables* II. Il nuovo cimento, 101B (1988) 19

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